Central Limit Theorems and Lee-Yang Zeros

Joel L. Lebowitz
Rutgers University
(Helsinki, June, 2014)

<u>Happy Birthday Antti</u>

We consider a sequence of increasing sets Λ_N, $N \to \infty$, and define the random variable $X_{\Lambda_N} \in \{0, 1, \ldots, N\}$ as the number of points (objects) in Λ_N. Λ_N could be a set in $\mathbb{R}^d, \mathbb{Z}^d$ or the edges of a graph G. In all cases, due to constraints, the number of points is at most N. Let

$$\text{Prob}(X = m) = p_m z^m / P(z), \quad m = 0, 1, \ldots, N$$

where $z \geq 0$, $p_0 > 0$, $p_N > 0$ and

$$P(z) = \sum_{m=0}^{N} p_m z^m$$

(I have left out the dependence of X, p_m and $P(z)$ on Λ_N and have assumed $p_0 > 0$ to avoid dealing with zeros of $P(z)$ at $z = 0$.)

$z \geq 0$ is called the fugacity in statistical mechanics and $P(z)$, with the proper definition of the p_m, is the grand canonical partition function. In some cases we shall consider, z does not enter into the probabilities and we simply have $\text{Prob}(X = m) = p_m/P(1)$. In all cases we can think of $P(z)$ as the generating function for the probabilities of having exactly m points in Λ_N.

Using the fundamental theorem of algebra we write

$$P(z) = C \prod_{j=1}^{N} (z + \zeta_j)$$

where $\{-\zeta_j\}$ are the zeros of $P(z)$.

Note that for finite N none of the zeros can be on the positive real axis. They must be on the negative real axis or come in complex conjugate pairs.

It follows from the definitions that

$$\langle X \rangle = \mu(z, \Lambda_N) = z\frac{\mathsf{d}}{\mathsf{d}z}\log P(z) = \sum_{j=1}^{N} \frac{z}{\zeta_j + z}.$$

We also have

$$\mathrm{Var}(X) = \sigma^2(z, \Lambda_N) = z\frac{\mathsf{d}}{\mathsf{d}z}\langle X \rangle = \sum_j \frac{z}{z + \zeta_j}\left(1 - \frac{z}{z + \zeta_j}\right).$$

We shall be interested in the asymptotic normality of the fluctuations of the number of objects in Λ_N when $N \to \infty$. We say that our system satisfies a CLT if

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \to 0, \quad \text{for } N \to \infty,$$

where

$$F(x) \equiv \frac{1}{P(z)} \sum_{m\leq\mu+\sigma x} p_m z^m = \text{Prob}\left\{\frac{X-\mu}{\sigma} < x\right\}$$

$$G(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2} \mathrm{d}u.$$

We say that our system satisfies the *stronger* local CLT (LCLT) if

$$\sup_{x\in\mathbb{R}} \left| F_\ell(x) - (2\pi)^{-1} e^{-x^2/2} \right| \to 0, \quad \text{for} N \to \infty,$$

where

$$F_\ell(x) = \sigma \text{Prob}\,(X = [\mu + \sigma x])$$

and the dependence on z and Λ_N has been suppressed.

Also suppressed is the fact that X can have only non-negative integer values.

I will now state several theorems which give sufficient conditions for a CLT or a stronger LCLT based on the location of the zeros ζ_j of $P(z : \Lambda_N)$. We assume in all cases that for some $z_0 > 0$ one has that $\mu(z_0, \Lambda_N)$ and $\sigma(z_0, \Lambda_N)$ approach ∞ as $N \to \infty$ and the statements below all refer to $z = z_0$.

Theorem 1. (Harper 67, Bender 73, Canfield 80, Pitmann 97)

When the zeros are all on the negative real axis, i.e. $\zeta_j > 0$, the system satisfies a LCLT.

Theorem 3. (L, Pittel, Ruelle, Speer, in preparation)

When Re$\zeta_j > 0$ and Imζ_j/Re$\zeta_j \le \sqrt{3}$ then the system satisfies a LCLT. (Note that Theorem 2 includes Theorem 1 as a special case.)

Theorem 3. (LPRS)

When Re$\zeta_j \geq 0$ the system satisfies a CLT.

Theorem 4. (LPRS)

Under the assumption that Re$\zeta_j \geq \delta > 0$ for all N and $p_1/[p_0 N^{2/3}] \to \infty$ as $N \to \infty$ the system satisfies a LCLT.

