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Steady State Self-Diffusion at Low Density

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We prove that the motion of a test particle in a hard sphere fluid in thermal equilibrium converges, in the Boltzmann–Grad limit, to the stochastic process governed by the linear Boltzmann equation. The convergence is in the sense of weak convergence of the path measures. We use this result to study the steady state of a binary mixture of hard spheres of different colors (but equal masses and diameters) induced by color-changing boundary conditions. In the Boltzmann–Grad limit the steady state is determined by the stationary solution of the linear Boltzmann equation under appropriate boundary conditions.

KEY WORDS: Test particle in a hard sphere fluid; Boltzmann–Grad limit; convergence to the Markov process; governed by the linear Boltzmann equation.

1. INTRODUCTION

In a preceeding paper[1] we investigated self-diffusion for a classical fluid by considering it as composed of two components which are mechanically identical but differ by their color, either black or white. If color-changing boundary conditions are imposed at a slab in such a way that there is a constant incoming flux of black particles from one side and of white particles from the other, then as $t \to \infty$ a steady state is established.

The steady state correlation functions are given by exit probabilities of test particle processes. In Ref. 1 it was shown that in the hydrodynamic scaling the expected properties of the color profile, e.g., linearity, follow from the assumption that test particles behave asymptotically as independent Brownian particles.

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The purpose of this paper is to investigate the steady state color profile for a hard sphere gas at low density, i.e., in the Boltzmann–Grad limit. It follows from our previous work that the time-dependent color profile is governed in this limit by the linear Boltzmann equation.[2] We will prove here that the steady state color profile is deterministic in the Boltzmann–Grad limit and is given by the stationary solution of the linear Boltzmann equation under appropriate boundary conditions. This kind of situation is expected to be true under more general circumstances: If there is a scaling limit such that the time-dependent phenomena are described by a certain kinetic equation, then a microscopic steady state should converge, in the same limit, to a stationary solution of that kinetic equation under appropriate boundary conditions. The proof is, however, difficult in the general case involving in some sense the interchange of the limit $t \to \infty$ and the scaling limit. This requires rather good control over the dynamics.

In Ref. 3 we studied the problem of heat transport in the Lorentz gas in the Boltzmann–Grad limit. In the Lorentz gas the only transport mechanism is diffusion. Our results in Ref. 2 apply therefore also to a noninteracting gas of particles carrying color in the external potential created by the scatterers. In this sense Ref. 3 is extended here to an interacting system.

The paper consists of two distinct parts. In Section 2 we prove that in the Boltzmann–Grad limit the stochastic process of a single test particle is governed by the linear Boltzmann equation. In Ref. 2 it was shown that the distribution of the test particle at a single time converges in the Boltzmann–Grad limit to the solution of the linear Boltzmann equation. This result was trivially extended in Ref. 4 to events depending only on a finite number of times. The exit probability is, however, an event which depends on all times. Hence to show the convergence of steady states we have to prove a stronger result.

Physically the result obtained here is very natural. So let us take the time to explain the content of Theorem 1, to be proved, in a nontechnical way. We consider a system of hard spheres of diameter ϵ and unit mass. They move inside a box with periodic boundary conditions. The particles are distributed according to the grand canonical equilibrium measure with inverse temperature β and fugacity $z_\epsilon = \epsilon^{-2} z$. (The limit $\epsilon \to 0$ is the Boltzmann–Grad limit.) We regard the particle with label 1 as our test particle. Its initial velocity, v_1, has a Maxwellian distribution and its initial position is uniformly distributed over the box. After a certain (random) time t_1 the test particle suffers a first collision. After this collision the test particle has the (random) velocity v_2. After another (random) time span t_2 the test particle suffers a second collision, etc. Let us assume that we observe the test particle only over a time span $0 \leqslant t \leqslant T$ with T arbitrary but finite. Then within that time span the test particle has only a finite (but arbitrary) number of collisions. We consider now the joint distribution

$$P^{\epsilon}(dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1})$$

of the random variables $v_1, t_1, v_2, \ldots, t_n, v_{n+1}$, $n = 1, 2, \ldots$. This distribution depends on ϵ because the dynamics and the initial distribution do so. We believe, although we have no proof, that $P^{\epsilon}(dv_1\, dt_2 \ldots dt_n\, dv_{n+1})$ has in fact a density. In any event P^{ϵ} can always be written as the sum of a regular part, which has a density, and a singular part,

$$\begin{aligned} &P^{\epsilon}(dv_1 dt_1 dv_2 \ldots dt_n dv_{n+1}) \\ &\quad = p^{\epsilon}(v_1, t_1, v_2, \ldots, t_n, v_{n+1}) dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1} \\ &\quad\quad + \bar{P}^{\epsilon}(dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1}) \end{aligned}$$

The linear Boltzmann equation describes the motion of the test particle in the following way: The initial velocity of the test particle is distributed according to a Maxwellian with inverse temperature β. The time t_1 up to the first collision has an exponential distribution with a parameter depending on v_1. The distribution of v_2 given v_1 is obtained through the collision with a fluid particle in equilibrium. The time t_2 up to the second collision has an exponential distribution with a parameter depending on v_2, etc. This defines then the joint distribution

$$p(v_1, t_1, v_2, \ldots, t_n, v_{n+1})\, dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1}$$

which can be written down quite explicitly [cf. (2.4) and (2.5)].

We show that

(i) $$\lim_{\epsilon \to 0} \int \bar{P}^{\epsilon}(dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1}) = 0, \qquad n = 1, 2, \ldots$$

The total weight of the singular part of the distribution vanishes as $\epsilon \to 0$;

(ii) $$\begin{aligned} &\lim_{\epsilon \to 0} \int dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1}\, g(v_1, t_1, v_2, \ldots, t_n, v_{n+1}) \\ &\quad \times p^{\epsilon}(v_1, t_1, v_2, \ldots, t_n, v_{n+1}) \\ &\quad = \int dv_1\, dt_1\, dv_2 \ldots dt_n\, dv_{n+1}\, g(v_1, t_1, v_2, \ldots, t_n, v_{n+1}) \\ &\quad \times p(v_1, t_1, v_2, \ldots, t_n, v_{n+1}) \end{aligned}$$

$n = 1, 2, \ldots,$ for all bounded functions g. The joint distribution of collision times and postcollisional velocities tends to the one computed from the linear Boltzmann equation.

To prove (i) and (ii) we rely again on the perturbational technique for the BBGKY hierarchy as developed by O. E. Lanford[5,6] (cf. also Ref. 7). The details of the proof differ from the one in Ref. 2.

In the second part of the paper the convergence (i) and (ii) is applied to the steady state problem. We also investigate local equilibrium. We keep this part rather brief, since in essence it follows Ref. 1 with the appropriate modifications.

