

# Microscopic Origins of Macroscopic Behavior

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(Much of the work described here was done jointly with S. Goldstein, R. Tumulka, D. Huse and N. Zanghi)

Let me begin my talk by quoting Freeman Dyson, an earlier recipient of this prize, about his definition of mathematical physics:

*“Mathematical physics is the discipline of people who try to reach a deep understanding of physical phenomena by following the rigorous style and method of mathematics.”*

— Freeman Dyson, *From Eros to Gaia*, p 164-165

Freeman was a good friend and this talk is dedicated to his memory as well as to the memory of my teachers Melba Phillips and Peter Bergmann.

My own area of mathematical physics is statistical mechanics which is concerned with the microscopic origin of macroscopic behavior. Since our mathematical abilities for dealing with strongly interacting many particle systems is quite limited it is fortunate that many striking features of macroscopic systems can be obtained from simplified microscopic models.

We therefore often take as our lowest level starting point an idealized description of atoms. As put by Feynman:

*If in some cataclysm all of scientific knowledge was to be destroyed, . . . , what statement would contain the most information in the fewest words? I believe it is . . . that all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.*

- Unfortunately even such idealized systems are too difficult to deal with in any detail. In fact many details would just be confusing. We need therefore to focus on those features of macroscopic systems which are independent of many details: they are typical of real macroscopic systems.
- I will therefore focus my talk on explaining, in a qualitative way, the microscopic origin of behavior of isolated macroscopic systems both in equilibrium and out of it and the fact that this behavior is *typical*.

- Thus many properties of equilibrium macroscopic systems hold for an overwhelming majority of the microstates in the micro-canonical ensemble.
- In fact, the fraction of systems with noticeable macroscopic deviations from the average behavior, computed in such an ensemble, is *typically* exponentially small in the number of degrees of freedom of such systems.
- An analogous statement holds for these (exponentially small) subsets of the micro-canonical ensemble which describe systems in nonequilibrium macrostates.

This property of typicality is true both classically and quantum mechanically. It explains why these ensembles can be used to predict the observed behavior of individual macroscopic systems *and* not just some average behavior.

This includes the time asymmetric approach to equilibrium, encoded in the second law, and observed in individual macroscopic systems. This behavior does not require explanations based on ergodicity, time averaging, or subjective information theory.

# Classical Systems

- In classical mechanics, the microstate of a system of  $N$  particles confined to a region  $V$  in  $\mathbb{R}^d$  is a point  $X$  in the  $2dN$ -dimensional phase space,

$$X = (\vec{r}_1, \vec{p}_1, \dots, \vec{r}_N, \vec{p}_N), \quad \vec{r}_i \in V \subset \mathbb{R}^d, \quad \vec{p}_i \in \mathbb{R}^d$$

- Its time evolution is given by a Hamiltonian  $H(X)$  which conserves energy, so  $X(t)$  will be confined to  $\Gamma_E$ , a thin shell surrounding the energy surface  $H(X) = E$ .
- A macroscopic system is one with “very large”  $N$ , say  $N \gtrsim 10^{20}$ .

# Macrostates

- To describe the macroscopic state of such a system of  $N$  particles in a box  $V$ , we make use of a much cruder description than that provided by the microstate  $X$ . We shall denote by  $M$  such a macroscopic description:  $M(X)$  is the macrostate of the system in the microstate  $X$ .
- As an example we may divide  $V$  into  $K$  cells, where  $K$  is large but still  $K \ll N$ , and specify the number of particles, the momentum and the amount of energy in each cell, with some tolerance.
- Clearly there are many  $X$ 's (in fact a continuum) which correspond to the same  $M$ . Let  $\Gamma_M$  be the region in  $\Gamma_E$  consisting of all microstates  $X$  corresponding to a given macrostate  $M$  and denote by  $|\Gamma_M|$  its Liouville volume.

