The Structure Function of Random Point Processes: Fluctuations and Rigidity

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Abstract

We consider a translation invariant point process in $\mathbb{R}^d$ or Z^d. Let $V(N_B)$ be the variance in the number of points, N_B, in a ball B of volume $|B|$. Generally, such as when particles with short range interactions are distributed according to a Gibbs measure, $V(N_B)/|B| > 0$. There are however many interesting cases when $V(N_B)/|B| \to 0$ as $|B| \to \infty$. Such processes are called hyperuniform (or superhomogeneous). This occurs when the structure function $S(k)$, the Fourier transform of the "full" pair correlation function, $G(r) = n\delta(r) + n^2[g(r) - 1]$, n the density, which is always non-negative, vanishes at $k = 0$, $S(k) = 0$. Just how fast $V(N_B)/|B|$ goes to zero depends on the way $S(k)$ behaves as $k \to 0$. I will discuss examples of such hyperuniform systems both old (Coulomb systems) and recent (facilitated point process). When $S(k)$ vanishes in an open set M in k-space (which may or may not include the origin) the system is maximally "rigid". Rigidity describes the amount of information about the points in B given the configuration of points outside of B. This can be zero as zero in a Poisson process or "maximal" where the exact position of the points in B are determined by the configuration outside B. Such systems also have other "crystalline" properties. (This is joint work with Subhro Gosh).

Fluctuations

Consider a translation invariant ergodic point process in $\mathbb{R}^d$ or $\mathbb{Z}^d$. Let N_Λ be the number of particles in a region Λ, with $\langle N_\Lambda \rangle = \rho|\Lambda|$.

For "most" systems, including those described by a Gibbs measure with "short range" interactions, the size of fluctuations in N_Λ, as measured by the variance $\mathrm{Var}(N_\Lambda) \equiv \langle N_\Lambda^2 \rangle - \langle N_\Lambda \rangle^2$ will grow like $|\Lambda|$ as in a Poisson process, or faster as at a liquid-vapor critical point.

There are, however, many interesting cases where the variance is sub-extensive, i.e.

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\mathrm{Var}(N_\Lambda)}{|\Lambda|} \to 0. \tag{1}$$

Such systems are called hyperuniform or superhomogeneous.

Variance and the pair correlation function

Note that $N_\Lambda = \sum_{i=1}^{\infty} \chi_\Lambda(\underline{r}_i)$, where $\{\underline{r}_i\}$ are the positions of the point processes, and

$$\chi_\Lambda(\underline{y}) = \begin{cases} 1 & \underline{y} \in \Lambda, \\ 0 & \underline{y} \notin \Lambda. \end{cases} \tag{2}$$

For a translation invariant system we then have

$$\mathrm{Var}(N_\Lambda) = \int_\Lambda \int_\Lambda \mathrm{d}\underline{r}_1 \mathrm{d}\underline{r}_2 G(\underline{r}_1 - \underline{r}_2) \tag{3}$$

$$= |\Lambda| \int_{\mathbb{R}^d} G(\underline{r}) \mathrm{d}\underline{r} - \int_{\mathbb{R}^d} G(\underline{r}) \alpha_\Lambda(\underline{r}) \mathrm{d}\underline{r}, \tag{4}$$

where $G(\underline{r})$ is the truncated "full" pair correlation function,

$$G(\underline{r}_1 - \underline{r}_2) = \rho\delta(\underline{r}_1 - \underline{r}_2) + \rho_2(\underline{r}_1 - \underline{r}_2) - \rho^2, \tag{5}$$

$$\alpha_\Lambda(\underline{r}) = \int \chi_\Lambda(\underline{r} + \underline{r}_1)[1 - \chi_\Lambda(\underline{r}_1)] \mathrm{d}\underline{r}_1, \tag{6}$$

