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Linear stability of electron flow produced by field emission

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A linear stability analysis of the planar one dimensional space charge limited flow is performed when the current is determined by a current-field relation, e.g., the Fowler-Nordheim or any other emission model. The initial velocity is assumed the same for all emitted electrons. The flow is shown to be stable with decaying oscillations depending on the nature of the emission law, including in some situations non-oscillating slowly decaying modes. When the emission variations are due only to changes of the initial flow velocity, the time of decay can be much longer than the electron transit time for a given flow setup. © *2013 American Institute of Physics*.
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I. INTRODUCTION

In this work, we extend previous analysis[1–4] of the linear stability of the space charge limited flow to the case when the current is determined by a current-field relation at the emitter. We consider a planar diode with flat parallel electrodes, which allows a one dimensional treatment. Using dimensionless units,[4] we set the inter-electrode distance and the anode voltage equal to 1, the electron mass to 2, and the grounded cathode is placed at $x=0$. Let $\varphi(x,t)$, $j(x,t)$, $v(x,t)$, and $\rho(x,t)\geq 0$ be the potential, current density, electron velocity, and charge density, respectively, inside the diode as functions of x and time t. We assume that the current at the cathode is determined by a non-negative function of the cathode electric field $f(t)$, i.e.,

$$j(0,t)=F[f(t)],\quad f(t)=\frac{\partial\varphi}{\partial x}(0,t)=E(0,t), \tag{1}$$

where

$$j(x,t)=\rho(x,t)v(x,t)$$

and $E(x,t)$ is the electric field at x. The function $F(f)$ can be the Fowler-Nordheim (FN) law[5] or some simpler models, which will be studied here. Our system is described by the following set of equations:

$$\frac{\partial^2\varphi}{\partial x^2}(x,t)=\rho(x,t), \tag{2a}$$

$$\frac{\partial j}{\partial x}(x,t)=-\frac{\partial\rho}{\partial t}(x,t), \tag{2b}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{1}{2}\frac{\partial\varphi(x,t)}{\partial x}. \tag{2c}$$

The boundary conditions (BC) are

$$\varphi(0,t)=0,\quad \varphi(1,t)=\varphi_0=1,\quad v(0,t)=\omega(t). \tag{3}$$

By substituting Eq. (2a) into Eq. (2b) and then integrating from 0 to x, we get a system of two coupled equations: Eq. (2c) and

$$v\frac{\partial^2\varphi}{\partial x^2}(x,t)+\frac{\partial^2\varphi}{\partial x\partial t}(x,t)=j(t)+\frac{\partial f}{\partial t}(t). \tag{4}$$

We have set $j(t)=j(0,t)$. The last derivative in Eq. (4) is a new term compared with[1–3] where the right side of Eq. (4) was just written as an integration constant, which depends only on t while the possibility of the current dependence on $f(t)$ was not considered.

We will use, see Ref. 1, along with the Euler variables (x,t), the Lagrangian variables (τ,t), where τ is the time of emission of an electron whose Eulerian coordinates coincide with the point (x,t). The derivatives can be expressed as

$$\left(\frac{\partial}{\partial\tau}\right)=\left(\frac{\partial x}{\partial\tau}\right)\frac{\partial}{\partial x},\quad \left(\frac{\partial}{\partial t}\right)=\left(\frac{\partial x}{\partial t}\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}, \tag{5}$$

where the partial derivatives in (τ,t) are in round parentheses. The electron coordinate in the new variables is $x(\tau,t)$ while its velocity (which is $v=\dot{x}$ in the Eulerian variables) will be

$$v(\tau,t)=\left(\frac{\partial x}{\partial t}\right). \tag{6a}$$

Equations (2c) and (4) can be rewritten now as

$$\left(\frac{\partial v}{\partial t}(\tau,t)\right)=\frac{E(\tau,t)}{2},\quad \left(\frac{\partial E}{\partial t}(\tau,t)\right)=j(t)+\left(\frac{\partial f(t)}{\partial t}\right). \tag{6b}$$

We keep here and below the same letters $E(\tau,t)$, $f(t)$, $\varphi(\tau,t)$, $v(\tau,t)$ for the flow characteristics as in Eq. (4). Thus, we arrive to the equation

$$2\left(\frac{\partial^3 x}{\partial t^3}(\tau,t)\right)=j(t)+\left(\frac{\partial f(t)}{\partial t}\right). \tag{6c}$$

This is similar to the equation derived by Lomax in Ref. 1 with the explicit additional term (derivative of f) on the right, whose origin comes from using in Eq. (4) the current

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dependence on the cathode field. Equation (6c) can be solved because its right side depends on t only as in Eq. (4). The BC (3) at $x=0$ (i.e., $t=\tau$) for $x(\tau,t)$ and its t-derivatives are

$$x(\tau,\tau)=0, \quad \left(\frac{\partial x}{\partial t}\right)(\tau,\tau)=\omega(\tau),$$
$$\left(\frac{\partial^2 x}{\partial t^2}\right)(\tau,\tau)=E(\tau,\tau)/2=f(\tau)/2. \tag{7}$$

The cathode emission $j(0,\tau)$ is determined by Eq. (1) via $E(\tau,\tau)$.

