Stability and Variability

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Dreams of Earth and Sky: A Celebration for Freeman Dyson

September 2013

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Quoting Dyson:

"To me the most astonishing fact in the universe, . . . , is the power of mind which drives my fingers as I write these words. Somehow by natural processes still mysterious, a million butterfly brains working together in a human skull have the power to dream, to calculate, . . . , to translate thoughts and feelings into marks on paper which other brains can interpret. . . ."

"... It appears to me that the tendency of mind to infiltrate and control matter is a law of nature. ... Mind is patient. Mind has waited for 3 billion years on this planet before composing its first string quartet. It may have to wait for another 3 billion years on this planet before it spreads all over the galaxy. Ultimately late or soon, mind will come into its heritage."

Freeman Dyson in
Infinite in all Directions, p. 118

“To make clear the real and lasting importance of unfashionable science, I return to the field in which I am an expert, namely mathematical physics. Mathematical physics is the discipline of people who try to reach a deep understanding of physical phenomena by following the rigorous style and method of mathematics. It is a discipline that lies at the border between physics and mathematics. The purpose of mathematical physicists is not to calculate phenomena quantitatively but to understand them qualitatively. They work with theorems and proofs not with numbers and computers. Their aim is to qualify with mathematical precision the concepts upon which physical theories are built. ”

Freeman Dyson in
From Eros to Gaia, pp. 164-165

Nature has a hierarchical structure, with time, length and energy scales ranging from the submicroscopic to the super galactic. Surprisingly, it is possible and, as emphasized by Freeman, in many cases essential to discuss these levels independently. Quarks are irrelevant for understanding protein folding and atoms are a distraction when studying tsunamis. Nevertheless, it has been a widespread dogma of science, very successful in the past four hundred years, that there are no new fundamental laws, only new phenomena, as one goes up the hierarchy. Explanations are therefore always looked for in the smaller scales.

Whether this paradigm will continue to hold as we try to fit general relativity with elementary particle physics, or even put mental states of consciousness into the framework of our current physical theories is still a mystery. But as noted by Ed Witten, “how [will] we know except by trying?”

Be that as it may this reductionism is certainly applicable when it comes to describing the properties of inanimate objects in terms of electrons and nuclei. This is the subject of statistical mechanics, which provides a framework for relating mesoscopic and macroscopic thermal phenomena to the microscopic world of atoms and molecules. Fortunately, many striking features of macroscopic systems, such as the abrupt change of properties of a substance at a phase transition, like the boiling and freezing of water, can be obtained from simplified microscopic models.

Statistical mechanics therefore often takes as its lowest level starting point Feynman's description of atoms as "little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another." Sometimes we even use cruder models such as particles on a lattice.

The reason why such crude models often work so well lies in the large disparity in the spatial and temporal scales between the world of atoms and the world of macroscopic objects. This not only necessitates a statistical theory which ignores many details but also assures, in analogy to the law of large numbers in probability theory, that such a theory will give predictions precise enough to have the force of "law", as in the second law of thermodynamics: the entropy of an isolated macroscopic system never decreases.

Statistical mechanics thus aims to explain how the cooperative behavior of many individual entities can give rise to new phenomena having no counterpart in the properties or dynamics of the separate entities. The nature of these entities can vary widely: in the traditional studies they are atoms in a fluid, spins in a magnet, electrons in a metal, etc. In more recent applications they can also be birds in a flock, people in a soccer stadium or at a demonstration.

The twentieth century saw the development of the subject into a physically very successful and mathematically very beautiful theory of systems in thermal equilibrium. Freeman's work played an essential role in this achievement.

The development of a comparable theory for the more complex world of nonequilbrium phenomena, ranging from heat conduction in metals to transport in living cells remains a challenge.

Stability of Matter

My first direct contact with Freeman's scientific work came in 1968 when I was working with Elliott Lieb on showing in a mathematical physics sense that statistical mechanics can provide a basis for the equilibrium thermodynamics of real matter consisting of electrons and nuclei interacting via Coulomb forces. A very crucial ingredient in our analysis was Dyson's proof with Andrew Lenard (1967) of the stability against collapse of macroscopic Coulomb systems. To quote from the (1968) paper with Lieb: "The Dyson-Lenard theorem is as fundamental as it is difficult."

Before going into the specifics of that problem, I will give a birds-eye view of how statistical mechanics of equilibrium systems connects the microscopic with the macroscopic.