Hyperuniform systems

When $\Lambda \to \mathbb{R}^d$ in a self similar way α_Λ will grow like the surface area $|\partial\Lambda|$ (with $|\partial\Lambda| = 2$ for $d = 1$). Dividing $\mathrm{Var}(N_\Lambda)$ by $|\Lambda|$, we get

$$\lim \frac{\mathrm{Var}(N_\Lambda)}{|\Lambda|} = \int_{\mathbb{R}^d} G(\underline{r})\mathrm{d}\underline{r} \geq 0. \tag{7}$$

Hyperuniform systems are those for which the variance is subextensive, i.e.

$$\int_{\mathbb{R}^d} G(\underline{r})\mathrm{d}\underline{r} = 0. \tag{8}$$

This occurs when the Fourier transform of $G(\underline{r})$, aka structure function,

$$S(k) = \int e^{i\underline{k}\cdot\underline{r}} G(\underline{r})\mathrm{d}\underline{r} \tag{9}$$

vanishes when $k = 0$, i.e. $S(0) = 0$.

Taking Λ to be a sphere and averaging $G(\underline{r})$ over angles we obtain

$$\lim_{|\Lambda|\to\infty} \frac{\alpha_\Lambda(\underline{r})}{|\partial\Lambda|} = \alpha_d |\underline{r}|, \tag{10}$$

where α_d is a constant, which depends on dimension, $\alpha_1 = 1$, $\alpha_2 = 1/\pi$, $\alpha_3 = 1/4$.

For hyperuniform systems we have when letting $\Lambda \nearrow \mathbb{R}^d$

$$\lim \frac{\mathrm{Var}(N_\Lambda)}{|\partial\Lambda|} = -\alpha_d \int_0^\infty r^d G(r)\mathrm{d}r = -\alpha_d \int_0^\infty r^d [\rho_2(r) - \rho^2]\mathrm{d}r, \tag{11}$$

with the integral proportional to the first moment of G. When the integral in (11) is finite the variance of a hyperuniform system, $V(N_\Lambda)$, will grow like $|\partial\Lambda|$. When the integral in (11) is infinite, $\mathrm{Var}(N_\Lambda)$ will grow faster than $|\partial\Lambda|$ but slower than $|\Lambda|$. This happens, for example, in cases where $\rho_2(r) - \rho^2 \sim r^{-\gamma}$, for $r \to \infty$ with $\gamma \leq d+1$, with the variance depending on γ.

In general the rate of growth of the variance, $\mathrm{Var}(N_\Lambda)$ with Λ depends for a hyperuniform system on how $S(k)$ goes to zero as $k \to 0$.

An example of a hyperuniform system with variance growing like the surface is obtained by perturbing the position of a system with a particle at each lattice site, by a random displacement r according to a probability distribution $h(r)\mathrm{d}r$. The system will remain hyperuniform if $\int rh(r)\mathrm{d}r < \infty$.

Question: Can the variance grow slower than $|\partial\Lambda|$?

The answer by J. Beck (1987) is “no” after averaging over rotation as we do in (11) (or Λ is a sphere). This implies that for a hyperuniform system the integral $\int_0^\infty r^d G(r)\mathrm{d}r < 0$, which corresponds to an effective repulsion between the particles.

The variance in a ball of radius R for a translationally invariant point process will then grow at least like R^{d-1}, the surface area.

It is still an open question how small this variance can be and whether it attains its minimum value for a regular lattice $d > 2$. In $d = 1$ this is given by $\mathbb{Z}$ and in $d = 2$ by the triangular lattice.

Equilibrium systems

Question: Can equilibrium systems be hyperuniform? For equilibrium systems described by a Gibbs grand canonical ensemble at fugacity z in a region $\Lambda \subset \mathbb{R}^d$ we have generally,

$$\frac{\mathrm{Var}(N_\Lambda)}{|\Lambda|} = z\frac{\mathrm{d}\rho(\Lambda)}{\mathrm{d}z}, \tag{12}$$

where $\rho(\Lambda) = \langle N_\Lambda \rangle / |\Lambda|$.