2. WEAK CONVERGENCE OF THE TEST PARTICLE PROCESS

We consider a system of hard spheres of diameter ϵ and unit mass. They move inside the box Λ of side length L with periodic boundary conditions. The phase space is the grand canonical phase space $\Gamma = \bigcup_{n\geqslant 1}(\Lambda \times \mathbb{R}^3)^n$. A point $\gamma \in \Gamma$ is specified by the number, n, of particles and their positions and momenta $(q_1, p_1, \ldots, q_n, p_n)$. We use the short-hand $x_j = (q_j, p_j)$. Let $d\gamma$ be the Lebesgue measure defined by $d\gamma \restriction (\Lambda \times \mathbb{R}^3)^n = (1/n!)\,dx_1 \ldots dx_n$.

For later convenience we define certain subsets of Γ. Let $\Gamma(n) = (\Lambda \times \mathbb{R}^3)^n$ and $\Gamma(n;\epsilon) = \{x_1, \ldots, x_n \in \Gamma(n) \,|\, |q_i - q_j| \geqslant \epsilon$ for any pair $i \neq j,\ i, j = 1, \ldots, n\}$. Furthermore, with $t \geqslant 0$, let $\Gamma(n; 0, t) = \{x_1, \ldots, x_n \in \Gamma(n) \,|\, q_j - p_j s \neq q_i - p_i s$, mod Λ, for any pair $i \neq j,\ i, j = 1, \ldots, n$ and all times $0 \leqslant s \leqslant t\}$; i.e., point particles which start at $(x_1, \ldots, x_n) \in \Gamma(n; 0, t)$ and evolve backwards in time have no spatial coincidence during the time span t. We set $\Gamma(\epsilon) = \bigcup_{n\geqslant 1}\Gamma(n;\epsilon)$ and $\Gamma(0,t) = \bigcup_{n\geqslant 1}\Gamma(n;0,t)$. $\Gamma\backslash\Gamma(0, t)$ is a set of $d\gamma$-measure zero.

Let μ^ϵ be the grand canonical equilibrium measure on Γ with fugacity $z_\epsilon = \epsilon^{-2}z$ and inverse temperature β. The Maxwellian at this temperature is denoted by h_β. Let $T_t^\epsilon : \Gamma \to \Gamma$ be the dynamical flow of hard spheres. T_t^ϵ exists μ^ϵ–a.s.[8]

We define the stochastic process of a single test particle for the time interval $[0, T]$, $0 < T < \infty$. Let Ω denote the path space for the momentum of the test particle. A path $\omega \in \Omega$ is a piecewise constant, right continuous function $\omega : [0, T] \to \mathbb{R}^3$. ω is specified by $(t_1, v_1, \ldots, t_k, v_k), t_j > 0$, $\sum_{j=1}^k t_j = T$, $v_j \in \mathbb{R}^3$, $j = 1, \ldots, k$, $k = 1, 2 \ldots$. t_j is the time between the $(j-1)$th and jth collision during which the velocity of the particle is v_j, i.e., $\omega(t) = v_1$ for $0 \leqslant t < t_1$ and $\omega(t) = v_j$ for $t_1 + \cdots + t_{j-1} \leqslant t < t_1 + \cdots + t_j$, $j = 2, \ldots, k$.

In this way Ω is identified with a subset of $\bigcup_{k\geqslant 1}\mathbb{R}^{4k}$. Ω is equipped with the Euclidean topology inherited from $\bigcup_{k\geqslant 1}\mathbb{R}^{4k}$. Let $\mathfrak{B}$ denote its Borel σ algebra and let $\mathfrak{B}_f$ denote the algebra of events depending only on a finite number of times. Ω, $\mathfrak{B}$, and $\mathfrak{B}_f$ depend on T. If needed, we add explicitly the time interval under consideration as, e.g., $\Omega[\tau, T]$. We regard the particle with label 1 as the test particle. Let $p^\epsilon(t, \gamma)$ denote the momentum of the first particle at time t for initial conditions $\gamma \in \Gamma$. Let

$G:\Gamma\to\Omega$ be the map

$$G:\gamma\to\{t\mapsto p^{\epsilon}(t,\gamma),\ 0\leqslant t\leqslant T\}\tag{2.1}$$

G is defined μ^{ϵ}–a.s. and induces a probability measure P^{ϵ} on Ω by

$$P^{\epsilon}(A)=\mu^{\epsilon}(G^{-1}A)\tag{2.2}$$

for all $A\in\mathfrak{B}$. We denote by $p^{\epsilon}(t)$ the momentum process of the test ($\equiv$ first) particle considered as a random variable on Ω.

Let $P^{\epsilon}(\cdot\,|p)$ be P^{ϵ} conditioned on $p^{\epsilon}(0)$. $P^{\epsilon}(\cdot\,|p)$ is defined dp–a.s. since in equilibrium $p_1=p^{\epsilon}(0)$ is distributed as $h_\beta(p_1)\,dp_1$ and $P^{\epsilon}(d\omega)=\int P^{\epsilon}(d\omega\,|p)h_\beta(p)\,dp$.

Because of periodic boundary conditions $p^{\epsilon}(t)$ conditioned on (q_1,p_1) is independent of the conditioning on q_1. The position process of the test particle is then given by

$$q^{\epsilon}(t)=q+\int_0^t ds\,p^{\epsilon}(s),\qquad \operatorname{mod}\Lambda\tag{2.3}$$

We describe now the limiting Markov jump process. Let $p(t)$ be the Markov jump process on Ω with inverse waiting time

$$\lambda(p)=\pi z\int dp'\,|p-p'|h_\beta(p')\tag{2.4}$$

and jump probability

$$\begin{aligned}K(p'\,|p)\,dp'&=\Big[\lambda(p)^{-1}\pi z\int dp''\,h_\beta(p'')\delta\big((p'-p)\cdot(p'-p'')\big)\Big]dp'\\&=\left[\lambda(p)^{-1}\pi z\int_{E(p,p')}de\,h_\beta(e)\frac{1}{|p-p'|}\right]dp'\end{aligned}\tag{2.5}$$

Here $E(p,p')$ is the plane through p' orthogonal to $p'-p$ and de is the two-dimensional Lebesgue measure on $E(p,p')$. Using the conservation of energy in a collision, it is shown in Ref. 4 that $p(t)$ is well defined, in the sense that $p(t)$ has a finite number of jumps in any finite time interval. Let

$$q(t)=q+\int_0^t ds\,p(s),\qquad \operatorname{mod}\Lambda\tag{2.6}$$

be the position process. The forward equation of the Markov process $(q(t),p(t))$ is the *linear Boltzmann equation*

$$\begin{aligned}\frac{\partial}{\partial t}f(q,p,t)&=-p\frac{\partial}{\partial q}f(q,p,t)+\pi z\int_{\hat\omega\cdot(p-p_1)\geqslant 0}dp_1\,d\hat\omega\,\hat\omega\cdot(p-p_1)\\&\quad\times\big[h_\beta(p_1')f(q,p')-h_\beta(p_1)f(q,p)\big]\\&\equiv -p\frac{\partial}{\partial q}f(q,p,t)+(Cf)(q,p,t)\end{aligned}\tag{2.7}$$

Here $\hat{\omega}$ is a vector on the unit sphere and the outgoing pair (p, p_1) is related to the incoming pair (p', p'_1) through a collision as

$$\begin{aligned} p' &= p - \hat{\omega}[\hat{\omega}\cdot(p-p_1)] \\ p'_1 &= p_1 + \hat{\omega}[\hat{\omega}\cdot(p-p_1)] \end{aligned} \tag{2.8}$$

Let $P(\cdot\,|\,p)$ denote the path measure of $p(t)$ on Ω conditioned on $p(0)$.