Skipping **many** steps: For a multi-species system of $\underline{N}$ particles in a region V with a microscopic "effective" Hamiltonian H, $H = K + U$, kinetic plus potential energy, in equilibrium at temperature $T = \beta^{-1}$, we define the canonical partition function of a quantum system as

$$Z(\beta, \underline{N}; V) = \sum_{\alpha} \mathrm{exp}[-\beta E_{\alpha}],$$

where the sum is over the energy eigenvalues of H. For classical systems the sum is replaced (up to some constants) by an integral of $\mathrm{exp}[-\beta U(X_{\underline{N}})]$ over the configurations of the system, denoted by $X_{\underline{N}}$.

We consider now sequences of (regular) domains V_j, $j = 1, 2 \ldots$, such that as j increases $V_j \to \infty$, $\underline{N}_j \to \infty$, $\underline{N}_j / V_j \to \rho$. Taking the "$j \to \infty$ limit" of the sequence,

$$f_j \equiv -(\beta V_j)^{-1} \log Z(\beta, \underline{N}_j, V_j) \to f(\beta, \underline{\rho})$$

we identify $f(\beta, \underline{\rho})$ with the Helmholtz free energy of the macroscopic system. To make this connection with thermodynamics work we have to show for "realistic" effective Hamiltonians describing macroscopic matter that the limit $f_j \to f(\beta, \underline{\rho})$ exists and has the right (convexity) properties required of a thermodynamic free energy of a homogenous system.

It may be worth noting here a quote from Onsager relevant to the "thermodynamic limit," $j \to \infty$. "The notion of homogeneous thermodynamic systems is valid for systems large compared to the size of molecules and small compared to the size of the moon."

It is not that real systems are infinite, they are not, but that the correct idealization for bulk properties of macroscopic systems containing about 10^{20} - 10^{30} atoms is an infinite system from which gravitational forces are excluded.

By the mid-sixties many authors, particularly Fisher and Ruelle had developed techniques for establishing the existence and properties of this thermodynamic limit when the interaction potential, U, satisfies two criteria:

a) H-stability

$$U(X_{\underline{N}_j}) \geq -BN \quad \text{(Classical mechanics)}$$

$$E_0 \geq -BN \quad \text{(Quantum mechanics)}$$

where E_0 is the ground state energy of H and N is the total number of particles.

b) Tempering: The interaction between two sets of particles separated by a distance R is bounded below by $-R^{-(d+\varepsilon)}$, where d is the dimension of the system.

It was further known that both these conditions hold for the case where U is the “effective” potential between neutral atoms or molecules, e.g. the Lenard-Jones pair potential but clearly neither a) nor b) is satisfied for classical systems containing both positive and negative charges e_α interacting via a Coulomb pair-potential $e_\alpha e_\gamma |\mathbf{r}_i - \mathbf{r}_j|^{-(d-2)}$. This interaction is unbounded below for $d \geq 2$ at short distances and decays too slowly at long distances.

The fact that Coulomb rather than nuclear interactions are relevant for ordinary matter follows, as pointed out by Dyson, from the energy scales involved.

Lars Onsager may have been the first to realize the importance of H-stability. He actually proved it in 1939 for systems which also have, in addition to the Coulomb interactions, hard core interactions between the particles.

But real systems do not have hard cores and so there was the problem of proving H-stability for quantum systems. Unlike classical systems, quantum systems do have a lower bound on the ground state energy for finite N, e.g. -13.5eV for one proton and one electron. What nobody knew however, at that time, was how that lower bound scales with N.

Here is where the heroic efforts of Dyson and Lenard (1967) came in - they proved that if either the negative (or positive) charged species were all Fermions then the quantum many body system is H-stable. Dyson later showed that having Fermions, which electrons are, is not only sufficient but also necessary for H-stability. When both charges are Bosons then $E_0 \leq -\text{const.}N^{+7/5}$. Such a system of charged bosons would therefore not be thermodynamically stable.

Using the Dyson-Lenard theorem for H-stability Elliott and I overcame the problem of slow decay by making use of the tendency of Coulomb systems to shield bare charges. This proved the existence and convexity of $f(\beta, \underline{\rho})$ for **neutral** Coulomb systems, i.e. $\sum e_\alpha \rho_\alpha = 0$, where e_α is the charge of species α and ρ_α its limiting density. The sum is over all species.

The situation is quite different when the system has a net charge.

What happens then is that the excess charge Q_j goes to the surface of the region V_j. The free energy f_j then depends on the amount of charge per surface area S_j of V_j, with f_j increasing with Q_j and going to infinity when $Q_j/S_j \to \infty$. This means that there is a strong inhibition against large charge fluctuations in Coulomb systems.

In fact if one considers the grand-canonical ensemble for Coulomb systems, where one does not fix the particle number $\underline{N}_j$ or the charge Q_j, then the thermodynamic quantities are the same as if one only considered a restricted grand canonical ensemble made up of neutral systems with $Q_j = 0$.