The question then is whether an equilibrium system can be hyperuniform in the $|\Lambda| \to \infty$ limit? (This would correspond in the thermodynamic limit to, $\frac{\mathrm{d}p}{\mathrm{d}\rho}$ being infinite at some temperatures and densities, e.g. as it would be at a phase transition where the pressure is a discontinuous function of the density.)

It was shown by Ginibre in 1967 that for systems with hard cores (or systems with positive pair potentials) which decay sufficiently rapidly this cannot be the case at any positive temperature away from the close packing density, corresponding to $z = \infty$.

It follows from the Ginibre Theorem that in equilibrium systems, like the (expected) crystalline phase of hard spheres in three dimensions, there must be enough defects per unit volume to make the variance extensive, i.e. not hyperuniform, when $z < \infty$.

Ginibre's results were extended to some lattice systems with many body interactions (L., Pittel, Ruelle, and Speer).

Presumably the results hold also for more general interactions, like the Lennard-Jones pair potential and for quantum systems.

(They do not hold for the ground state of an ideal Fermi gas.)

Coulomb Systems

The situation is more interesting when one considers systems with long long range Coulomb interactions. The Coulomb interaction between charges e_i, e_j at positions $\underline{r}_i$, $\underline{r}_j$ in $\mathbb{R}^d$ is $e_i e_j v_d(r)$, with $r = |\underline{r}_i - \underline{r}_j|$ and

$$v_d(r) = \begin{cases} -r & d = 1 \\ -\log(r) & d = 2 \\ +r^{2-d} & d \geq 3 \end{cases} \tag{13}$$

Fluctuations in Q_Λ, the net charge in Λ in a Coulomb system are the primary physical examples of hyperuniform processes.

One Component Plasma

To simplify matters, and relate Q_Λ to N_Λ, I shall consider from now on only the simplest kind of Coulomb system: the classical one component plasma (OCP). This model, also known as "Jellium", was introduced by Wigner (1934). It consists of particles with a positive charge e moving in a uniform background of negative charge with density $-\rho e$.

The uniform incompressible rotationally invariant background for a given origin produces an external potential equal to $\frac{1}{2}\rho e r^2$.

Setting $e = 1$, the potential energy of such a system of N particles in $\mathbb{R}^d$ (taken as a limit of spheres with radius R going to infinity) is given by

$$U(\underline{r}_1, \cdots, \underline{r}_N) = \sum_{i<j} v_d(\underline{r}_i - \underline{r}_j) + \frac{\rho}{2} \sum_{i=1}^{N} r_i^2. \tag{14}$$

The canonical equilibrium probability distribution of this system is given by

$$\mu_N \propto \exp[-\beta U]. \tag{15}$$

When $N \to \infty$, the measures μ_N have a limit μ, at least along subsequences, which describes a point process in $\mathbb{R}^d$ with average particle density ρ.

One Component Plasma: $d = 1$

The OCP is exactly solvable in $d = 1$: the extremal μ is periodic with period ρ^{-1}, for all $\beta > 0$ (Kunz 1994).

The probability distribution of $(N_\Lambda - \rho|\Lambda|)$, Λ an interval, has exponential decay as $|\Lambda| \to \infty$ (Martin, Yalcin 1980).

The variance of N_Λ is therefore bounded and is trivially proportional to $|\partial\Lambda| = 2$.

(The extremal measures for general 1D systems with bounded variance have a periodic component (Aizenman, Goldstein, L. 2001).)

One Component Plasma: $d = 2$

In $d \geq 2$, the OCP system is isotropic at "small" β.

For "large" β, the system is expected to form a periodic "Wigner crystal". Numerical simulations predict the formation of the Wigner crystal to be around $\beta = 140$, in $d = 2$.

In $d = 2$ this system is exactly solvable at $\beta = 2$. At this temperature the particles have the same distribution as the eigenvalues of an i.i.d. complex Gaussian matrix: the Ginibre ensemble scaled to have average density ρ.