Theorem 1. For any $T > 0$ and all $A \in \mathfrak{B}[0,T]$

$$\lim_{\epsilon\to 0} P^{\epsilon}(A\,|\,p) = P(A\,|\,p) \tag{2.9}$$

uniformly on compact sets of $\mathbb{R}^3$.

We have deliberately chosen the simplest possible setup. The proof works also for general domains with either deterministic or stochastic boundary conditions and in any dimension $d \geqslant 2$. The convergence for arbitrary times relies on the fact that the fluid is in thermal equilibrium. For other initial fluid states or for boundary conditions which do not preserve the equilibrium state one can still prove convergence for short times. The limit Markov process will, in general, be nonhomogeneous in time.[4]

To prove Theorem 1 we first give a series of definitions, establish some notation, and prove several lemmas. We then establish an expression for $P^{\epsilon}(A\,|\,p)$ and for certain correlation functions in terms of a perturbation series (BBGKY hierarchy). The proof of convergence of these correlation functions for short times and an iteration scheme then imply Theorem 1.

Let us define the space $\Delta(x_1, \ldots, x_m; \epsilon)$ of *collision histories*, $m = 1, 2, \ldots$. If $(x_1, \ldots, x_m) \in \Gamma(m)\backslash\Gamma(m;\epsilon)$, then $\Delta(x_1, \ldots, x_m;\epsilon) = \emptyset$. For given $(x_1, \ldots, x_m) \in \Gamma(m;\epsilon)$,

$$\Delta(x_1, \ldots, x_m;\epsilon) = \Delta(x_1, \ldots, x_m, 0;\epsilon) \cup \bigcup_{n\geqslant 1}\ \bigcup_{j_1=1,\ldots,m} \cdots \bigcup_{j_n=1,\ldots,m+n-1} \Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n;\epsilon)$$

with

$$\Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n;\epsilon) \subset \{t_1, \ldots, t_n \in \mathbb{R}\,|\,0 \leqslant t_n \leqslant \cdots \leqslant t_1 \leqslant T\} \times \mathbb{R}^{3n} \times (S^2)^n$$

A point in $\Delta(x_1, \ldots, x_m;\epsilon)$ is denoted by δ. We equip $\Delta(x_1, \ldots, x_m;\epsilon)$ with the Lebesgue measure $d\delta$, i.e., $d\delta \restriction \Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n;\epsilon) = dt_1 \ldots dt_n\, d\hat{p}_1 \ldots d\hat{p}_n\, d\hat{\omega}_1 \ldots d\hat{\omega}_n$. $\Delta(x_1, \ldots, x_m, 0;\epsilon)$ consists of a single point. The collision history corresponding to this point consists of m spheres starting at $(x_1, \ldots, x_m)$ at time T and evolving backwards in time for a time span T. If a triple collision occurs, i.e., if at least three spheres touch each other simultaneously, then the collision history terminates and remains undefined for previous times. For $n \geqslant 1$ the collision history corresponding to the point $\delta \in \Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n; \epsilon)$ is constructed in the following way: One starts with m nonoverlapping spheres at $(x_1, \ldots, x_m)$ at time T. These spheres evolve backwards in time for a time span $T - t$. If a triple collision occurs, then the collision history terminates and remains undefined for previous times. Let $q_j^\epsilon(t_1)$, $p_j^\epsilon(t_1)$, if defined, denote the position and momentum of the jth sphere at the end of the time span $T - t_1$. At time t_1 one adds an $(m+1)$th sphere to the system of m spheres at the point $q_{j_1}^\epsilon(t_1) + \epsilon\hat{\omega}_1$ with momentum $\hat{p}_1$. If this $(m+1)$th sphere overlaps with any of the other spheres already present, then δ does not belong to $\Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n; \epsilon)$. To the configuration $\{x_j^\epsilon(t_1) \,|\, j = 1, \ldots, m+1\}$ at time t_1, where $q_{m+1}^\epsilon(t_1) = q_{j_1}^\epsilon(t_1) + \epsilon\hat{\omega}_1$ and $p_{m+1}^\epsilon(t_1) = \hat{p}_1$, one associates the configuration $\{x_j^\epsilon(t_1 -) \,|\, j = 1, \ldots, m+1\}$ just before the collision. In particular,

$$\begin{aligned} p_{j_1}^\epsilon(t_1 -) &= p_{j_1}^\epsilon(t_1) - \left[\hat{\omega}_1 \cdot \left(p_{j_1}^\epsilon(t_1) - p_{m+1}^\epsilon(t_1)\right)\right]\hat{\omega}_1 \\ p_{m+1}^\epsilon(t_1 -) &= p_{m+1}^\epsilon(t_1) + \left[\hat{\omega}_1 \cdot \left(p_{j_1}^\epsilon(t_1) - p_{m+1}^\epsilon(t_1)\right)\right]\hat{\omega}_1 \end{aligned} \tag{2.10}$$

if $\hat{\omega}_1 \cdot (p_{m+1}^\epsilon(t_1) - p_{j_1}^\epsilon(t_1)) > 0$, and

$$\begin{aligned} p_{j_1}^\epsilon(t_1 -) &= p_{j_1}^\epsilon(t_1) \\ p_{m+1}^\epsilon(t_1 -) &= p_{m+1}^\epsilon(t_1) \end{aligned} \tag{2.11}$$

if $\hat{\omega}_1 \cdot (p_{m+1}^\epsilon(t_1) - p_{j_1}^\epsilon(t_1)) \leqslant 0$. Then the configuration $\{x_j^\epsilon(t_1 -) \,|\, j = 1, \ldots, m+1\}$ evolves backwards in time for a span $t_1 - t_2$. If a triple collision occurs, then the collision history terminates. At time t_2 one adds a $(m+2)$th sphere to the system of $(m+1)$ spheres at the position $q_{j_2}^\epsilon(t_2) + \epsilon\hat{\omega}_2$ with momentum $\hat{p}_2$, etc. Finally, at time $t_n -$ one has a configuration of $(m+n)$ spheres which evolve then backwards in time up to $t = 0$. $\Delta(x_1, \ldots, x_m, j_1, \ldots, j_n; \epsilon)$ is the set of points such that no overlap occurs upon adding extra spheres. $dx_1 \ldots dx_m$-almost surely, for $d\delta$-almost all $\delta \in \Delta(x_1, \ldots, x_m; \epsilon)$ the collision history is defined. We denote by $\{x_j^\epsilon(t, x_1, \ldots, x_m, \delta)\}$ the configuration of particles at time t of the collision history corresponding to $\delta \in \Delta(x_1, \ldots, x_m; \epsilon)$.