In the unrestricted ensemble

$$\frac{\langle Q_j \rangle}{V_j} \to 0, \quad \frac{\langle Q_j^2 \rangle}{V_j} \to 0$$

We can also ask this question of fluctuations in a different way. Suppose we have a very large neutral system in a domain V (V will generally be taken to be infinite) then what are the charge fluctuations in a subdomain $\Lambda \subset V$.

I will now consider this question as well as similar fluctuations in the eigenvalue distribution of random matrices, which Dyson brilliantly connected with Coulomb systems.

Fluctuations

To fluctuate is normal and in most cases fluctuations are themselves normal, by which I mean that they scale like the square root of typical values as in a Poisson process. There are however many very interesting exceptions where the fluctuations are subnormal. These range from charge fluctuations in Coulomb systems to energy fluctuations in the early universe. I will now describe some exact results for such systems, many related in one way or another to work done or inspired by Freeman.

To make things simple I will only consider systems with one type of particle. So for Coulomb systems I will consider only the one component plasma (OCP) or jellium model introduced by Wigner in the '30s. In this system positive point charges are immersed in a uniform continuum neutralizing negative background so fluctuations in particle number are the same as fluctuations in charge.

The results extend to charge fluctuations in multi-species systems.

Consider particle configurations in $\mathbb{R}^d$ with a translation invariant measure μ on configurations $X = (\mathbf{x}_1, \mathbf{x}_2, \dots)$. Let $N_\Lambda(X) =$ number of particles in region $\Lambda \subset \mathbb{R}^d$ with volume $|\Lambda|$. The average number of particles in Λ is then given, for a system with density ρ, by

$$\langle N_\Lambda \rangle = \rho |\Lambda|$$

and the variance by

$$\mathcal{V}_\Lambda = \langle (N_\Lambda - \langle N_\Lambda \rangle)^2 \rangle \sim |\Lambda|^\alpha$$

- $\alpha = 1$, Normal, always true for short range interaction at high temperatures.

- $\alpha > 1$, Critical point.

- $\alpha < 1$, Coulomb systems, eigenvalues of Gaussian random matrices, cosmology, lattice points, etc.

The variance $\mathcal{V}_\Lambda$ is expressible in terms of the pair correlation function of the infinite system,

$$\begin{aligned}\mathcal{V}_\Lambda &= \int_\Lambda \int_\Lambda \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 G(\mathbf{r}_1 - \mathbf{r}_2) \\ &= |\Lambda| \int_{\mathbb{R}^d} G(\mathbf{r}) \mathrm{d}\mathbf{r} - \int_{\mathbb{R}^d} G(\mathbf{r}) \alpha_\Lambda(\mathbf{r}) \mathrm{d}\mathbf{r},\end{aligned}$$

where

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \left\langle \sum_{i,j} \delta(\mathbf{r}_1 - \mathbf{x}_i) \delta(\mathbf{r}_2 - \mathbf{x}_j) \right\rangle - \rho^2,$$

$$\alpha_\Lambda(\mathbf{r}) = \int \chi_\Lambda(\mathbf{r} + \mathbf{r}_1)[1 - \chi_\Lambda(\mathbf{r}_1)] \mathrm{d}\mathbf{r}_1$$

$$\chi_\Lambda(\mathbf{y}) = \begin{cases} 1 & \mathbf{y} \in \Lambda \\ 0 & \mathbf{y} \notin \Lambda \end{cases}$$

What happens when $\Lambda \uparrow \mathbb{R}^d$ in a self similar way? α_Λ will then grow like the surface area $|\partial\Lambda| \sim |\Lambda|^{(d-1)/d}$ with $|\partial\Lambda| = 2$ for $d = 1$. Averaging $\alpha_\Lambda(\mathbf{r})/|\partial\Lambda|$ over rotations we obtain

$$\lim_{|\Lambda|\to\infty} \frac{\alpha_\Lambda(\mathbf{r})}{|\partial\Lambda|} = \alpha_d \; |\mathbf{r}|,$$

where

$$\alpha_d = \begin{cases} 1/2 & d = 1 \\ 1/\pi & d = 2 \\ \ldots & \end{cases}$$

We set $\lim \frac{1}{|\Lambda|}\mathcal{V}_\Lambda = \int_{\mathbb{R}^d} G(\mathbf{r}) \mathrm{d}\mathbf{r} = b$.

For critical systems $b = \infty$.

We are interested in cases where $b = 0$.

Examples where $b = 0$ are:

1) One component Coulomb systems with pair interactions $\varphi(r)$ and uniform charge background of density ρ. Then generally $\mathcal{V}_\Lambda \sim$ surface area of Λ.