One Component Plasma: $d = 2$, $\beta = 2$

The correlations of the exactly solvable model have super-good clustering properties (Ginibre, Jancovici) with the truncated pair correlation function

$$\rho_2(r) - \rho^2 = -\rho^2 e^{-\pi\rho r^2}, \quad r = |\underline{r}_1 - \underline{r}_2|. \tag{16}$$

Higher order truncated correlations also decay like $e^{-\gamma D^2}$, where D is the distance between groups of particles.

Integrating (16), one sees that $\int G(r)\mathrm{d}r = 0$, so this system is hyperuniform. $S(k) \sim k^2$ as $k \to 0$. The variance in a domain Λ for the system grows like $|\partial\Lambda|$ with the coefficient given by (11). The same is expected to be true, with $\langle Q_\Lambda^2 \rangle \sim |\partial\Lambda|$ for all charge neutral systems, $\langle Q_\Lambda \rangle = 0$, in $d \geq 2$.

Let me now mention some other examples of hyperuniform processes. These interpolate in some ways between random fluids and ordered crystals.

I will begin with a natural system which appears to exhibit hyperuniformity.

I will then consider determinantal processes of much current interest, a large class of which exhibit hyperuniformity.

Following this I will describe a process which exhibits hyperuniformity at a nonequilibrium "critical point".

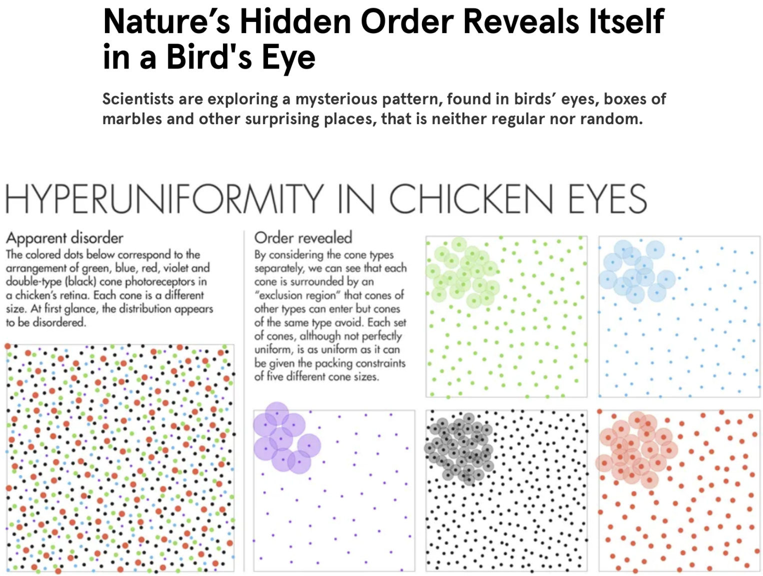

Figure from article by Natalie Wolchover. Wired, July 15, 2016.

Determinantal Processes

Determinantal process are ones for which the k-point correlation

$$\rho_k(x_1, \ldots, x_k) = \det[K(x_j, x_l)]_{j,l=1,\ldots,k} \tag{17}$$

$$= \det \begin{bmatrix} K(x_1, x_1) & \cdots & K(x_1, x_k) \\ \vdots & \ddots & \vdots \\ K(x_k, x_1) & \cdots & K(x_k, x_k) \end{bmatrix}. \tag{18}$$

This determines a point process, with $K(x, x) = \rho_1(x)$ the density at x, when K is Hermitian and all its eigenvalues are in $[0, 1]$. In particular for $k = 2$ and translation invariant processes, $K(x, y) = K(x - y)$,

$$\rho_2(x_1, x_2) = \rho^2 - |K(x_1 - x_2)|^2. \tag{19}$$

Determinantal point processes whose kernels are projection operators are hyperuniform.

Key Examples of Hyperuniform Determinantal Processes

Distribution of eigenvalues of the Ginibre ensemble, which, as already stated, is the same as the 2D OCP at $\beta = 2$.