Every collision in a collision history is called a *recollision* with the exception of those between a just added particle and particles already present at the time of adding. It is also convenient to refer to an overlap at the time of adding as a recollision. The essence of the Boltzmann–Grad limit is that recollisions disappear as $\epsilon \to 0$.

For $\epsilon = 0$ the space of collision histories, $\Delta(x_1, \ldots, x_m)$, is defined in the analogous way. The time evolution is now the one of free particles. The kth extra particle is added at time t_k at $q_{j_k}(t_k)$ with momentum $\hat{p}_k$. For given $\hat{\omega}_k$ the configuration $\{x_j(t_k) \mid j = 1, \ldots, m+k\}$ is related to the configuration $\{x_j(t_k -) \mid j = 1, \ldots, m+k\}$ just before the collision by transforming the outgoing pair of momenta $(p_{j_k}(t_k), p_{m+k}(t_k))$ to the incoming pair of momenta $(p_{j_k}(t_k -), p_{m+k}(t_k -))$ according to (2.10) and (2.11). Every collision history is defined and $\Delta(x_1, \ldots, x_m, n, j_1, \ldots, j_n) = \{t_1, \ldots, t_n \in \mathbb{R} \mid 0 \leqslant t_n \leqslant \cdots \leqslant T\} \times \mathbb{R}^{3n} \times (S^2)^n$. We denote by $\{x_j(t, x_1, \ldots, x_m, \delta)\}$ the configuration of particles at time t of the collision history corresponding to $\delta \in \Delta(x_1, \ldots, x_m)$.

As a subset of $\Delta(x_1, \ldots, x_m)$ we define the space of "good" collision histories, $\Delta(x_1, \ldots, x_m; 0)$. This set is obtained by removing from $\Delta(x_1, \ldots, x_m)$ those points for which the corresponding collision history has a spatial coincidence, i.e., $q_i(t) = q_j(t)$ for some t, $0 \leqslant t \leqslant T$, and some pair $i \neq j$, excluding those spatial coincidences which have to occur upon adding the extra particles. A good collision history has no recollisions between point particles. $\Delta(x_1, \ldots, x_m) \backslash \Delta(x_1, \ldots, x_m; 0)$ is a hypersurface of $d\delta$-measure zero.

$\Delta(-;\epsilon)$, $\Delta(-,0)$ and $\Delta(-)$ depend on T. If needed, we add explicitly the time interval under consideration as, e.g., $\Delta(-;\epsilon,[\tau, T])$.

We define $F_\epsilon : \Delta(x_1, \ldots, x_m; \epsilon) \to \Omega$ by

$$F_\epsilon : \delta \to \{t \mapsto -p_1^\epsilon(T-t, x_1, \ldots, x_m, \delta) \mid 0 \leqslant t \leqslant T\} \tag{2.12}$$

and $F : \Delta(x_1, \ldots, x_m) \to \Omega$ by

$$F : \delta \to \{t \mapsto -p_1(T-t, x_1, \ldots, x_m, \delta) \mid 0 \leqslant t \leqslant T\} \tag{2.13}$$

Let $\rho^\epsilon = (\rho_1^\epsilon, \rho_2^\epsilon, \ldots)$ and $r = (r_1, r_2, \ldots)$ be some vector of correlation functions. We define $H_\epsilon(\rho^\epsilon) : \bigcup_{(x_1, \ldots, x_m) \in \Gamma(m)} \Delta(x_1, \ldots, x_m; \epsilon) \to \mathbb{R}$ by

$$\begin{aligned} H_\epsilon(\rho^\epsilon)&(x_1, \ldots, x_m, \delta) \\ &= \prod_{k=1}^{n} \{\hat{\omega} \cdot (\hat{p}_k - p_{j_k}^\epsilon(t_k, x_1, \ldots, x_m, \delta))\} \epsilon^{2(n+m)} \\ &\quad \times \rho_{n+m}^\epsilon(x_1^\epsilon(0, x_1, \ldots, x_m, \delta), \ldots, x_{n+m}^\epsilon(0, x_1, \ldots, x_m, \delta)) \end{aligned} \tag{2.14}$$

and $H(r) : \bigcup_{(x_1, \ldots, x_m) \in \Gamma(m)} \Delta(x_1, \ldots, x_m) \to \mathbb{R}$ by

$$\begin{aligned} H(r)&(x_1, \ldots, x_m, \delta) \\ &= \prod_{k=1}^{n} \{\hat{\omega}_k \cdot (\hat{p}_k - \hat{p}_{j_k}(t_k, x_1, \ldots, x_m, \delta))\} \\ &\quad \times r_{n+m}(x_1(0, x_1, \ldots, x_m, \delta), \ldots, x_{n+m}(0, x_1, \ldots, x_m, \delta)) \end{aligned} \tag{2.15}$$

Let $\tau \geqslant 0$ and $T > \tau$. For $A \in \mathfrak{B}[\tau, T]$ we define the time reversed and shifted set A' by

$$A' = \{\omega' \in \Omega[0, T-\tau] \mid \omega'(t) = -\omega(T-t) \quad \text{for} \quad 0 \leqslant t \leqslant T-\tau \quad \text{with} \quad \omega \in A\} \tag{2.16}$$

Let $\chi(\cdot)$ denote the indicator function of a set. Let ρ_A^ϵ be the correlation functions of the nonnormalized measure $\chi(G^{-1}A')\mu^\epsilon$, i.e., of the restriction of μ^ϵ to those phase points for which the path of the first particle is in A'. [$\chi(G^{-1}A')\mu^\epsilon$ is not symmetric with respect to x_1. We still define the correlation functions as usual by $\rho_{A,m}^\epsilon(x_1, \ldots, x_m) = \sum_{n=0}^{\infty}(1/n!) \int dx_1' \ldots dx_n' f_{n+m}^A(x_1, \ldots, x_m, x_1', \ldots, x_n')$, where f_n^A are the densities of $\chi(G^{-1}A')\mu^\epsilon$ with respect to $d\gamma$.] Let $A \in \mathfrak{B}[0,T]$. Any path is written as (ω_1, ω_2) with $\omega_1 \in \Omega[0,\tau]$ and $\omega_2 \in \Omega[\tau, T]$. We define then

$$A(\omega_1) = \{\omega_2 \mid (\omega_1, \omega_2) \in A\} \subset \Omega[\tau, T] \tag{2.17}$$

Let $\rho_{\text{eq}}^\epsilon$ be the vector of correlation functions of the equilibrium measure μ^ϵ.