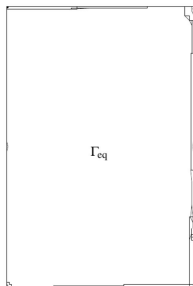
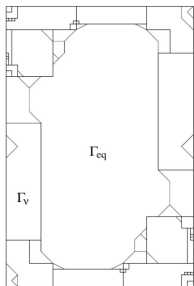
- It can be proven that, generally, in every  $\Gamma_E$  of a macroscopic system, there is one region  $\Gamma_M$  which has most of the volume of  $\Gamma_E$ . This is called the equilibrium macrostate  $M_{\text{eq}}$ .

$$\frac{|\Gamma_{M_{\text{eq}}}|}{|\Gamma_E|} = 1 - \varepsilon \quad (1)$$

with  $\varepsilon \ll 1$ . When  $M(X)$  specifies a nonequilibrium state,  $|\Gamma_M|$  is much smaller.

- Thus for a gas consisting of  $N$  particles in a volume  $V$  the ratio of  $|\Gamma_M|$ , a macrostate  $M$  in which all the particles are in the left half of the box, and  $|\Gamma_{M_{\text{eq}}}|$ , the macrostate  $M_{\text{eq}}$  in which there are  $(\frac{1}{2} \pm 10^{-10})N$  particles in the left half of the box, is of order  $2^{-N}$ .

## Pictorially



The second picture is slightly more faithful. Neither shows the topology or differences in relative sizes of the different  $\Gamma_M$ 's. In general, the closer  $M$  to  $M_{eq}$  the larger  $\Gamma_M$ .

- A system is then in macroscopic thermal equilibrium (MATE) iff  $X \in \Gamma_{M_{\text{eq}}}$ .

- Relevant properties of macroscopic systems depend only on sums of functions which depend only on the coordinates and momenta of a few particles.

The values of these functions are approximately the same for almost all  $X \in \Gamma_{M_{\text{eq}}}$ , hence they are observed for “typical” systems in equilibrium. In fact since  $|\Gamma_{M_{\text{eq}}}| \sim |\Gamma_E|$  they are also typical of  $X \in \Gamma_E$ .

- This justifies the use of the microcanonical ensemble to compute relevant properties of an equilibrium system: independent of whether the dynamics is ergodic in a mathematical sense.

## Nonequilibrium states

- Thus, most microstates in  $\Gamma_E$  of a macroscopic system correspond to the system being in equilibrium.
- A similar statement is true for most wave functions, in fact also for most energy eigenstates, in  $\mathcal{H}_E$ , the energy shell of the Hilbert space.
- Fortunately there are also microstates which correspond to macroscopic systems which are out of equilibrium (or we would not be here).
- Given an  $X(t_0)$  in such a  $\Gamma_M$ ,  $M \neq M_{\text{eq}}$ , at an “initial” time  $t_0$ , we would like to know how the properties of a macroscopic system, isolated for  $t > t_0$ , change with time.

## Approach to Equilibrium

- Boltzmann (also Maxwell, Kelvin, . . . ) argued that given this disparity in the sizes of the  $\Gamma_M$  corresponding to the various macrostates, the evolution of a “typical” microstate  $X$ , specified to be at  $t = t_0$  in the phase space region  $\Gamma_M$ , will be such that  $|\Gamma_{M(X(t))}|$  will not decrease (on a macroscopic scale) for  $t > t_0$ .
- In fact for any  $\Gamma_M$  the relative volume of the set of microstates  $X \in \Gamma_M$  for which this is false over some time period  $\tau$  during which the macrostate undergoes a macroscopically noticeable change, but not longer than the age of the universe, goes to zero exponentially in the number of atoms in the system.
- This explains and describes the evolution towards equilibrium of macroscopic systems which start in the macrostate  $\Gamma_M$ ,  $M \neq M_{\text{eq}}$ , and are kept (effectively) isolated afterwards.