Let $T(\tau)$ be the transit time for an electron injected at time τ to cross the diode. We solve now Eq. (6) by integrating it from τ to t, $\tau \le t \le T$ three times. Using Eq. (7) and some straightforward manipulations yields

$$x(\tau,t)=\int_\tau^t \left[j(t')\frac{(t-t')^2}{4}+f(t')\frac{t-t'}{2}+\omega(t')\right]dt'. \tag{8}$$

In the case when only $\rho(0,t)$ is time dependent and $v(0,t)=\omega_0$ is fixed, one should replace $\omega(\tau)$ in Eq. (7) and $\omega(t')$ in Eq. (8) with ω_0.

Note that the treatment in Lagrangian variables does not give directly the macroscopic flow characteristics in space and time, but they can be expressed in terms of $v(\tau,t)$ and its time derivatives using Eqs. (6a) and (6b).

We are interested in the stability of the steady state of this system. This is different from the cases studied in Refs. 1–3 where the current is specified a priory by being injected externally at the cathode.

II. STEADY STATE REGIME

For the time independent state, the flow parameters such as cathode field f_0, current j_0, and T are fixed, the value of τ is irrelevant and here it is taken to be zero. Thus, Eq. (8) implies the following relations:

$$x(t)=\frac{j_0 t^3}{12}+\frac{f_0 t^2}{4}+\omega_0 t, \tag{9}$$

$$v(t)=\frac{j_0 t^2}{4}+\frac{f_0 t}{2}+\omega_0, \tag{10}$$

where $\omega_0=\omega(0)$ while the variable t can be treated as the usual Euler one for an individual electron emitted at $t=0$. Equations (9) and (10) allow to reduce the evaluation of the current density and f_0 to straightforward computations for any current-field law. Substituting $t=T$ in Eqs. (9) and (10) and using also Eq. (1), one comes to the relations

$$x_0=\frac{j_0 T^3}{12}+\frac{f_0 T^2}{4}+\omega_0 T=1,$$
$$v_0=\frac{j_0 T^2}{4}+\frac{f_0 T}{2}+\omega_0=\sqrt{1+\omega_0^2}, \quad j_0=F(f_0), \tag{11}$$

where $x_0=x(T)=1$ is the anode coordinate and $v_0=v(T)=\sqrt{1+\omega_0^2}$ is the electron velocity at the anode whose potential is 1.

Equations (9)–(11) define the potential $\varphi(x)$ implicitly in a quite complicated way, but the current parameters can be found by solving Eqs. (11) for j, f, and T numerically. In the cases when this can be done analytically (for example when $\omega_0=0$ and $j=\alpha f$ or $j=\beta f^2$), the results are the same as in Ref. 6.

A remarkable property of the electron motion in the steady state found by Lomax and exhibited here by Eqs. (9) and (10): no matter what are the flow parameters and the source of electrons, if they all have the same initial velocity, the distance from the emitter of an individual particle is always a cubic parabola as a function of time of its travel while the speed is a quadratic one. This comes from the fact that, as seen in Eq. (6b), the electric field in Lagrangian units is proportional to t in the steady state.

Before proceeding to the flow stability, we describe some additional properties of stationary systems and their new features connected with the origin of emission. Equations (11) involve four parameters j_0, f_0, ω_0, and T, which completely determine the flow in our setup. In fact these parameters are not independent in view of Eqs. (11). For example, if ω_0 is given, one easily finds by elementary manipulations from Eq. (11) that

$$j_0=\frac{12}{T^2}\left(\sqrt{1+\omega_0^2}+\omega_0-\frac{2}{T}\right),$$
$$f_0=-4T\left(\sqrt{1+\omega_0^2}+2\omega_0-\frac{3}{T}\right). \tag{12}$$

The transit time can be found from Eq. (12) in the following form:

$$T=\frac{6}{2\omega_0+\sqrt{1+\omega_0^2}+\sqrt{(2\omega_0+\sqrt{1+\omega_0^2})^2+3f_0}}, \tag{13}$$

which implies that for the field emission case when $f_0 \ge 0$, the maximum value of T is 3, which corresponds to the limiting Child-Langmuir (CL) regime $f_0=\omega_0=0$.[7] This result and Eqs. (12) will be used later. The minimum transit time for $\omega_0=0$ is 2 as it follows from Eq. (13) when the space charge is absent and $f_0=1$.