1D bulk eigenvalue limit of the Gaussian or the Circular Unitary ensembles, a.k.a. the sine kernel process of the Dyson log gas, a Coulomb system with 2D logarithmic interactions, confined to a line, at inverse temperature $\beta = 2$. $K(x - y) = [\sin \pi(x - y)]/\pi(x - y)$.

For this system the variance of N_Λ grows like $\log |\Lambda|$, which is faster than $|\partial\Lambda|$ but slower than $|\Lambda|$. $G(r)$ decays like r^{-2} so its first moment is infinite.

The ground state of an ideal Fermi gas of N particles in a region Λ in any dimension has a wave function given by a Slater determinant. This can be shown to have a probability distribution given by a determinantal process with a projection kernel and thus be hyperuniform.

Facilitated Exclusion Process (FEP)

In (the continuous time symmetric version of) this model a site $x \in \mathbb{Z}^d$ can be occupied by at most one particle. Each particle can jump to a neighboring empty site only if it has also an occupied neighbor site (to facilitate it).

Starting with an initial random (Bernouli) state at density ρ the system approaches, as $t \to \infty$, either a frozen state in which all particles are isolated or an active stationary state in which there is a finite density of active particles. The transition occurs in $d = 1$ at $\rho_c = 1/2$ at which density the stationary state is periodic.

In $d = 2$ numerical investigations show that $\rho_c \approx .3309$, much less than the maximal density for the frozen state, $\rho = .5$. The critical density ρ_c is even smaller for the discrete time version of the model in $d = 2$. This small value of ρ_c is surprising.

Even more surprising is the finding by Hexner and Levine that the stationary state at $\rho = \rho_c$ is hyperuniform, (Hexner and Levine, 2015). This seems to be a universal feature of these type of lattice nonequilibrium trapping phase transitions.

Using extensive computer simulations, Hexner and Levine found that the variance of N in a square of area ℓ^2 grows approximately like $\ell^{1.5}$ which is faster than the perimeter but slower than the volume. This means, according to our previous analysis, that the first moment of $G(r)$ diverges at ρ_c. Further assumptions suggest that $G(r) \sim r^{-1.45}$ as $r \to \infty$ and $S(k) \sim k^{.45}$ as $k \to 0$.

(There is no proof at the present time that for $d > 1$, ρ_c is greater than any ϵ, say $\epsilon = 10^{-20}$.)

Stealthy Hyperuniform Systems

As already noted the way $S(k) \to 0$ as $k \to 0$ determines many of the properties of hyperuniform systems.

An extreme case of this occurs when $S(k) = 0$ in an open set M in $\underline{k}$-space ($\mathbb{R}^d$ for continuous systems in $\mathbb{R}^d$, $\mathbb{T}^d$ for systems in $\mathbb{Z}^d$) containing the origin $\underline{k} = 0$ the process is called stealthy hyperuniform (SH) by Torquato, et al.

The name stealthy comes from the fact that for waves with wave vector $\underline{k} \in M$ the system is transparent just like it is for crystals when $\underline{k}$ is between Brillouin bands. Such systems therefore have properties which lie between those of a random fluid and an ordered crystal. These can be exploited for practical applications, e.g. they are isotropic for wave transmission.

It turns out that SH systems also have many other crystalline like properties.

In work with Subhro we proved a conjecture by Zhang, et al. [2019] that SH system have zero probability of holes larger than a certain diameter D_0, i.e. there are no spherical regions with diameter $D > D_0$ in which there are no particles. For "typical" systems including equilibrium systems with hard cores (and decaying interation) the probability of such an event is very small but not zero. For Poisson processes the probability that a region Λ is empty, is given e.g. $P_0 \sim e^{-\rho|\Lambda|}$, for Jellium the probability is even smaller. For SH systems D_0 is proportional to the size of the gap M.