Theorem 5. (Iaglonitzer and Souillard, 79)

Suppose there exists a disc of radius $\delta > 0$ centered around $z_0 > 0$ such that, uniformly in N, there are no zeros of $P(z)$ in this disc. Then if $\sigma^2(z_0, \Lambda_N)/N^{2/3} \to \infty$, the system satisfies a CLT for $z = z_0$.

Note: The conditions of Theorem 4 also imply that $\sigma^2(z_0, \Lambda_N)/N^{2/3} \to \infty$.

Example 1: Determinantal point processes

Determinantal point processes are prominent structures in the theory of random matrices as well as in many other contexts. These are processes for which the k-point correlation function in a set Λ can be written as a $k \times k$ determinant,

$$\rho_k(x_1, \ldots, x_k) = \det[K(x_j, x_l)]_{j,l=1,\ldots,k}, \quad x_i \in \Lambda_N$$

where $K(x, y)$ — referred to as the correlation kernel — is independent of k. Here K (viewed as the kernel for an integral operator supported on Λ_N) is Hermitian, and all its eigenvalues are discrete and lie between zero and one. The ρ_k depend on Λ_N but have well defined limits as $N \to \infty$.

In particular,

$$\rho_1(x) = K(x,x),$$
$$\rho_2(x_1,x_2) = \rho_1(x_1)\rho_1(x_2) - |K(x_1,x_2)|^2 \le \rho_1(x_1)\rho_1(x_2)$$

In general

$$\rho_k(x_1,\ldots,x_k) \le \prod_{j=1}^{k} \rho_1(x_j),$$

so these processes are "repulsive".

In terms of the correlation functions, the mean and variance of the number of points is $\Lambda \subset \Lambda_N$, $\mu(\Lambda)$ and $\sigma^2(\Lambda)$, are given by the integrals (or sums),

$$\mu(\Lambda) = \int_\Lambda \rho_1(x)\,dx,$$
$$\sigma^2(\Lambda) = \int_\Lambda dx_1 \int_\Lambda dx_2 \,\{\rho_2(x_1,x_2) - \rho_1(x_1)\rho_1(x_2) + \rho_1(x_1)\delta(x_1-x_2)\}\,.$$

It can be shown, using Mobius formula to compute Prob($X_\Lambda = m$) $= p_m/P(1)$ from the correlation functions (see e.g. Forrester book, 2010), that

$$P(1-z;\Lambda) = 1+\sum_k \frac{(-1)^k}{k!} z^k \int_\Lambda \mathrm{d}x_1 \ldots \int_\Lambda \mathrm{d}x_k \, \rho_k(x_1, \ldots, x_k) = \prod (1-z\lambda_\ell),$$

where the λ_ℓ are the eigenvalues of K acting on $L^2(\Lambda)$.

We then have for any determinantal process with Hermitian kernel:

$$P(z;\Lambda) = C \prod_{l=0}^{\infty} (\zeta_l + z), \qquad \zeta_l = \frac{1-\lambda_l(\Lambda)}{\lambda_l(\Lambda)}$$

where the $\lambda_l(\Lambda)$ are eigenvalues of the integral operator K supported on Λ, with $\lambda_\ell \in [0,1]$ so the ζ_l are all on the negative real axis.

Theorem 1 then gives a LCLT for all such determinantal point processes (Forrester and L, (FL) 2014).

Random Matrices

One of the best known examples of a determinantal point process is given by the eigenvalues of the random matrices specified by the Gaussian Unitary Ensemble (GUE): a Gaussian probability measure on the space of complex $N \times N$ Hermitian matrices M with Prob$(M) \sim$ exp$(-c\text{Tr}MM^{\dagger})$ which is unitary invariant and thus unchanged by conjugation by unitary matrices.

The probability distribution of eigenvalues $\{\gamma_j\}$ of M on $\mathbb{R}$ is given by (Wigner, Dyson)

$$\prod(\gamma_j - \gamma_k)^\beta \exp\left(-c\sum_{n=1}^{N}\gamma_n^2\right),$$

where $\beta = 2$. By scaling the eigenvalues so that the mean density is unity and taking $N \to \infty$, one obtains a translation invariant determinantal point process specified by the so-called sine kernel $K(x, y) = \sin\pi(x-y)/\pi(x-y)$, for which $\mu(\Lambda) = |\Lambda|$ and $\sigma^2(\Lambda) \sim \frac{1}{\pi}\log|\Lambda|$, where $|\Lambda|$ is the length of an interval in $\mathbb{R}$.