· $\underline{i}$. $\varphi(r) = -r$ for $d = 1$. Bounded variance, i.e. $\mathcal{V}_\Lambda \to$ constant; Wigner crystal at all temperatures, both classical and quantum (Jansen and Jung). Thus there is no decay of $G(\mathbf{r})$ in $d = 1$.

This is not true in higher dimensions at high temperatures. One can prove that Coulomb systems have good decay of correlations: exponential decay classically (with hard cores) and power law decay, r^{-6}, quantum mechanically in $d = 3$. (The reason for the slower quantum decay is that the quantum fluctuations interfere with the shielding. The r^{-6} is related to the van-der-Waals forces between atoms.)

· <u>*ii*</u>. $\varphi(r) = -\log r$ for $d = 2$.

$\mathcal{V}_\Lambda \sim$ surface area, expect Wigner crystal at low temperature.

There is an exact solution for the distribution of particles of the OCP at $\beta e^2 = 2$. This distribution is isomorphic to the distribution of zeroes for Gaussian random matrices <u>without</u> symmetry, (Ginibre, Jancovici).

· <u>*iii*</u>. $\varphi(r) = r^{-(d-2)}$ for $d \geq 3$.

$\mathcal{V}_\Lambda \sim$ surface area.

2) "Coulomb systems" in $\mathbb{R}^d$ with $d+1$ dimensional interactions, e.g. particles on a line with $-\log r$ interaction. This corresponds, as shown by Dyson, to the suitably normalized (to have density ρ) eigenvalue distribution of random Gaussian matrices, with symmetry, GOE, GUE, GSE.

We then have for the variance of the number of eigenvalues in an interval of length L

$$\mathcal{V}_L = \langle (N_L - \rho L)^2 \rangle \sim \log L.$$

Interestingly in all the Coulomb cases considered above the deviation from the average divided by the square root of the variance gives

$$\frac{(N_\Lambda - \langle N_\Lambda \rangle)}{\sqrt{\mathcal{V}_\Lambda}} \to \xi,$$

a standard Gaussian random variable. This was proven for the eigenvalue distribution on the interval by Costin and L with an assist from Dyson.

3) Harrison-Zeldovitch distribution for the initial radiation (matter) density in universe.

Let us consider now the "surface" term for the case of subnormal fluctuations, i.e. when $b = 0$. We then have

$$\kappa = \lim \frac{\mathcal{V}_\Lambda}{|\partial\Lambda|} = -\alpha_d \int_{\mathbb{R}^d} rG(r)\mathrm{d}r \geq 0,$$

where we have sphericalized G. κ could be infinite in which case variance $\mathcal{V}_\Lambda$ grows faster than "surface" area.

Question: Given $b = 0$, can $\kappa = 0$? I.e. can the variance grow slower than $|\partial\Lambda|$. The answer by J. Beck is "no" if the distribution is rotational invariant (or Λ is a sphere). It is still an open question how small κ can be and whether is attains its minimum value for a regular lattice.

Large deviations for the 2D OCP with $\rho = 1$, $e = 1$

Let $n(R)$ be the number of particles in a disc of radius R. Then $\langle n(R) \rangle = \pi R^2$ and $\langle (n(r) - \pi R^2)^2 \rangle \sim cR$ where $c(\beta)$ is a constant computable for $\beta = 2$. It was then shown by Jancovici, L, Manificat (using physical arguments) that for every $\alpha > 1/2$

$$\text{Probability}\left\{ |n(r) - \pi R^2| > R^\alpha \right\} \sim \exp\left\{ -R^{\theta(\alpha)} \right\},$$

where

$$\theta(\alpha) = \begin{cases} 2\alpha - 1, & 1/2 \le \alpha \le 1 \\ 3\alpha - 2, & 1 \le \alpha \le 2 \\ 2\alpha, & \alpha \ge 2 \end{cases}$$

As already noted the distribution of particles in the OCP at $\beta = 2$ is the same as that of the eigenvalues of random matrices with independent complex Gaussian entries. It also turns out to be the same as the distribution of zeroes of the Gaussian Entire Function (GEF)

$$f(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}},$$

where $\zeta_0, \zeta_1, \ldots$ are Gaussian i.i.d. complex random variables. The large deviation formula for GEF was proven rigorously by Nazarov, Sodin, and Volberg.

Distribution of lattice points:

Gauss problem: consider a two dimensional square lattice $\mathbb{Z}^2$. Take a circle with radius R centered at the origin. Find a bound on the fluctuations of $\#\{\mathbf{n} \in \mathbb{Z}^2 | |\mathbf{n}| < R\} \equiv N_0(R)$, i.e. find γ such that

$$\left| N_0(R) - \pi R^2 \right| < C R^{\gamma+\epsilon}, \qquad \gamma = ?$$

Gauss: $\gamma = 1$, 1788.