# Boltzmann's Entropy

- To make a connection with the Second Law, Boltzmann defined the (Boltzmann) entropy of a macroscopic system in a microstate  $X$  as

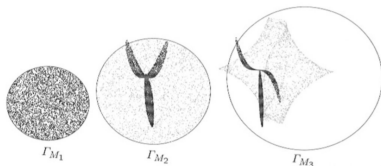
$$S_B(X) = \log |\Gamma_{M(X)}| = S_B(M).$$

- Thus calling  $S(E) = \log |\Gamma_E| \sim \log |\Gamma_{M_{\text{eq}}}|$ , Boltzmann showed that this definition agrees with the Clausius thermodynamic entropy of a gas.

- The above heuristic argument, based on relative phase space volume, is the correct explanation for the behavior typically observed in actual macroscopic systems. It is, however, very far from a mathematical theorem and contains no quantitative information about time scales.
- A desirable result would be the rigorous derivation of hydrodynamic equations from the microscopic dynamics. This is a “holy grail” of our research.

# Hydrodynamic Equations

- We can say more mathematically *if* the time evolution of the macrostate  $M$ , given by  $M(X(t)) = M_t$ , effectively satisfies an autonomous deterministic equation, such as the Navier-Stokes equation or the heat equation.
- Such an equation means that if  $M_{t_1} \rightarrow M_{t_2}$ ,  $t_2 > t_1$ , and  $M_{t_2} \rightarrow M_{t_3}$ ,  $t_3 > t_2$ , then the microscopic dynamics  $T_t$  carries  $\Gamma_{M_{t_1}}$  inside  $\Gamma_{M_{t_2}}$ , i.e.  $T_{t_2-t_1}\Gamma_{M_{t_1}} \subset \Gamma_{M_{t_2}}$  and  $T_{t_3-t_2}\Gamma_{M_{t_2}} \subset \Gamma_{M_{t_3}}$  with *negligible error*.



In particular a typical phase point in  $\Gamma_{M_1}$  will go to  $\Gamma_{M_2}$  and then to  $\Gamma_{M_3}$ , i.e.  $T_{t_3-t_1}\Gamma_{M_{t_1}} \subset \Gamma_{M_{t_3}}$ .

- The fact that phase space volume is conserved by the Hamiltonian time evolution implies that  $|\Gamma_{M_t}| \leq |\Gamma_{M_{t'}}|$ ,  $t < t'$ , and thus that  $S_B(M_{t'}) \geq S_B(M_t)$  for  $t' \geq t$ . Boltzmann's argument then becomes rigorous.
- What is crucial here is that the phase points in the region in  $\Gamma_2$  coming from  $\Gamma_1$  behave, forward in time, as if they were typical of  $\Gamma_2$ .
- They are, however, very atypical backwards in time: if we reverse all the velocities in  $\Gamma_2$ , then at a later time,  $t' = t_2 + (t_2 - t_1)$  all of the points initially in  $\Gamma_{M_1}$  will again be in  $\Gamma_{M_1}$  (with their velocities reversed) a smaller region than  $\Gamma_{M_2}$ .
- It is here that the crucial time asymmetry, expressed in the hydrodynamic equation, comes in.

- The reason for this asymmetry in typical behaviors is due to initial conditions.
- That is, when nature or the experimentalist who is part of nature, starts out with a nonequilibrium system in an initial state  $X \in \Gamma_M$  we can assume that  $X$  is typical of  $\Gamma_M$ , and continues to be so in the forward time direction.
- But how did all this get started?