With these definitions we have the following:

Lemma 1. For any $0 < \tau < T$ and all $A \in \mathfrak{B}[0,T]$

$$P^\epsilon(A \mid -p_1) = \left[\epsilon^2 \rho_{\text{eq},1}^\epsilon(x_1)\right]^{-1} \int_{\Delta(x_1;\,\epsilon,[0,\tau])} d\delta\, H_\epsilon(\rho_{A(F_\epsilon(\delta))}^\epsilon)(x_1, \delta) \tag{2.18}$$

Proof. Let $0 < s_1 < \cdots < s_k < \tau < s_{k+1} < \cdots < s_{k+l} < T$. Let us choose $A \in \mathfrak{B}[0,T]$ to be of the form $A = \times_{j=1}^{k+l}\{p^\epsilon(s_j) \in B_j\}$ with Borel sets $B_j \subset \mathbb{R}^3$. We set $A_1 = \times_{j=1}^{k}\{p^\epsilon(s_j) \in B_j\} \in \mathfrak{B}[0,\tau]$ and $A_2 = \times_{j=k+1}^{k+l}\{p^\epsilon(s_j) \in B_j\} \in \mathfrak{B}[\tau, T]$. Let us set $f_j(p) = \chi_{B_j}(-p)$, $j = 1, \ldots, k+l$. By $\gamma \to f_j(p_1)$ we identify them with functions on Γ. Let f_{eq}^ϵ be the densities of the equilibrium measure μ^ϵ on Γ with respect to $d\gamma$ and let $E^\epsilon(\cdot \mid p)$, $E^\epsilon(\cdot)$ denote the expectation with respect to $P^\epsilon(\cdot \mid p)$, $P^\epsilon(\cdot)$. Then

$$\begin{aligned} &E^\epsilon(f_1(p^\epsilon(-s_1)) \cdots f_{k+l}(p^\epsilon(-s_{k+l}))) \\ &\quad = \int d\gamma \left[(f_1(\cdots (f_{k+l}(f_{\text{eq}}^\epsilon \circ T_{-T+s_{k+l}}^\epsilon)) \right. \\ &\qquad\qquad \left. \circ\, T_{-s_{k+l}+s_{k+l-1}}^\epsilon \cdots)) \circ T_{-s_1}^\epsilon \right](\gamma) \end{aligned} \tag{2.19}$$

Let $V^\epsilon(t)$ be the evolution operator for the correlation functions, i.e., if ρ is the sequence of correlation functions of some nice initial measure, then $V^\epsilon(t)\rho$ are the correlation functions of the time-evolved measure at time t. By linearity $V^\epsilon(t)$ extends to arbitrary vectors of functions which are bounded by the correlation functions of some equilibrium measure. If $\rho = \{\rho_1(x_1), \rho_2(x_1, x_2), \ldots\}$, then let $f\rho$ be shorthand for $\{f(x_1)\rho_1(x_1), f(x_1)\rho_2(x_1, x_2), \ldots\}$. Conditioning in (2.19) on the initial momentum and using time reversal invariance, we obtain

$$\begin{aligned} &E^\epsilon(f_1(-p^\epsilon(s_1)) \cdots f_{k+l}(-p^\epsilon(s_{k+l})) \mid -p_1) \\ &\quad = \left[\rho_{\text{eq},1}^\epsilon(x_1)\right]^{-1} \left[V^\epsilon(s_1) f_1 V^\epsilon(s_2 - s_1) \cdots f_k \right. \\ &\qquad\qquad \left. \times V^\epsilon(\tau - s_k)\rho_{A_2}^\epsilon \right]_1(x_1) \end{aligned} \tag{2.20}$$

$V^\epsilon(t)\rho^\epsilon$ is given through a perturbation series[5] as

$$\epsilon^{2m}\left[V^\epsilon(t)\rho^\epsilon\right]_m(x_1, \ldots, x_m) = \int_{\Delta(x_1, \ldots, x_m; \epsilon)} d\delta\, H_\epsilon(\rho^\epsilon)(x_1, \ldots, x_m, \delta) \tag{2.21}$$

If we expand in (2.20) all V^ϵ's in their perturbation series (2.21), then

$$P^\epsilon(A \mid -p_1) = \left[\epsilon^2\rho^\epsilon_{\mathrm{eq},1}(x_1)\right]^{-1}\int_{\Delta(x_1;\, \epsilon, [0,\tau]) \cap F_\epsilon^{-1}A_1} d\delta\, H_\epsilon(\rho^\epsilon_{A_2})(x_1, \delta)$$

$$= \left[\epsilon^2\rho^\epsilon_{\mathrm{eq},1}(x_1)\right]^{-1}\int_{\Delta(x_1;\, \epsilon, [0,\tau])} d\delta\, H_\epsilon(\rho^\epsilon_{A(F_\epsilon(\delta))})(x_1, \delta) \tag{2.22}$$

In the last step we used that for the A chosen $A(F_\epsilon(\delta)) = A_2$ if $F_\epsilon(\delta) \in A_1$ and $A(F_\epsilon(\delta)) = \emptyset$ if $F_\epsilon(\delta) \notin A_1$.

Taking finite union and intersections (2.22) holds for all $A \in \mathfrak{B}_f$. Since $\mathfrak{B}_f$ generates $\mathfrak{B}$, (2.18) follows. ■

By the same argument one obtains the following:

Lemma 2. For any $0 < \tau < T$ and all $A \in \mathfrak{B}[0, T]$,

$$\epsilon^{2m}\rho^\epsilon_{A,m}(x_1, \ldots, x_m) = \int_{\Delta(x_1, \ldots, x_m;\, \epsilon, [0,\tau])} d\delta\, H_\epsilon(\rho_{A(F_\epsilon(\delta))})(x_1, \ldots, x_m, \delta) \tag{2.23}$$

(2.23) is the starting point for investigating the limit $\epsilon \to 0$.

Lemma 3. There is a constant $c > 0$ such that for any $T > 0$, all $A \in \mathfrak{B}[0, T]$ and all $m = 1, 2 \ldots,$

$$0 \leqslant \epsilon^{2m}\rho^\epsilon_{A,m}(x_1, \ldots, x_m) \leqslant c\prod_{j=1}^{m} z h_\beta(p_j) \tag{2.24}$$

provided ϵ is sufficiently small.

Proof. From the definition below (2.16) it follows that

$$0 \leqslant \rho^\epsilon_{A,m} \leqslant \rho^\epsilon_{\mathrm{eq},m} \tag{2.25}$$

One knows that the equilibrium correlation functions satisfy the bound (2.24) for sufficiently small ϵ.[9] ■

Lemma 4. Let $0 < \tau < T$ with $\tau = 0.1\sqrt{\beta}\,/z$. Let us assume that for any $B \subset \mathfrak{B}[0, T - \tau]$ and all $m = 1, 2, \ldots,$

$$\lim_{\epsilon \to 0} \epsilon^{2m}\rho^\epsilon_{B,m}(x_1, \ldots, x_m) = P(B \mid -p_1)\prod_{j=1}^{m}\{z h_\beta(p_j)\} \tag{2.26}$$

uniformly on compact sets of $\Gamma(m; 0, T - \tau)$. Then for any $A \in \mathfrak{B}[0, T]$ and all $m = 1, 2, \ldots,$

$$\lim_{\epsilon \to 0} \epsilon^{2m} \rho^{\epsilon}_{A,m}(x_1, \ldots, x_m) = P(A \mid -p_1) \prod_{j=1}^{m} \{ z h_\beta(p_j) \} \tag{2.27}$$

uniformly on compact sets of $\Gamma(m; 0, T)$.