There were many incremental improvements on this over the years. The latest bound as far as I know is

Huxley: $\gamma = 131/208$, 2003.

Conjecture (Hardy): $\gamma = 1/2$.

Correct variance ?: $R^{1/2}(\log R)^{\delta}$.

The Gauss problem is related to the distribution of energy eigenvalues of a particle in a unit torus. In the early '90s Pavel Bleher, Zheming Cheng, Freeman Dyson and I considered the following more general problem. Let $\mathbf{a} \in \{0,1\}^2$ and define

$$N_{\mathbf{a}}(R) = \#\{\mathbf{n} \in \mathbb{Z}^2 | |\mathbf{n} - \mathbf{a}| \leq R\},$$

so the Gauss problem corresponds to $\mathbf{a} = \mathbf{0}$

So far no randomness. From the point of view of energy level statistics we are interested in the behavior of

$$F_{\mathbf{a}}(R) \equiv [N_{\mathbf{a}}(R) - \pi R^2]/\sqrt{R}$$

as the energy or R varies over some range, e.g. R varies uniformly between 1 and T.

Following ideas by Heath-Brown, Bleher, Cheng, Dyson, L proved the following result:

The probability that $F_{\mathbf{a}}(R)$ lies in the interval $(x, x+\mathrm{d}x)$ approaches $p_{\mathbf{a}}(x)\mathrm{d}x$ weakly as $T \to \infty$, with $p_{\mathbf{a}}(x) \sim C_{\mathbf{a}}\,\mathrm{exp}[-x^4]$.

It was further shown by Bleher and Dyson that $p_{\mathbf{a}}(x)$ is a very singular function of $\mathbf{a}$. In fact they showed that the second moment

$$D(\mathbf{a}) = \int_{-\infty}^{\infty} x^2 p_{\mathbf{a}}(x) \mathrm{d}x$$

has a sharp local maximum at every rational point $\mathbf{a}$.

Here follow some numerical graphs of $p_{\mathbf{a}}(x)$ taken from a review by Bleher.