The same holds for the Circular Unitary Ensemble (CUE), consisting of unitary matrices chosen according to the Haar measure, where all the γ_i lie on a circle.

Costin and L. [CL, 95] proved a CLT for this system. This was done by showing that as a consequence of the property that $\sigma_\Lambda \to \infty$ as $|\Lambda| \to \infty$, all properly normalized cumulants beyond the second vanish for $|\Lambda| \to \infty$.

In fact the proof of CL makes no explicit use of the particular determinantal point process for the GUE, requiring only that the corresponding kernel be locally trace class and self-adjoint, and that the variance tends to infinity. As noted in CL this was pointed out by H. Widom.

Theorem 1 then shows that the fluctuation in the eigenvalue distribution of the bulk GUE (and CUE) actually satisfy a LCLT.

CL also proved a CLT for the Gaussian Orthogonal Ensemble (GOE), $\beta = 1$, and the Gaussian Symplectic Ensemble (GSE), $\beta = 4$, neither of whose eigenvalues are determinantal processes. This has been strengthened by FL to a LCLT for the GSE but not so far for the GOE.

The proof of CL and FL are both based on known interpolation formulas between the eigenvalues of the GUE and those of the GOE and GSE.

The eigenvalue distribution of both the GOE and GSE are given by a Pfaffian process as are many other useful distributions. It is therefore an interesting open (as far as I know) question if there are any general statements one can make about fluctuations of Pfaffian processes.

(For Pfaffian processes the kernel $K(x, y)$ is given by a 2×2 anti-symmetric matrix.)

Ginibre Ensemble and Coulomb Systems

The Ginibre ensemble consists of non-Hermitian matrices with i.i.d. complex Gaussian entries. The eigenvalues form a determinantal point process with a complex Hermitian kernel: in the limit $N \to \infty$ this is given by $K(w,z) = \frac{1}{\pi} e^{-(|w|^2+|z|^2)/2} e^{w\bar{z}}$ where z and w are complex. Here $\Lambda_N \subset \mathbb{R}^2$ and we again scale the eigenvalues so that when $N \to \infty$ and $\Lambda_N \to \mathbb{R}^2$ we have a translation and rotation invariant point process with density 1, $\rho_1(x) = 1$.

The distribution of eigenvalues of the Ginibre ensemble in the complex z plane is exactly the same as that of the one component plasma (OCP), also known as jellium, in which N positively charged particles interacting with a pair potential, $-e^2 \log|r_i - r_j|$, in a uniform negative background at reciprocal temperature $\beta = 2$, charge $e = 1$. This Coulomb system, which is well defined for all β, is exactly solvable at $\beta = 2$ with the decay of correlations between regions separated by a distance r being proportional to $\sim e^{-cr^2}$.

The variance $\sigma^2(\Lambda) \sim |\partial\Lambda|$. This system is thus, according to Theorem 1, asymptotically normal.

In fact we know more: let $\mathbb{R}^2$ be divided into squares Γ_j of area L^2 whose centers are located $L\mathbb{Z}^2$.

$$\xi_j = X(\Gamma_j)/\sigma(\Gamma_j), \quad \sigma(\Gamma_j) = KL^{1/2}.$$

Then the joint distribution of the $\{\xi_j\}$ approaches as $L \to \infty$ a Gaussian measure with covariance

$$C_{j,k} = \left[\delta_{j,k} - \frac{1}{4}\sum_e \delta_{j-k,e}\right] = \frac{1}{4}\,[-\Delta]_{j,k}\,, \qquad (*)$$

where e is the unit lattice vector and Δ is the discrete Laplacian. This means that the charge fluctuations in $\Gamma_{j,L}$ are compensated by the opposite charges in neighboring squares.

This is actually a special case of a more general result about charge fluctuations in Coulomb systems (proved by L (1983) for $d \geq 2$).

The formula $(*)$ also true for the eigenvalues of the GUE with $\sigma^2(L) \sim \frac{1}{\pi} \log L$ whose distribution corresponds to that of a one dimensional OCP with logarithmic (2 dimensional) Coulomb interactions between the charges at $\beta = 2$ (CL).

The strong screening of Coulomb systems expressed by $(*)$ gives the charges an almost crystal like rigidity. This is manifested by the remarkable recent result of Gosh and Peres (2013) for the Ginibre ensemble: Suppose we are given the location of all the charges (particles, eigenvalues) outside some region Λ. Then the number of particles inside Λ is uniquely determined. Nevertheless the distribution of points inside Λ is absolutely continuous wrt Lesbesgue measure.