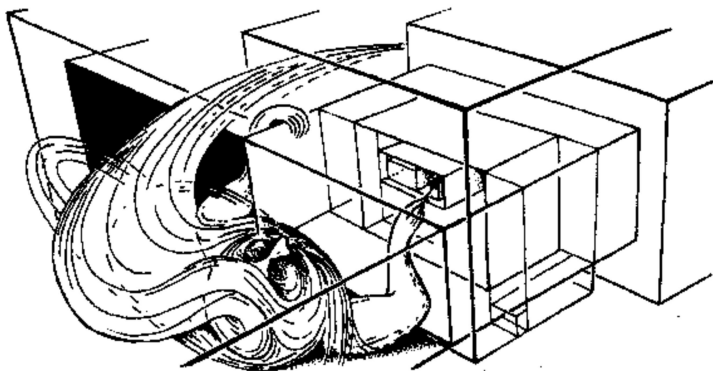
## Initial Conditions

*From the fact that the differential equations of mechanics are left unchanged by reversing the sign of time without changing anything else, Herr Ostwald concludes that the mechanical view of the world cannot explain why natural processes always run preferentially in a definite direction. **But such a view appears to me to overlook that mechanical events are determined not only by differential equations, but also by initial conditions.** In direct contrast to Herr Ostwald I have called it one of the most brilliant confirmations of the mechanical view of Nature that it provides an extraordinarily good picture of the dissipation of energy, as long as one assumes that the world began in an initial state satisfying certain conditions. I have called this state an improbable state.*

— L. Boltzmann

*It is necessary to add to the physical laws the hypothesis that, in the past the universe was more ordered in the technical sense, [i.e. low  $S_B$ ] than it is today ... to make an understanding of irreversibility."*

— R. P. Feynman



Creation of the universe: a fanciful description! The Creator's pin has to find a tiny box, just 1 part in  $10^{10^{123}}$  of the entire phase-space volume, in order to create a universe with as special a Big Bang as that we actually find.

R. Penrose, *The Emperor's New Mind*

The "tiny box" is a macrostate with low  $S_B$ .

## Refining Macrostates: Dilute Gases

- Following Boltzmann, we refine the description of a macrostate  $M$  by noting that the microstate  $X = \{\mathbf{r}_i, \mathbf{v}_i\}$ ,  $i = 1, \dots, N$ , can be considered as a set of  $N$  points in six dimensional  $\mu$ -space. We then divide up this  $\mu$ -space into  $\tilde{J}$  cells  $\tilde{\Delta}_\alpha$ , centered on  $(\mathbf{r}_\alpha, \mathbf{v}_\alpha)$ , of volume  $|\tilde{\Delta}_\alpha|$ . A macrostate  $\tilde{M}$  is then specified by the (coarse grained) number of particles in each  $\tilde{\Delta}_\alpha$ ,

$$\tilde{M} = \{N_\alpha\}, \quad \alpha = 1, \dots, \tilde{J} \ll N.$$

- Using  $\tilde{M}$ , we can associate with a typical  $X$  a coarse grained density  $f_X \sim N_\alpha/|\tilde{\Delta}_\alpha|$  in  $\mu$ -space, i.e. such that  $N_\alpha = \int_{\tilde{\Delta}_\alpha} d\mathbf{x} d\mathbf{v} f_X(\mathbf{x}, \mathbf{v})$ .

- Computing the phase space volume associated to the macrostate  $f_X$ ,  $|\Gamma_{f_X}|$ , Boltzmann showed that, up to constants, this is given for a dilute gas, by

$$S_{\text{gas}}(f_X) = -k \int_V d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} f(\mathbf{x}, \mathbf{v}) \log f(\mathbf{x}, \mathbf{v})$$

- Not to be confused with the Gibbs-Shannon entropy for a product measure.
- The maximum of  $S_{\text{gas}}(f)$  over all  $f$  with a given energy, which is here just the kinetic energy, is given by the Maxwell distribution

$$f_{\text{eq}} = \frac{N}{V} (2\pi kT/m)^{-3/2} \exp[-m\mathbf{v}^2/2kT]$$

where  $kT = 2/3(E/N)$ .

- In this case

$$S_{\text{gas}}(f_{\text{eq}}) = S(E, N, V) = Nk \left[ \frac{3}{2} \log T - \log \frac{N}{V} + \text{Const.} \right]$$

- When  $f \neq f_{\text{eq}}$  then  $f$  and consequently  $S_{\text{gas}}(f)$  will change in time.
- The second law, now says that for *typical*  $X$ ,  $f_{X_t}(\mathbf{x}, \mathbf{v}) = f_t(\mathbf{x}, \mathbf{v})$  has to be such that  $S_{\text{gas}}(f_t) \geq S_{\text{gas}}(f_{t'})$ , for  $t \geq t'$ .
- This is exactly what happens for a dilute gas described by the Boltzmann equation.

$$\frac{d}{dt} S_{\text{gas}}(f_t) \geq 0, \quad \text{Boltzmann's } \mathcal{H}\text{-theorem}$$

- A rigorous derivation of the Boltzmann equation (in a suitable limit) was given by Oscar Lanford (1975): a “tour de force” of mathematical physics.