Another way of the flow characterization by a single parameter is through A, introduced in Ref. 1, which in our units has the form

$$A=\frac{2}{j_0 T^3}. \tag{14}$$

A defines the flow completely. Using all three Eqs. (11), we obtain a system

$$T(\omega_0+\sqrt{1+\omega_0^2})=2+\frac{1}{6A}, \quad T^3 F(f_0)=2/A, \tag{15a}$$

$$f_0=(4-4\omega_0 T-2/3A)/T^2, \tag{15b}$$

which allows to evaluate both T and ω_0 in terms of A. Our analysis above is not restricted to non-negative f_0 as we did not use yet the assumptions $f_0 \ge 0$ in $j=F(f_0)$. The

requirement that $j_0, f_0 \geq 0$ implies, in view of Eqs. (12), that solutions, which violate the inequalities

$$\frac{2}{\omega_0 + \sqrt{1+\omega_0^2}} \leq T \leq \frac{3}{2\omega_0 + \sqrt{1+\omega_0^2}}, \tag{15c}$$

should be dropped. Equations (15a) and (15c) show that $A \geq 1/6 + (\sqrt{1+\omega_0^2} + \omega_0)/2$ in this case. The value $A = 1/6$ is achieved in the case of the original CL flow, when $\omega_0 = f_0 = 0$, $j_0 = 4/9$, $T = 3$.

The solution of Eqs. (15) is shown in Fig. 1 for the case when the emitted current is proportional to the electric field $F(f_0) = \alpha f_0$, for $\alpha = 10$. The parameter A cannot be smaller than ~0.184 for this $F(f_0)$ with $f_0 \geq 0$.

An emission model with $F(f_0) = \beta f_0^2$ exhibits a similar picture with the minimum $A \approx 0.915$. A smaller A makes $f_0 < 0$ giving rise to the appearance of a virtual cathode. We do not consider this situation here.

Figs. 1 and 2 show that increasing the parameter A is always accompanied by the decrease of the time of crossing the diode by emitted electrons, while the initial electron velocity and current density are increasing if A grows.

III. LINEAR STABILITY ANALYSIS OF TIME DEPENDENT PERTURBATION

We consider (first in Eulerian x, t variables) a perturbation of the steady state (j_0, f_0, ω_0, and T) when by the cathode current has the form

$$j(0,t) = j_0 + j_1 e^{ikt}, \tag{16}$$

with $j_1/j_0 \ll 1$. The perturbation j_1 will in general consist of two terms, a perturbation of the velocity ω_0 and/or of the charge density at the cathode ρ_0

$$j_1 = \rho_0 \omega_1 + \rho_1 \omega_0,$$

depending on the nature of emission. The corresponding variation of the initial electron velocity and cathode electric

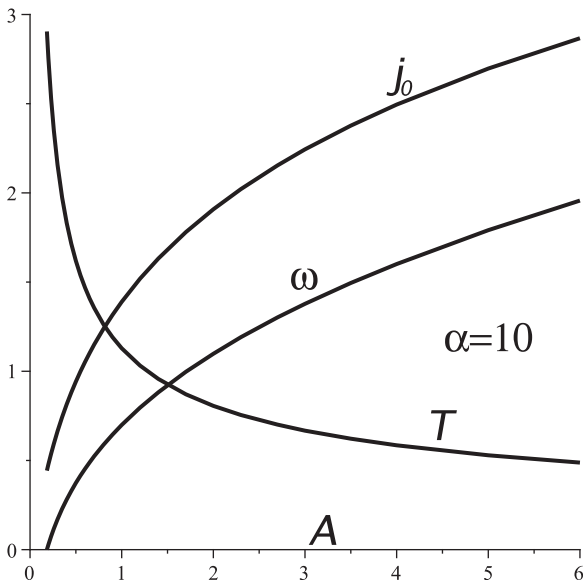

FIG. 1. Stationary flow parameters vs. A for $F(f_0) = \alpha f_0$.