Interestingly there is also another translation and rotational invariant distribution of points in the plane which also has the rigidity property (Gosh and Peres, 2013). These are the zeros of the so called Gaussian Analytic Function (GAF), also sometimes referred to as the Weyl polynomial,

$$f(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$$

where the ξ_k are i.i.d. standard complex Gaussians.

It turns out that the distribution of zeros of the GAF have the same (qualitative) variance and large deviation function (LDF) as that of the Ginibre ensemble, i.e. the OCP at $\beta = 2$.

The LDF, which can be computed for general β from electrostatic energy considerations (L, Jancovici, Manificat, 1993), reflects the strong repulsion between the charges. Roughly speaking, the probability of having $n(R)$ particles in a disc of radius R behaves as

$$\text{Prob}\left\{|n(R) - \pi R^2| > R^\alpha\right\} \sim \exp\left[-c_\alpha R^{\phi(\alpha)}\right],$$

with

$$\phi(\alpha) = \begin{cases} 2\alpha - 1 & , \quad \frac{1}{2} < \alpha \leq 1 \\ 3\alpha - 2 & , \quad 1 \leq \alpha \leq 2 \\ 2\alpha & , \quad \alpha \geq 2 \end{cases}$$

This probability is much smaller than the large deviations for systems with short range interactions where for $\alpha = 2$ one would get e^{-cR^2} instead of e^{-cR^4}.

For CUE (and bulk GUE) the probability of a “gap” of size L (in the scaling with unit density) is given by (Dyson 1962)

$$\mathrm{Prob}(X_L = 0) \sim \exp[-\frac{1}{8}\pi L^2]$$
$$\mathrm{Prob}(X_L = 1) \sim \mathrm{Prob}(X_L = 0)\exp[\pi L]$$

For a nice review of these results see Forrester (2013).

I note finally that the zeros of the GAF do not form a determinantal process so the correlation cannot be computed explicitly. On the other hand the zeros of the Kac polynomial

$$f_1(z) = \sum \xi_k z^k$$

(without the $(k!)^{-1/2}$) do form a determinantal process (Peres and Virag, 2004). These correspond in the scaling limit to charges confined to a disc interacting with the logarithmic potential at $\beta = 2$. The charges get pushed to the boundary of the Disc (Forrester, 2010).

There are other well known examples of determinantal processes in statistical physics and mathematics. Spin polarized free fermions in dimension d provide examples of determinantal point processes in higher dimensions. For Λ a sphere of radius R, σ_Λ^2/R^{d-1} is proportional to $\log R$ in the limit $R \to \infty$, and in particular σ_Λ^2 diverges in this limit so that we have a LCLT. The kernel $K(x,y)$ for each of these systems is proportional to the Bessel function $J_{d/2}(x-y)/|x-y|^{d/2}$. In fact this system (with Slater determinantal wave functions) was the first of such processes to be investigated (Macke, 1975).

A possible example, conditional upon the validity of the Montgomery-Odlyzko law, is the set of Riemann zeros for large modulus. The Montgomery-Odlyzko law states that certain statistical properties of the latter, upon appropriate scaling, coincide with the bulk scaled GUE and thus might form a determinantal point process. A proof that these zeros satisfy a LCLT is an open question, while the weaker statement of a CLT has now been proven (Bourgade, et al.) even without the Riemann hypothesis.

Example 2: Graph counting polynomials

Let G be a connected graph with a set E of edges each of which can be occupied by a particle or empty. We say that a configuration of occupied edges is admissible if the number of occupied edges attached to any vertex v, $d(v)$, is restricted to some set of natural numbers. The maximal total number of occupied edges in $E = \Lambda_N$ is N, and p_m is the number of permissible configurations in which there are exactly m occupied edges.

When $d(v) = \{0, 1\}$, $P(z)$ corresponds to the “matching polynomial” of the graph. This system has the statistical mechanical interpretation of placing “dimers” with fugacity z on the edges of the graph. The vertices to which no dimers are attached are called monomers. It was shown by Heilmann and Lieb (1972) that all the zeros of $P(z)$ for the dimer model lie on the negative real z-axis. The variance was shown by Godsil (1981) to go to infinity as $N \to \infty$ so the system satisfies a LCLT.