As put by Boltzmann:

*In one respect we have even generalized the entropy principle here, in that we have been able to define the entropy in a gas that is not in a stationary state.*

*On the basis of kinetic theory of gases Boltzmann had discovered that, aside from a constant factor, entropy is equivalent to the logarithm of the “probability” of the state under consideration. Through this insight he recognized the nature of course of events which, in the sense of thermodynamics, are “irreversible”. Seen from the molecular-mechanical point of view, however all courses of events are reversible. If one calls a molecular-theoretically defined state a microscopically described one, or, more briefly, micro-state, then an immensely large number ( $Z$ ) of states belong to a macroscopic condition.  $Z$  is then a measure of the probability of a chosen macro-state. This idea appears to be of outstanding importance also because of the fact that its usefulness is not limited to microscopic description on the basis of mechanics.*

— A. Einstein, Autobiographical notes

## Boltzmann vs. Gibbs Entropies

- Given an ensemble (probability) density  $\mu(X)$ , the Gibbs-Shannon entropy is given by

$$S_G \equiv -k \int_{\Gamma} \mu \log \mu \, dX. \quad (2)$$

- Clearly if  $\mu = \tilde{\mu}_M$ , where

$$\tilde{\mu}_M = \begin{cases} |\Gamma_M|^{-1}, & \text{if } X \in \Gamma_M; \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

then

$$S_G(\tilde{\mu}_M) = k \log |\Gamma_M| = S_B(M). \quad (4)$$

This is essentially the case for the microcanonical ensemble and thus the Gibbs and Boltzmann entropies are equal for equilibrium systems.

- However  $S_G(\mu)$  does not change in time for isolated systems and therefore is “useless” for such systems not in equilibrium, while  $S_B(M(X))$  captures the essence of typical macroscopic behavior.



Boltzmann's grave in Zentralfriedhof, Vienna, with bust and entropy formula.

# Quantum Systems

Let me turn now to quantum systems.

- I will take the microstate of a system to be its wave function  $\psi \in \mathcal{H}_E$ , where  $\mathcal{H}_E$  is a shell in Hilbert space of thickness  $\Delta E$ .
- This is not perfect but it will have to do for the present.
- The question then is which  $\psi$  correspond to the system being in macroscopic thermal equilibrium (MATE), i.e. what is the analog of a classical microstate  $X$  lying in  $\Gamma_{M_{\text{eq}}}$ .

- Following von Neumann\*, we take the macro-observables  $M_j$  to commute with each other. We assume that this can be achieved by suitably “rounding off”, i.e. coarse-graining, the operators representing the macro-observables.

- Taking  $\mathcal{H}_E$  to be an eigenspace of a “macro energy” operator, and thus to commute with the other macro-observables, all  $M_j$  can be regarded as operators on  $\mathcal{H}_E$ . Their joint spectral decomposition defines an orthogonal decomposition

$$\mathcal{H}_E = \bigoplus_{\nu} \mathcal{H}_{\nu}, \quad (5)$$

- The subspaces  $\mathcal{H}_{\nu}$  (“macro spaces”), the joint eigenspaces of the macro-observables, correspond to the different macro states. This corresponds to division of the classical energy shell  $\Gamma_E$  into disjoint regions  $\Gamma_M$ .