Proof. Let $K \subset \Gamma(m; 0, T)$ be compact. Let us set $r_{A(F(\delta)),m}(x_1, \ldots, x_m) = P(A(F(\delta)) \mid -p_1) \prod_{j=1}^{m} \{ z h_\beta(p_j) \}$. By Lemma 2 we have to show then that

$$\lim_{\epsilon \to 0} \sup_{(x_1, \ldots, x_m) \in K} \Bigg| \int_{\Delta(x_1, \ldots, x_m; \epsilon, [0,\tau])} d\delta \, H_\epsilon(\rho^{\epsilon}_{A(F_\epsilon(\delta))})(x_1, \ldots, x_m, \delta) - \int_{\Delta(x_1, \ldots, x_m; [0,\tau])} d\delta \, H(r_{A(F(\delta))}) \times (x_1, \ldots, x_m, \delta) \Bigg| = 0 \tag{2.28}$$

Using the bound (2.24) together with the invariance of the equilibrium measure it is shown in Ref. 7 that for any $A \in \mathfrak{B}[\tau, T]$

$$|H_\epsilon(\rho^{\epsilon}_{A})(x_1, \ldots, x_m, \delta)| \leqslant \prod_{j=1}^{m} \{ z' h_{\beta'}(p_j) \} \prod_{j=1}^{n} \{ z' h_{\beta'}(\hat{p}_j) \} \tag{2.29}$$

for some pair (z', β') independently of ϵ. Since $\tau = 0.1\sqrt{\beta}/z$, by assumption, the bound (2.29) is integrable with respect to $d\delta$ on $\Delta(x_1, \ldots, x_m; [0, \tau])$.

Let $\alpha > 0$. Because of the integrable bound (2.29) we can choose a compact set $K_1 \subset \bigcup_{(x_1, \ldots, x_m) \in K} \Delta(x_1, \ldots, x_m; 0, [0, \tau])$ such that $\int_{(\Delta(x_1, \ldots, x_m) \cap K_1)^c} d\delta \, |H_\epsilon(\rho_{A(F_\epsilon(\delta))})(x_1, \ldots, x_m, \delta)| \leqslant \alpha$ independently of $(x_1, \ldots, x_m) \in K$. Therefore

$$\sup_{(x_1, \ldots, x_m) \in K} \Bigg| \int_{\Delta(x_1, \ldots, x_m; \epsilon, [0,\tau])} d\delta \, H_\epsilon(\rho_{A(F_\epsilon(\delta))})(x_1, \ldots, x_m, \delta) - \int_{\Delta(x_1, \ldots, x_m; [0, ,\tau])} d\delta \, H(r_{A(F(\delta))})(x_1, \ldots, x_m, \delta) \Bigg|$$

$$= 2\alpha + \sup_{(x_1, \ldots, x_m) \in K} \Bigg| \int_{\Delta(x_1, \ldots, x_m; \epsilon, [0,\tau]) \cap K_1} d\delta \times H_\epsilon(\rho_{A(F_\epsilon(\delta))})(x_1, \ldots, x_m, \delta) - \int_{\Delta(x_1, \ldots, x_m; [0,\tau]) \cap K_1} d\delta \times H(r_{A(F(\delta))})(x_1, \ldots, x_m, \delta) \Bigg| \tag{2.30}$$

Because K_1 is compact we can choose ϵ small enough such that there are no recollisions for any collision history $(x_1, \ldots, x_m, \delta) \in K_1$. Then

$F_\epsilon(\delta) = F(\delta)$ and $\{p_j^\epsilon(t, x_1, \ldots, x_m, \delta)\} = \{p_j(t, x_1, \ldots, x_m, \delta)\}$, $0 \leqslant t \leqslant \tau$. Let us choose a compact set $K_2 \subset \Delta(x_1, \ldots, x_m; [0, \tau])$ such that $K_1 \subset K \times K_2$. Then the last term in (2.30) is bounded by

$$\int_{K_2} d\delta \left| \prod_{k=1}^{n} \{\hat{\omega}_k \cdot (\hat{p}_k - \hat{p}_{j_k}(t_{k_1}, x_1, \ldots, x_m, \delta))\} \right|$$
$$\times \sup_{(x_1, \ldots, x_m) \in K} \chi(\Delta(x_1, \ldots, x_m; [0, \tau]) \cap K_1)(\delta)$$
$$\times \Big| \rho^E_{A(F(\delta)), n+m}(x_1^\epsilon(0, x_1, \ldots, x_m, \delta), \ldots, x_{n+m}^\epsilon(0, x_1, \ldots, x_m, \delta))$$
$$- r_{A(F(\delta)), n+m}(x_1(0, x_1, \ldots, x_m, \delta), \ldots, x_{n+m}(0, x_1, \ldots, x_m, \delta)) \Big| \tag{2.31}$$

There exists some compact set $\tilde{K} \subset \Gamma(n + m; 0, T - \tau)$ such that the configuration at time zero is contained in $\tilde{K}$ for every $(x_1, \ldots, x_m, \delta) \in K_1$. By assumption the integrand of (2.31) vanishes then for every $\delta \in K_2$ as $\epsilon \to 0$. By Lebesgue-dominated convergence we conclude then that the last term in (2.30) can be made less than α for suitable small enough ϵ.

To identy the limit one chooses again first sets A which depend only on a finite number of times; cf. Ref. 4 for details of the calculation. ■

Lemma 5. Let $T = \tau$. Then for all $A \in \mathfrak{B}[0, \tau]$ and all $m = 1, 2, \ldots,$

$$\lim_{\epsilon \to 0} \epsilon^{2m} \rho^\epsilon_{A,m}(x_1, \ldots, x_m) = P(A \mid -p_1) \prod_{j=1}^{m} \{z h_\beta(p_j)\} \tag{2.32}$$

uniformly on compact sets of $\Gamma(m; \tau)$.