In current work by Pittel, L., Ruelle and Speer we consider the case of graph counting polynomials which arise when the restriction of $d(v) \in \{0, 1\}$ above is generalized to $d(v)$ some set of non-negative integers. We obtain a LCLT when $d(v) = \{0, 1, 2\}$ corresponding in statistical mechanics to "unbranched" polymers.

The result makes use of the fact that all the zeros of $P(z)$ have negative real parns* so that a LCLT follows from Theorem 4 once the additional properties on the location of the zeros, required by Theorem 4, have been verified. This is relatively easy when the maximum number of edges impinging on a vertex is bounded, but in general requires some additional conditions.

$*$ The proof of this follows from an extension of the work of Ruelle (99).

Before going on to give applications of Theorem 5, which come at this time exclusively from statistical mechanics, let me give some proofs. I will begin with Theorem 3 from which Theorem 2 and thus Theorem 1 will follow using an additional input.

Proof of Theorem 3

Let us split up the roots of $P(z)$ into those which are real (and therefore negative) and those which are complex. Then we can write

$$P(z) = C \prod^{(1)} (t_j + z) \prod^{(2)} (|\zeta_\alpha|^2 + 2x_\alpha z + z^2), \qquad z > 0,$$

where the first product is over all real roots , $t_j > 0$ while the second product is over all pairs of complex roots with real part $x_\alpha = \mathrm{Re}\zeta_\alpha$.

We now observe that each term $t_j + z$ is the partition or generating function of a Bernoulli random variable η_j,which takes the value 0 with probability $\frac{t_j}{t_j+z}$ and the value 1 with probability $\frac{z}{t_j+z}$.

Similarly each term in the second product $|\zeta_\alpha|^2 + 2x_\alpha z + z^2$ is, for $x_\alpha \geq 0$, the generating function for a random variable $\eta_\alpha = \{0, 1, 2\}$ with probabilities

$$w_\alpha(0) = \frac{|\zeta_\alpha|^2}{|z + \zeta_\alpha|^2}, \quad w_\alpha(1) = \frac{2x_\alpha z}{|z + \zeta_\alpha|^2}, \quad w_\alpha(2) = \frac{z^2}{|z + \zeta_\alpha|^2}.$$

Thus X can be written as a sum of independent, non-identical, random variables η_j and η_α. It follows then from results in Feller that X will satisfy a CLT whenever Var$(X_{\Lambda_N}) \to \infty$ as $N \to \infty$. This proves Theorem 3.

Proof of Theorem 1 and 2

To go from a CLT to a LCLT it is sufficient (Bender, 1973) that the p_m have the log concavity property, i.e. $p_m^2 \geq p_{m-1}p_{m+1}$. When all the roots are on the negative axis $\zeta_j > 0$, the concavity of $P(z)$ was proven by I. Newton.

For the more general case, $\text{Im}\zeta_j/\text{Re}\zeta_j \leq \sqrt{3}$, we use a result by Menon (1969) that the product of concave polynomials is also concave*. This is obviously true for the $\zeta_j > 0$ since $p_2 = 0$ and can be checked to be true for the generating function of the random variables η_α, taking on values $\{0,1,2\}$, whenever $\text{Im}\zeta_\alpha/\text{Re}\zeta_\alpha \leq \sqrt{3}$.

$*$ (Not true for convex polynomials.)

I should note here that for determinantal processes the fact that the distribution of the number of points is a sum of independent Bernoulli variables was proven by Shirai and Takahashi (2003) and in a different way by Ben Hough, et al. (2006). Neither of these proofs used the fact that L-Y zeros lie on the negative axis and did not (at least explicitly) prove a LCLT.

The proofs of Theorems 4 and 5 are a bit more complicated and I shall not go into them here.

Lee-Yang zeros for Ising Spins

The thermodynamic properties of an equilibrium system in a domain $\Lambda \subset \mathbb{Z}^d$ are given by given by the "grand canonical" partition function

$$P(\beta, z; \Lambda) = \sum_{m=0}^{N_{\text{max}}} z^m Q(\beta, m; \Lambda)$$

where $Q(\beta, m; \Lambda)$ is the "canonical" partition function for m particles in Λ, β is the reciprocal temperature, and $N_{\text{max}}(\Lambda) \leq |\Lambda|$.

Using spin language, $\sigma(x) = \pm 1$, $x \in \Lambda$, $m(\underline{\sigma}) = \sum \eta(x)$, $\eta(x) = \frac{1}{2}[1 + \sigma(x)] = \{0, 1\}$, $z = e^{2\beta h}$, the magnetic fugacity for h a uniform magnetic field, we have up to constant terms

$$P(\beta, z; \Lambda) = \sum_{\underline{\sigma}} z^m e^{-\beta U(\underline{\sigma})},$$

where $U(\underline{\sigma})$ is the interaction energy for a given configuration in Λ.