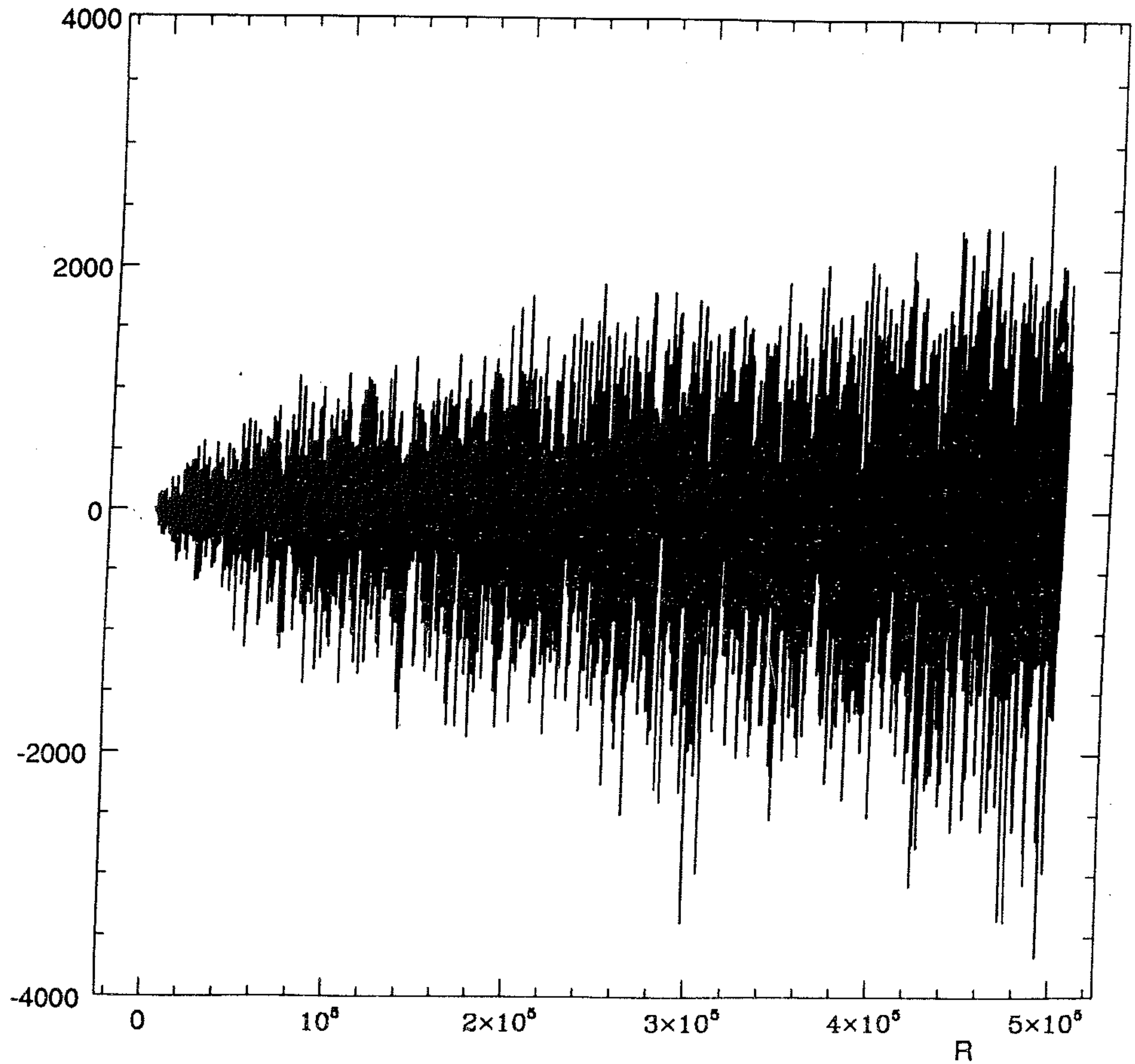

Fig. 1. The graph of the function $n(R^2) = \#\{n \in \mathbb{Z}^2 \colon |n| \le R\} - \pi R^2$.