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\* Chapter 5 of his book: Mathematical Foundations of Quantum Mechanics

- A system is in a macrostate  $M_\nu$  if its wave function  $\psi$  is “close” to  $\mathcal{H}_\nu$ , i.e.  $\langle \psi | P_\nu | \psi \rangle \geq 1 - \delta$ ,  $\delta \ll 1$ .
- The “volume” of each macro space  $\mathcal{H}_\nu$  is its dimension  $d_\nu$ .
- As in the classical case, it is generally true that one of the  $\mathcal{H}_\nu$ , denoted  $\mathcal{H}_{\text{eq}}$ , has most of the dimensions of  $\mathcal{H}_E$ , i.e.,

$$\frac{\dim \mathcal{H}_{\text{eq}}}{\dim \mathcal{H}_E} = 1 - \varepsilon \quad (6)$$

with  $\varepsilon \ll 1$ .

- A macroscopic system is in MATE if

$$\langle \psi | P_{\nu_{\text{eq}}} | \psi \rangle \geq 1 - \delta, \quad (7)$$

with  $P_{\nu_{\text{eq}}}$  being the projection to  $\mathcal{H}_{\text{eq}}$ .

- The Boltzmann entropy  $S_B(\{\psi\})$  of a system in a macrostate  $M$  is then given by the log of the dimension of the macro space  $\mathcal{H}_\nu$ ;  
 $S_B(M_\nu) = \log d_\nu$ :  $d_\nu$  is the quantum analogue of  $|\Gamma_{M_\nu}|$ .
- Most everything we said about isolated classical macroscopic systems then also holds for quantum systems. In particular for  $\varepsilon \ll \delta$ , most pure states in  $\mathcal{H}_{\text{eq}}$  are in MATE.
- The Boltzmann argument for increase in entropy of isolated macroscopic systems out of equilibrium is then similar to that in the classical case.
- However, unlike classical systems where any subsystem of a system in a pure state is also in a pure state, a subsystem  $S$  of a quantum system with a wave function  $\psi$  will be described by a density matrix  $\rho_S$ .

- We can now define a system with wave function  $\psi$  to be in microscopic thermal equilibrium (MITE) if, for any not-too-large subsystem  $S$ , say subsystems with linear dimension  $\ell < \ell_0$ , the reduced density matrix of  $S$  is close to the thermal equilibrium density matrix of  $S$

$$\rho_S^\psi \approx \rho_S^{\text{mc}} \quad (8)$$

where

$$\rho_S^\psi = \text{tr}_{S^c} |\psi\rangle\langle\psi| \quad (9)$$

is the reduced density matrix of  $S$  obtained by tracing out the complement  $S^c$  of  $S$ , and

$$\rho_S^{\text{mc}} = \text{tr}_{S^c} \rho^{\text{mc}} \quad (10)$$

- $\rho^{\text{mc}}$  is the microcanonical density matrix corresponding to a uniform distribution over energy eigenstates in  $\mathcal{H}_E$ .

- The distinction between MITE and MATE is particularly relevant for systems with many-body localization (MBL) for which the energy eigenfunctions fail to be in MITE while necessarily most of them, but not all, are in MATE.
- The argument for most energy eigenfunctions being in MATE is based on the fact that, calling  $D$  the dimension of  $\mathcal{H}_E$ , we have for energy eigenfunctions  $|n\rangle$

$$\frac{1}{D} \sum_{n=1}^D \langle n | P_{\nu_{\text{eq}}} | n \rangle = \frac{1}{D} \text{tr}(P_{\nu_{\text{eq}}}) = 1 - \varepsilon$$

Noting that  $\langle n | P_{\nu_{\text{eq}}} | n \rangle \leq 1$ , the average being close to 1 means that most eigenstates are close to  $\mathcal{H}_{\nu_{\text{eq}}}$ . This is consistent with the Eigenfunction Thermalization Hypothesis (ETH).

- In fact for generic macroscopic systems, including those with MBL, most wave functions in an energy shell are in both MATE and MITE.

This follows from the following result.