We also proved upper and lower bounds on the density of particles in a volume $\omega \in \mathbb{R}^d$. This shows that most configurations of SH systems in $\mathbb{R}^d$ (almost all in $\mathbb{Z}^d$) are Delone sets (no more than one particle in balls of size d_0 and no holes larger than D_0).

Rigidity

In the same work with Subhro we generalized the notion of stealthy process to consider the situation where $S(k)$ vanishes in an open set M in $\underline{k}$-space which need <u>not</u> include the origin, $\underline{k} = 0$. We call these generalized stealthy (GS) processes and show that GS systems have remarkable <u>rigidity</u> properties. To describe these let me go back to general point processes in $\mathbb{R}^d$ or $\mathbb{Z}^d$.

So far we have discussed fluctuations of particles, or charges, in a region Λ without saying anything about the configuration of particles/charges outside Λ, i.e. in $\Lambda^c = \mathbb{R}^d \setminus \Lambda$. We ask now: what can we say about the distribution of points inside Λ given the configuration in Λ^c, i.e. we want the conditional probability $\mu(X_\Lambda | X_{\Lambda^c})$ of a configuration in Λ given X_{Λ^c}.

For equilibrium Gibbs measures μ of particle systems on $\mathbb{R}^d$ the answer to this is given by the DLR (Dobrushin, Lanford, Ruelle) equations.

$$\mu(\underline{x}_1, \ldots, \underline{x}_N | X_{\Lambda^c}) = \frac{\exp[-\beta U(X_\Lambda | X_{\Lambda^c})]}{\int e^{-\beta U(X_\Lambda | X_{\Lambda^c})} \mathrm{d} X_\Lambda}, \tag{20}$$

where $U(X_\Lambda | X_{\Lambda^c})$ is the potential energy of a configuration in Λ given the configuration in $\Lambda^c = \mathbb{R}^d \setminus \Lambda$.

When the interaction U decays sufficiently rapidly with distance and μ is ergodic, the behavior of $\mathrm{Var}(N_{\Lambda'})$ is, for Λ' far from the boundaries of Λ, similar to the unconditional case, and the Ginibre lower bound on the variance holds.

This is however not the case for system with long range Coulomb interactions, where $U(X_\Lambda | X_{\Lambda^c})$ is not well defined.

Aizenman and Martin (AM 1981) proved, for $d = 1$ Coulomb system, that the charge in an interval $[a, b] = \Lambda$, which corresponds for the OCP to the number of particles in Λ, is uniquely specified by the configuration X_{Λ^c} for all typical configurations with respect to the infinite volume measure μ. (The set of atypical configurations has measure zero.)

The property that the measure $\mathbb{P}(N_\Lambda | X_{\Lambda^c})$ is concentrated at a single value of N_Λ has been called "number rigidity" by Ghosh and Peres (2012).

They showed that the Ginibre ensemble and the standard planar Gaussian zero process, $f(z) = \sum_{n=0}^{\infty} a_n z^n / \sqrt{n!}$, have this property ($a_n$ centered Gaussian random variable).

Ghosh (2012) also proved number rigidity for the GUE (and the CUE) point processes.

Both the Ginibre and the GUE ensemble correspond to Coulomb systems (with logarithmic interactions) at particular temperatures.

Ghosh and Peres also showed for these systems that while N_Λ is fixed by X_{Λ^c}, the distribution of points inside Λ is not rigid; in fact it is absolutely continuous with respect to Lebesgue measure. Same is true for the $d = 1$ Coulomb system considered by AM.

In work with Subhro, we show that number rigidity holds in fact for all hyperuniform processes in $d = 1, 2$, satisfying a decay assumption on ρ_2^T:

$$\rho_2^T(x, y) \leq \frac{C}{1 + |x - y|^2}, \quad \text{in } d = 1, \tag{21}$$

$$\rho_2^T(x, y) \leq \frac{C}{1 + |x - y|^{4+\epsilon}}, \quad \text{in } d = 2. \tag{22}$$

$$\rho_2^T(x, y) \equiv \rho_2(x, y) - \rho_1(x)\rho_2(y). \tag{23}$$

Rigidity of point processes in $d > 2$

Our proof of sufficiency does not generalize to $d > 2$.