As already mentioned earlier, for finite Λ there can be no zeros at $z \geq 0$, the physically relevant value of the fugacity. This means that the thermodynamic pressure defined as $\Pi(\beta, z; \Lambda) = \frac{1}{|\Lambda|} \log P(\beta, z; \Lambda)$ is real analytic for all fugacities and there can be no "phase transitions", defined as non-analycities in the pressure as a function of z.

The situation is different in the thermodynamic limit $\Lambda \to \mathbb{Z}^d$. This is the right model for a macroscopic system containing, say 10^{23} atoms, when we are not considering surface effects. In this limit the pressure is given by

$$\Pi(\beta, z) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{\log P(\beta, z; \Lambda)}{|\Lambda|}.$$

The zeros of $P(\beta, z; \Lambda)$ can now approach the positive z-axis and cause singularities in the pressure.

In a famous work Lee and Yang (1952) proved that for ferromagnetic pair interactions,

$$U(\underline{\sigma}) = -\sum_{x,y} J(x,y)\sigma(x)\sigma(y), \qquad J(x,y) \geq 0$$

all the zeros of P, $\{\zeta_j, j = 1, \ldots, |\Lambda|\}$, lie on the unit circle, $|\zeta_j| = 1$ corresponding, for $z > 0$, to zero field, $h = 0$. Theorem 5 by Iaglonitzer and Souillard then shows that the system satisfies a CLT for $h \neq 0$ and for $h = 0$ and β small. For β large the zeros do approach the real z-axis at $z = 1$ and the fluctuations are no longer Gaussian at the critical temperature, $\beta = \beta_{\mathrm{C}}$, for $d \leq 4$. For $\beta > \beta_{\mathrm{C}}$ the nature of the Gibbs measure when $\Lambda \to \mathbb{Z}^d$ depends on boundary conditions.

The fact that the variance $\sigma^2(z, \Lambda) \to \infty$ is true quite generally by Ginibre's theorem described later.

Ruelle (2010) gave a general characterization of polynomials satisfying the Lee-Yang property, $|\zeta_j| = 1$. He showed in particular that for Ising systems the <u>only</u> interactions $U(\underline{\sigma})$ for which $|\zeta_j| = 1$ for <u>all</u> β are ferromagnetic pair interactions.

In works by L. and Ruelle (2011) and by L., Ruelle and Speer (2012) we considered the behavior of the zeros for different interactions. We showed in particular, for certain classes of interactions $U(\underline{\sigma})$ that the zeros of $P(\beta, z; \Lambda)$ all lie in some cases in the left half of the complex z-plane. These systems satisfy the conditions of Theorem 4 and thus satisfy a LCLT.

An example of a system where we show that all the zeros lie on the negative real axis is the following.

Consider $\mathbb{Z}^2$ with anti-ferromagnetic interactions $J < 0$ between all pairs of vertices on alternating squares.

×		×	
	×		×
×		×	
	×		×
×		×	

The proof that the variance of $X_\Lambda \geq c|\Lambda|$, $c > 0$, under quite general conditions, is due to Ginibre.

Theorem 6 Ginibre (1967)
Let $P(X_\Lambda = m) = P_m/m!$. If $P_j = 0$ then $P_{j+k} = 0$ for all $k \geq 0$. Then if

$$P_{n+2}/P_{n+1} \geq P_{n+1}/P_n - A, \quad A > -1 \qquad (*)$$

one has that

$$\text{Var}(X_\Lambda) \geq \langle X_\Lambda \rangle/(1+A).$$

The proof is elementary. Write

$$\langle m \rangle^2 (1+A)^2 = \left\langle \left[\frac{P_{m+1}}{P_m} + mA \right] \right\rangle^2 \leq \left\langle \left[\frac{P_{m+1}}{P_m} + mA \right]^2 \right\rangle$$

and expand.

Ginibre then shows that for pair interaction satisfying some simple conditions $(*)$ holds. For lattice systems, $\sum_{x,y} |J(x,y)| < \infty$ is sufficient. In LPRS we extended Gitibre's proof to many body interactions and to graph counting polynomials whenever $d(v) = \{0, 1, 2, \ldots, k\}$ (with no gaps).