### Canonical Typicality

Consider an isolated system consisting of two parts. Call them system 1 and 2 or system and reservoir. Then,

#### Theorem

*Let  $H$  be the Hamiltonian of the whole system and let the number of particles in system 1 and 2 be  $N_1 \ll N_2$ . Let  $\mathcal{H}_E \subset \mathcal{H}_1 \otimes \mathcal{H}_2$  be an energy shell. Then for most  $\Psi \in \mathcal{H}_E$  with  $\|\Psi\| = 1$ ,*

$$\text{tr}_2 |\Psi\rangle\langle\Psi| \approx \text{tr}_2 \rho^{\text{mc}},$$

*where  $\rho_{\text{mc}}$  is the microcanonical density matrix of the whole system at energy  $E$ , i.e. equal weight to all energy eigenstates in  $\mathcal{H}_E$ .*

# Typicality

- The theorem says that most wave functions in the energy shell  $\mathcal{H}_E$  will be both in MATE and in MITE.
- When I say “for most  $\Psi$ ” of the composite system, I mean that  $\Psi$  is typical with respect to a uniform measure on the unit sphere in  $\mathcal{H}_E$ . This measure on “wave functions” was considered already by Schrödinger and particularly Felix Bloch. It yields the microcanonical measure  $\rho^{\text{mc}}$  but goes beyond it.
- In fact for macroscopic systems one can show that MITE implies MATE.

The opposite is however not true. This is particularly relevant when one considers energy eigenfunctions  $|n\rangle$ .

While most energy eigenstates including those with MBL must, as shown, generally be in MATE, most energy eigenfunctions for systems with MBL are not in MITE.

- There is no analog to MITE for a classical system where any subsystem of a composite system in state  $X^{(1,2)}$  is also in a unique state  $X^{(2)}$ .
- When the interaction between systems 1 and 2 is weak,  $H \approx H_1 \otimes I_2 + I_1 \otimes H_2$ , then, as is well known,

$$\text{tr}_2 \rho^{\text{mc}} \approx \frac{1}{Z} e^{-\beta H_1},$$

for  $\beta = \beta(E) = dS_{\text{eq}}(E)/dE$ .

If MATE-ETH holds strictly, i.e., if *all* energy eigenstates in  $\mathcal{H}_{\text{mc}}$  are in MATE, then every state  $\psi \in S(\mathcal{H}_{\text{mc}})$  will sooner or later reach MATE and spend most of the time in MATE in the long run. That is because, writing  $\overline{f(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int f(t) dt$  for time averages,  $|n\rangle$  for the energy eigenstate with eigenvalue  $E_n$  and  $\psi_t = e^{-iHt}\psi$ ,

$$\begin{aligned} \overline{\langle \psi_t | P_{\text{eq}} | \psi_t \rangle} &= \sum_{n, n'} \langle \psi | n \rangle \overline{e^{iE_n t} \langle n | P_{\text{eq}} | n' \rangle e^{-iE_{n'} t} \langle n' | \psi \rangle} \\ &= \sum_n |\langle \psi | n \rangle|^2 \langle n | P_{\text{eq}} | n \rangle \geq \sum_n |\langle \psi | n \rangle|^2 (1 - \delta) \\ &= 1 - \delta, \end{aligned}$$

provided  $H$  is non-degenerate, i.e.,  $E_n \neq E_{n'}$  for  $n \neq n'$  (using  $\overline{e^{iEt}} = 1$  if  $E = 0$  and  $= 0$  otherwise).

- A similar statement is true when there is degeneracy.

## Summary of Boltzmann's Ideas (also Maxwell, Kelvin, Feynman)

Time-asymmetric behavior as embodied in the second law of thermodynamics is observed in *individual macroscopic* systems. It can be understood as arising naturally from time-symmetric microscopic laws when account is taken of a) the great disparity between microscopic and macroscopic sizes, b) initial conditions, and c) that what we observe is “typical” behavior of real systems — not of all imaginable ones. Common alternate explanations, such as those based on equating irreversible macroscopic behavior with ergodic or mixing properties of ensembles (probability distributions) already present for chaotic dynamical systems having only a few degrees of freedom or on the impossibility of having a truly isolated system, are either unnecessary, misguided or misleading.