Proof. As in Lemma 2 we write

$$\epsilon^{2m} \rho^\epsilon_{A,m}(x_1, \ldots, x_m) = \int_{\Delta(x_1, \ldots, x_m; \epsilon, [0, \tau]) \cap F_\epsilon^{-1} A} d\delta \, H_\epsilon(\rho^\epsilon_{\text{eq}})(x_1, \ldots, x_m, \delta) \tag{2.33}$$

For the equilibrium correlation functions

$$\lim_{\rho \to 0} \epsilon^{2m} \rho^\epsilon_{\text{eq},m}(x_1, \ldots, x_m) = \prod_{j=1}^{m} \{z h_\beta(p_j)\} \tag{2.34}$$

uniformly on compact sets of $\Gamma(m; 0)$. By the argument given in the proof of Lemma 4 we conclude that (2.33) converges uniformly on compact sets of $\Gamma(m; \tau)$. The limit is identified by using sets A which depend only on a finite number of times. ■

Proof of Theorem 1. By Lemmas 4 and 5 for any $T > 0$ and all $A \in \mathfrak{B}[0, T]$

$$\lim_{\epsilon \to 0} \epsilon^{2m} \rho^\epsilon_{A,m}(x_1, \ldots, x_m) = P(A \mid -p_1) \prod_{j=1}^{m} \{z h_\beta(p_j)\} \tag{2.35}$$

uniformly on compact sets of $\Gamma(m; 0, T)$. By Lemmas 1 and 2

$$P^{\epsilon}(A \mid -p_1) = \left[\epsilon^2 \rho^{\epsilon}_{\mathrm{eq},1}(x_1)\right]^{-1} \rho^{\epsilon}_{A,1}(x_1) \tag{2.36}$$

By (2.35) this converges uniformly on compact sets of $\Gamma(1; [0, T]) = \Lambda \times \mathbb{R}^3$. ∎

The proof given can be extended to the infinite hard sphere systems, $\Lambda = \mathbb{R}^3$, and to the motion of several test particles. Without proof we state a theorem in the form needed for the discussion of the steady state. We denote by $P^{\epsilon}(\cdot \mid x_1, \ldots, x_n)$ the path measure on $(\mathbb{R}^3 \times \Omega)^n$ of the stochastic process of n test particles starting at $(x_1, \ldots, x_n)$.

Theorem 2. Let $\Lambda = \mathbb{R}^3$. For any $T > 0$, all $A \in \mathfrak{B}[0, T]^n$ and all $n = 1, 2, \ldots$ we have

$$\lim_{\epsilon \to 0} P^{\epsilon}(A \mid x_1, \ldots, x_n) = P \times \cdots \times P(A \mid x_1, \ldots, x_n) \tag{2.37}$$

and

$$\lim_{\epsilon \to 0} P^{\epsilon}\left(A \mid (q + \epsilon^{2/3} q_1, p_1, \ldots, q + \epsilon^{2/3} q_n, p_n\right)$$
$$= P \times \cdots \times P(A \mid q, p_1, \ldots, q, p_n) \tag{2.38}$$

uniformly on compact sets of $\Gamma(n; 0, T)$.

(2.37) means that any finite number of test particles move independently in the Boltzmann–Grad limit $\epsilon \to 0$. (2.38) means that even if the test particles start close ($\sim \epsilon^{2/3}$) to each other on the scale of a mean free path but far apart ($\sim \epsilon^{-1/3}$) on the scale of the hard sphere diameter, they still move independently as $\epsilon \to 0$.

3. CONVERGENCE OF THE STEADY STATES

We return to the steady state situation considered in Ref. 1. In addition to the mechanical degrees of freedom a sphere carries now a color σ. $\sigma = 0$ corresponds to white and $\sigma = 1$ to black. We assume that the hard sphere fluid is infinitely extended and that, ignoring color, it is in thermal equilibrium at fugacity $\epsilon^{-2} z$ and inverse temperature β. We imagine a slab, Λ, of width L perpendicular to the x axis and centered at $L/2$, whose sole purpose is to impose boundary conditions on the colors. All particles to the left of the slab are black and all particles to the right of the slab are white. If a particle inside the slab exits to the left, then its color is changed to (or remains) black and if it exits to the right, then its color is changed to (or remains) white.

If initially all particles inside Λ are black, then under these boundary conditions the colored fluid approaches a steady state as $t \to \infty$.[1] Let us define two events:

$$A(0) = \left\{ (q, \omega) \in \mathbb{R}^3 \times \Omega \mid \text{the path } t \to q + \int_0^t ds\, \omega(s) \right.$$
$$\left. \text{exits } \Lambda \text{ first to the right} \right\}$$

$$A(1) = \left\{ (q, \omega) \in \mathbb{R}^3 \times \Omega \mid \text{the path } t \to q + \int_0^t ds\, \omega(s) \right.$$
$$\left. \text{exits } \Lambda \text{ first to the left or never exits } \Lambda \right\}$$

(If $q \notin \Lambda$, this is considered as an exit.) Then the steady state correlation functions are given by

$$\rho_n^\epsilon(x_1, \sigma_1, \ldots, x_n, \sigma_n; L) = \rho_{\mathrm{eq},n}^\epsilon(x_1, \ldots, x_n)$$
$$\times P^\epsilon\left(\bigtimes_{j=1}^n A(\sigma_j) \mid q_1, -p_1, \ldots, q_n, -p_n \right) \quad (3.1)$$

We want to investigate the Boltzmann–Grad limit, $\epsilon \to 0$, of the steady state correlation functions. For the equilibrium correlation functions one knows that

$$\lim_{\epsilon \to 0} \epsilon^{2n} \rho_{\mathrm{eq},n}^\epsilon(x_1, \ldots, x_n) = \prod_{j=1}^n \{ z h_\beta(p_j) \} \quad (3.2)$$

uniformly away from points of spatial coincidence.

We start by investigating the convergence of the first correlation function. Since it is given through the motion of a single test particle, we can use Theorem 1. As before Ω is the path space of the momentum process of the test particle and $P^\epsilon(\cdot \mid p)$ denotes the path measure. We set $T = \infty$. Let us define four events in Ω:

$$A(q, \sigma, T) = \left\{ \omega \in \Omega \mid \text{the path } t \mapsto q + \int_0^t ds\, \omega(s) \right.$$
$$\text{exits } \Lambda \text{ first to the right, if } \sigma = 0, \text{ and exits } \Lambda \text{ first}$$
$$\left. \text{to the left, if } \sigma = 1, \text{ during the time interval } [0, T] \right\}$$

$$\bar{A}(q, 0, T) = A(q, 0, T)$$

$$\bar{A}(q, 1, T) = \left\{ \omega \in \Omega \mid \text{the path } t \mapsto q + \int_0^t ds\, \omega(s) \right.$$
$$\text{exits } \Lambda \text{ first to the left or stays inside } \Lambda \text{ during}$$
$$\left. \text{the time interval } [0, T] \right\}$$

Then

$$P^{\epsilon}\big(\bar{A}(q_1,\sigma,\infty)\,|\,p_1\big) = P^{\epsilon}\big(A(\sigma)\,|\,q_1,p_1\big) \tag{3.3}$$

By Theorem 1 for all T

$$\lim_{\epsilon\to 0} P^{\epsilon}\big(A(q,\sigma,T)\,|\,p\big) = P\big(A(q,\sigma,T)\,|\,p\big) \tag{3.4}$$