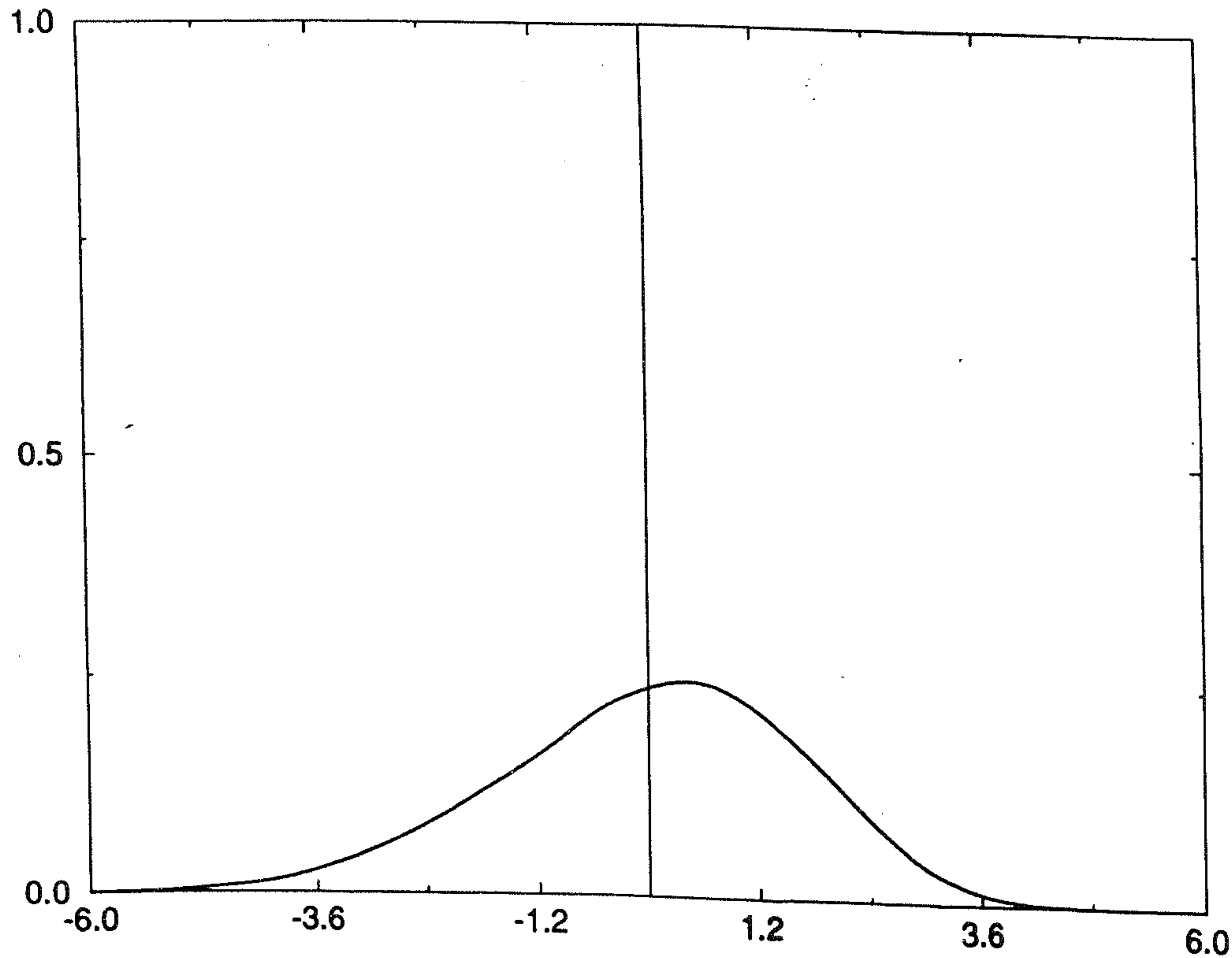

Fig. 2. The density $p_\alpha(x)$ of the limit distribution of the normalized error function $F_\alpha(R)$ given in (2.14) with $\mu = 1$ (circle) and $\alpha = (0,0)$.

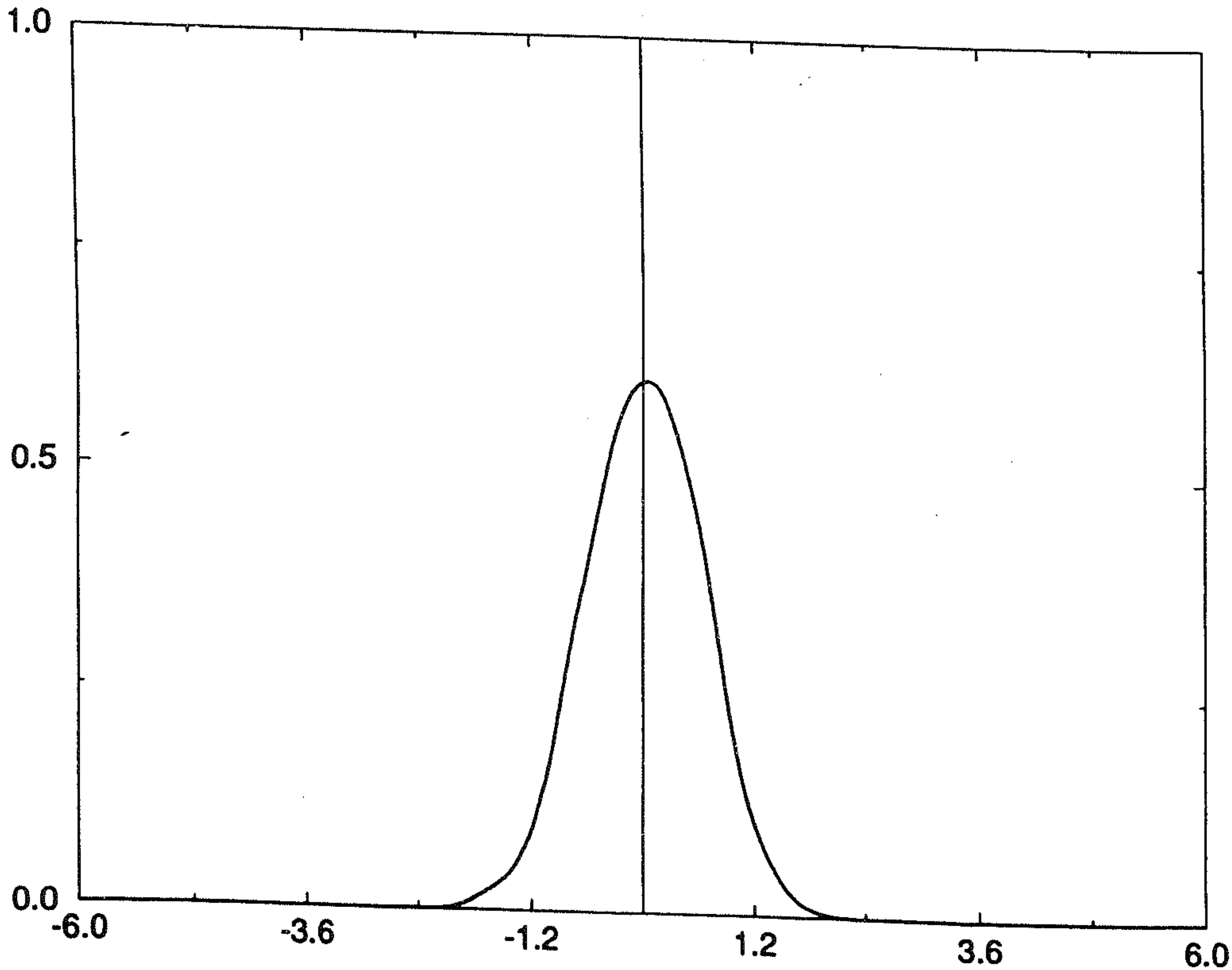

Fig. 5. $\mu = 1$, $\alpha = (0.2, 0.2)$.