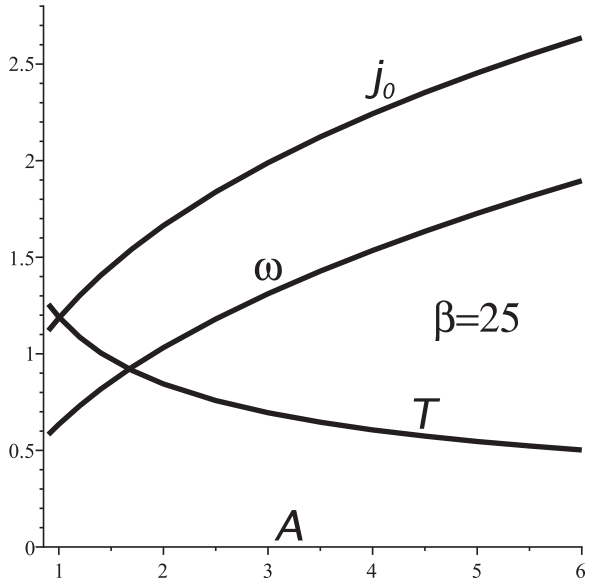

FIG. 2. Stationary flow parameters for $F(f_0) = \beta f_0^2$.

field (which is the origin of perturbation) can be written in the form

$$\omega(0,t) = \omega_0 + q\frac{j_1\omega_0}{j_0}e^{ikt};$$
$$E(0,t) = f_0 + \frac{j_1}{F'}e^{ikt}, \quad \text{where} \quad F' \equiv \frac{dF(f_0)}{df_0}. \tag{17}$$

In Eq. (17), $q = \frac{\omega_1}{\omega_0}/\frac{j_1}{j_0}$. When q varies from zero to 1, the contribution from changes of the initial velocity to j_1 varies in the same range. As a result of the perturbation (we come back to variables τ, t), the electron transit time is modified $T \to T + \vartheta(t)$, where clearly $\vartheta(t) \ll T$ is of order of j_1. Now the perturbation terms Eqs. (16) and (17) will transform Eq. (9) into the following form:

$$x(\tau,t) = \frac{1}{12}j_0(t-\tau)^3 + \frac{1}{4}f_0(t-\tau)^2 + \omega_0(t-\tau)$$
$$+\frac{j_1 i e^{ik\tau}}{4k^3}\left\{k^2(t-\tau)^2 - 2p[ik(t-\tau) + 1 - e^{ik(t-\tau)}]\right\}$$
$$+ q\omega_0\frac{j_1 i e^{ik\tau}}{j_0 k}[1 - e^{ik(t-\tau)}], \tag{18}$$

where $p(k, f_0) = 1 + ik/F'$.

Using BC Eq. (3) at the anode, we obtain for the electron arriving at the anode at time t

$$x_0 = x(t - T - \vartheta, t) = 1, \quad \varphi_0 = \varphi(t - T - \vartheta, t) = 1. \tag{19}$$

Keeping only terms linear in j_1 in Eqs. (17) and using Eq. (11), we obtain

$$x_0 - 1 = \vartheta(t)v_0 + j_1 i e^{ik(t-T)}$$
$$\times\left[\frac{k^2T^2 - 2p(1 - e^{ikT} + ikT)}{4k^3} + q\omega_0\frac{1 - e^{ikT}}{j_0 k}\right] = 0.$$

This equation determines $\vartheta(t)$ and shows that it is proportional to j_1.

To satisfy Eq. (19), we express[2] the fixed anode potential as

$$\varphi(1,t) = 1 = \int_0^1 E(x,t)dx.$$

This can be rewritten in the Lagrangian variables, using Eq. (6), as follows:

$$1 = 2\int_t^{t-T-\vartheta} \frac{\partial^2 x}{\partial t^2}(\tau,t)\frac{\partial x}{\partial \tau}(\tau,t)d\tau. \quad (20)$$

We integrate Eq. (20) by parts to get

$$1 = 2x_0\frac{\partial^2 x}{\partial t^2}(t-T-\vartheta,t) - 2\int_t^{t-T-\vartheta} x(\tau,t)\frac{\partial^3 x}{\partial \tau \partial t^2}(\tau,t)d\tau. \quad (21)$$

From Eqs. (7), (16), and (17), one has

$$2\frac{\partial^2 x}{\partial t^2}(\tau,t) = j_0(t-\tau) + f_0 + j_1\left(\frac{e^{ikt}-e^{ik\tau}}{ik} + \frac{e^{ikt}}{F'} + 2ikq\omega_0\frac{e^{ikt}}{j_0}\right)$$

and, therefore,

$$2\frac{\partial^3 x}{\partial \tau \partial t^2}(\tau,t) = -j_0 - j_1 e^{ik\tau}.$$