Since $A(q,\sigma,T)$ is an increasing sequence of events as $T\to\infty$,

$$\liminf_{\epsilon\to 0} P^{\epsilon}\big(A(q,\sigma,\infty)\,|\,p\big) \geqslant P\big(A(q,\sigma,\infty)\,|\,p\big) \tag{3.5}$$

For the limiting Markov process

$$\sum_{\sigma=0,1} P\big(A(q,\sigma,\infty)\,|\,p\big) = 1 \tag{3.6}$$

Therefore

$$\lim_{\epsilon\to 0} P^{\epsilon}\big(A(q,\sigma,\infty)\,|\,p\big) = P\big(A(q,\sigma,\infty)\,|\,p\big) \tag{3.7}$$

By the definition of $\bar{A}(q,\sigma,T)$ this implies then

$$\lim_{\epsilon\to 0} P^{\epsilon}\big(\bar{A}(q,\sigma,\infty)\,|\,p\big) = P\big(A(q,\sigma,\infty)\,|\,p\big) \tag{3.8}$$

Therefore we conclude that

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon^2\rho_1^{\epsilon}(x_1,\sigma_1;L) &= zh_{\beta}(p_1)P\big(A(q_1,\sigma,\infty)\,|\,-p_1\big)\\ &\equiv zh_{\beta}(p_1)f_L(q_1,-p_1,\sigma_1)\end{aligned} \tag{3.9}$$

The exit probability $f_L(q,p,\sigma)$ satisfies the backward equation

$$p\frac{\partial}{\partial q}f_L(q,p,\sigma) + (C^*f_L)(q,p,\sigma) = 0 \tag{3.10}$$

for $q\in\Lambda$ with the boundary conditions

$$f_L(q,p,\sigma) = \begin{cases}\sigma & \text{for}\quad q_x = 0 \quad\text{and}\quad \hat{n}\cdot p > 0\\ 1-\sigma & \text{for}\quad q_x = L \quad\text{and}\quad \hat{n}\cdot p < 0\end{cases} \tag{3.11}$$

Here $\hat{n}$ is the unit vector pointing along the positive 1-direction. By definition $f_L(q,p,\sigma) = \sigma$ for $q_x < 0$ and $f_L(q,p,\sigma) = 1-\sigma$ for $q_x > L$.

Extending somewhat the analysis in Ref. 10 one obtains bounds which ensure that $f_L(q,p,\sigma)$ is linear in q up to errors of the order $1/L$. Also the steady state current, $j_L(q,\sigma) = \int dp\, pf_L(q,p,\sigma)$, equal D/L up to errors of the order $1/L^2$ with D the diffusion coefficient as computed from the linear Boltzmann equation (2.7).

Using Theorem 2 and the analog of the above argument, we obtain the following:

Theorem 3. The steady state correlation functions converge in the Boltzmann–Grad limit,

$$\lim_{\epsilon\to 0}\epsilon^{2n}\rho_n^\epsilon(x_1,\sigma_1,\ldots,x_n,\sigma_n;L) = \prod_{j=1}^{n}\{zh_\beta(p_j)f_L(q_j,-p_j,\sigma_j)\}\,dx_1\ldots dx_n\text{-a.s.} \tag{3.12}$$

The factorization (3.12) is equivalent to the fact that the color random field becomes deterministic as $\epsilon\to 0$. Let $n^\epsilon(\Lambda_0,\sigma)$ be the number of particles of color σ in the bounded region $\Lambda_0\subset\mathbb{R}^3\times\mathbb{R}^3$. Then

$$\lim_{\epsilon\to 0}\epsilon^2 n^\epsilon(\Lambda_0,\sigma) = \int_{\Lambda_0} dq\,dp\,zh_\beta(p)f_L(q,-p,\sigma) \tag{3.13}$$

in probability.

4. LOCAL EQUILIBRIUM

On the scale measured in units of the *mean free path* the microscopic structure is completely lost in the Boltzmann–Grad limit. Any finite volume on this scale eventually contains an infinite number of particles. One may partially recover this microscopic structure by considering the system on a scale of a constant *interparticle distance*. On this scale the mean free path grows as $\epsilon^{-2/3}$. Also the system size has to increase as $\epsilon^{-2/3}$, e.g., in the steady state setup $L\sim\epsilon^{-2/3}$. On the other hand, on this scale the hard sphere diameter is $\epsilon^{1/3}$ and therefore still goes to zero in the Boltzmann–Grad limit $\epsilon\to 0$. Therefore locally the system resembles an ideal gas. Since density and velocity distribution change only over distances of the order $\epsilon^{-2/3}$, locally the state of the system approaches that of an infinitely extended ideal gas in equilibrium (constant density and arbitrary velocity distribution). The crucial point here is that the state of the system has the identical structure even when evolved over several mean free times and that the parameters determining the local equilibria are governed by the Boltzmann equation.

It is convenient to go back to the scale which we used all the way long, namely, the one measured in units of the mean free path. We investigate then the distribution of particles in a small (of the order $\epsilon^{2/3}$, equivalently of the order one on the scale of a constant interparticle distance) neighborhood around the point q. We define the *local state at* q through its correlation functions

$$\rho_n^\epsilon(x_1,\sigma_1,\ldots,x_n,\sigma_n;L,q) = \rho_n^\epsilon(q+\epsilon^{2/3}q_1,p_1,\sigma_1,\ldots,q+\epsilon^{2/3}q_n,p_n,\sigma_n;L) \tag{4.1}$$

Comparing with (3.1) one has to study now test particle processes, where the initial positions of the test particles are separated on the order $\epsilon^{2/3}$, but are still far apart, $\epsilon^{-1/3}$, on the scale set by the hard sphere diameter. For the equilibrium correlation functions

$$\lim_{\epsilon\to 0} \rho^{\epsilon}_{\mathrm{eq},n}\left(q + \epsilon^{2/3}q_1, p_1, \ldots, q + \epsilon^{2/3}q_n, p_n\right) = \prod_{j=1}^{n} \left\{ zh_\beta(p_j) \right\} \tag{4.2}$$

Therefore, using Theorem 2 and the argument given in the previous section, we obtain for $q_x \neq 0, L$

$$\lim_{\epsilon\to 0} \rho^{\epsilon}_{n}(x_1, \sigma_1, \ldots, x_n, \sigma_n; q) = \prod_{j=1}^{n} \left\{ zh_\beta(p_j) f_L(q, -p_j, \sigma_j) \right\} dx_1 \ldots dx_n\text{-a.s.} \tag{4.3}$$

In the Boltzmann–Grad limit the local state of the system at the point $q \in \mathbb{R}^3$ is the equilibrium state of an infinitely extended, colored ideal gas: The gas has constant density z and a Maxwellian velocity distribution at inverse temperature β. Independently of its location a particle with momentum p is black with probability $f_L(q, -p, 1)$ and white with probability $f_L(q, -p, 0) = 1 - f_L(q, -p, 1)$.

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