In fact, there is a strong evidence based on the work of Peres and Sly that no conditions on the decay of correlations would work for $d \geq 3$.

Peres and Sly (2014) show that the i.i.d. perturbed lattice point process, i.e. a particle at each lattice point $y = \mathbb{Z}^d$ is moved to $y + x$ with probability $h(x)\mathrm{d}\underline{x}$:

- is rigid in 1D if the perturbations have finite first moment (consistent with our sufficiency conditions)
- is rigid in 2D if the perturbations have finite second moment (consistent with our sufficiency conditions)
- Exhibits a remarkable phase transition in its rigidity behavior for Gaussian perturbations $h(x) = (2\pi\sigma)^{-d/2} \exp[-x^2/2\sigma]$ in $d \geq 3$: there is a critical σ_c such that the process is rigid if the variance σ of $h(x)$ satisfies $\sigma < \sigma_c$, and is not rigid if $\sigma > \sigma_c$, although there is superhomogeneity as well as fast (Gaussian) decay of correlations in the latter case.

Going beyond number rigidity Ghosh and Krishnapur defined a hierarchy of rigidities by checking the number moments of the distribution of particles in Λ which are determined by the configuration, X_{Λ^c}, in Λ^c.

Number rigidity only requires that the zeroth moment, N_Λ, be determined by X_{Λ^c}. Maximal rigidity corresponds to the actual configurations in Λ, X_Λ, be determined with probability 1 by X_{Λ^c}. This means that $\mu(X_\Lambda | X_{\Lambda^c})$ is essentially a delta function. This is the case for "perfect" crystals.

Now comes the punch line: we prove [Ghosh, L] that all GS processes are maximally rigid. N.B. This can happen even for systems which have normal fluctuations, as would be the case when M does not include the origin and $S(0) > 0$.

One can construct GS systems for Gaussian processes on a lattice.

Proof: Basic idea

The basic idea of [GP] and [G] to prove number rigidity is to find a sequence of functions $\phi_\epsilon(x)$, such that $\phi_\epsilon(x) = 1$ for $x \in \Lambda$ and $\mathrm{Var}(\phi) \leq \epsilon$, for any $\epsilon > 0$. Then in the limit $\epsilon \to 0$:

$$\sum \phi(x_i) = \sum \chi_\Lambda(x_i) + \sum \chi_{\Lambda^c}(x_i)\phi(x_i) \tag{24}$$

$$= N_\Lambda + \sum \chi_{\Lambda^c}(x_i)\phi(x_i) \tag{25}$$

$$\to \left\langle \sum \phi(x_i) \right\rangle = \int \rho(x)\phi(x)\mathrm{d}x, \tag{26}$$

where $\chi_\Lambda(x)$ is the characteristic function of the set Λ. This determines N_Λ given X_{Λ^c}.

This is accomplished in our case by choosing a sequence $\phi_R(x) = \phi(x/R)$ with an appropriate $\phi(x)$.

For references see:

- *Number rigidity in superhomogeneous random point fields*, J Stat Phys (2017) 166:1016–1027 (2017), arXiv:1601.04216 (S. Ghosh and J. Lebowitz)
- *Fluctuations, large deviations and rigidity in hyperuniform systems: a brief survey*, arXiv:1608.07496, Indian Journal of Pure and Applied Mathematics 2017, Volume 48, Issue 4, pp 609–631 (S. Ghosh, J. Lebowitz)
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- *Fluctuation and Entropy in Spectrally Constrained random fields*, arXiv 2002.10087, Communications in Mathematical Physics volume 386, pages749–780 (2021) (Kartick Adhikari, Subhroshekhar Ghosh, Joel L. Lebowitz)