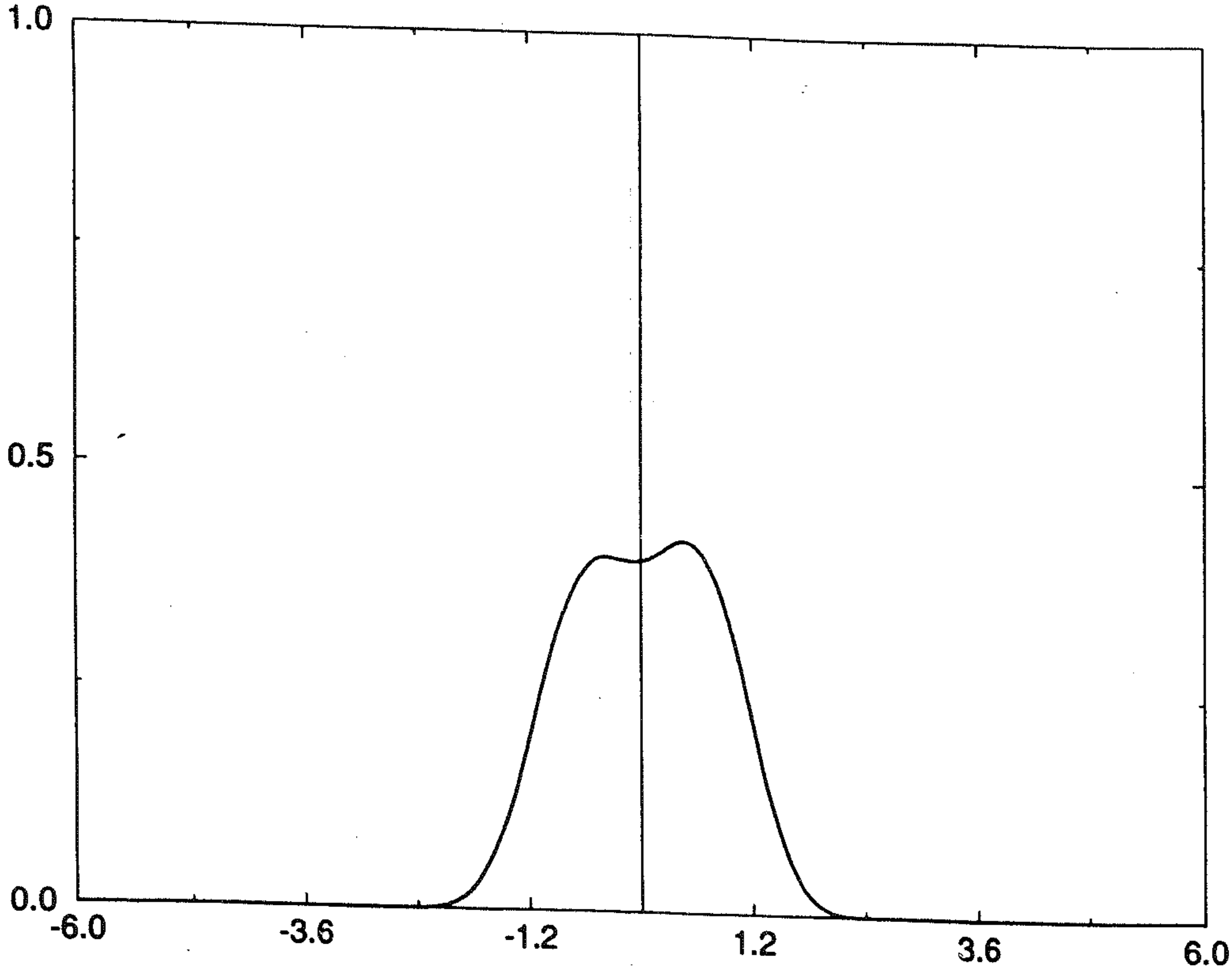

Fig. 8. $\mu = 1$, $\alpha = (0.3437, 0.4304)$.

Time is getting short so let me end my talk with a quote from Schrödinger, which I believe expresses ideas similar to those of Freeman's:

"I am born into an environment - I know not whence I came nor whither I go nor who I am. This is my situation as yours, every single one of you. The fact that everyone always was in this same situation, and always will be, tells me nothing. Our burning question as to the whence and whither - all we can ourselves observe about it is the present environment. That is why we are eager to find out about it as much as we can. That is science, learning, knowledge, this is the true source of every spiritual endeavour of man."


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