Thus, the linear approximation of the first term in Eq. (21) is

$$Tj_0 + f_0 + \vartheta j_0 + j_1 e^{ikt}\left(\frac{p - e^{-ikT}}{ik} + 2ikq\omega_0/j_0\right).$$

The same approximation allows to rewrite Eq. (21) in the form

$$Tj_0 + f_0 - \frac{j_0T^2}{2}\left(\frac{j_0T^2}{24} + \frac{f_0T}{6} + \omega_0\right) + j_0(1-x_a)\vartheta + \frac{j_1e^{ikt}}{2k^4}R(k,T) = 1. \quad (22)$$

The zero order term in Eq. (22) can be easily shown to be equal to 1 by substituting j_0 and f_0 from Eqs. (12), while the term with $\vartheta(t)$ clearly is zero. Thus, the BC (we denote $s = ikT$) require the function

$$R(k,T) = j_0 p\left[2 - s - (2+s)e^{-s} + \frac{2s^3}{j_0T^3}\right] + s\frac{f_0(s+1)e^{-s} - f_0}{T} + \frac{2s^2\omega_0}{T^2}\left[e^{-s} - 1 + q\left(\frac{2s^3}{j_0T^3} + 1 - s - e^{-s}\right)\right],$$

to vanish. This determines the permitted values of the wave parameter k, i.e., provides the dispersion relation

$$(2+s)e^{-s} = 2 - s + As^3 + \frac{s}{pj_0T}\left\{f_0(1+s)e^{-s} - f_0 + \frac{2s\omega_0}{T}[(q-1)(1-e^{-s}) + q(As^3 - s)]\right\}. \quad (23)$$

In Eq. (23), the term in curly brackets is new compared with Refs. 1–3 where the current j_0 was independent of the cathode field, which meant $F' = 0$ and $p = \infty$. The parameter $A = 2/j_0T^3$ represents the same quantity as in Ref. 1.

Note that for a given flow regime, we have in Eq. (23) only one free parameter among f_0, j_0, ω_0 or A. If one choses f_0 for instance, then $\omega_0 = 1/T - j_0T^2/12 - f_0T/4$, $j_0 = F(f_0)$, see eq. (11), and $T = \left[\sqrt{f_0^2 + 4j_0(\sqrt{1+\omega_0^2} - \omega_0)} - f_0\right]/j_0$. If only ω_0 is given, one can find T from Eqs. (12) by using $F(f_0)$ and then evaluate j_0, f_0.

A. Case of a very weak emitter response to the electric field

When the diode current is independent from the cathode electric field, i.e., $dj/df = F' = 0$, the parameter p becomes infinitely large and we return to the injected current case studied in Refs. 1–3. In many situations with a realistic field emission law, the function F' is expected to be very small. In the case $|p| \gg T^2$, our new term in Eq. (23) will be a small correction compared with the term $2s^3/j_0T^3$. Thus, the results[1–4] should be valid here. Even in the cases when $|p|$ is not very large (this corresponds usually to very strong electric fields), solutions of Eq. (23) would be distributed differently from the case with injected current, but the flow stays stable.

If the current perturbations are caused by internal processes in the conducting emitter, then the electric field at the emitter surface can stay independent of them and be the same as in the steady state. In this case, the linear analysis would be identical to that with an externally injected current studied in Refs. 1–3 with the same conclusion about the flow stability for $A > 1/6$.

B. Flow stability in the general case of field emission

We consider now the dispersion relation (23) in two extreme situations when (i) $j(0, t)$ variations are originated by changing $\rho(0,t)$ only while $\omega(t) = \omega_0$ or (ii) by changing $\omega(t)$ and keeping $\rho(0,t) = \rho_0$ constant. They correspond to $q = 0$ and $q = 1$, respectively. We start from the case (i) which looks more suitable for semiconductor emitters, metallic ones are probably described better with fixed ρ_0 since the field does not penetrate the cathode.

1. Fixed initial velocity ω_0

We can use Eqs. (18)–(23) setting $q = 0$ in this case. It was shown in Refs. 1–3 that the injected current is unstable only when $A < 1/6$, while for $A > 1/6$ and even in the CL regime when $A = 1/6$, the flow is stable.[4] An analysis of Eq. (23) similar to one made by Lomax in Ref. 2 shows that it has only one real solution $s = 0$, which is time independent and belongs to the stationary state. In addition, Eq. (15) immediately implies $A > 1/6(1 - \omega_0 T)$ and $\omega_0 T < 1$ when f_0 positive.

The numerical solutions of Eq. (23) for the linear emission model are shown in Fig. 3 as locations of the smallest, by their absolute value, roots of k in the left part of the complex plane. (There is always also another set of roots

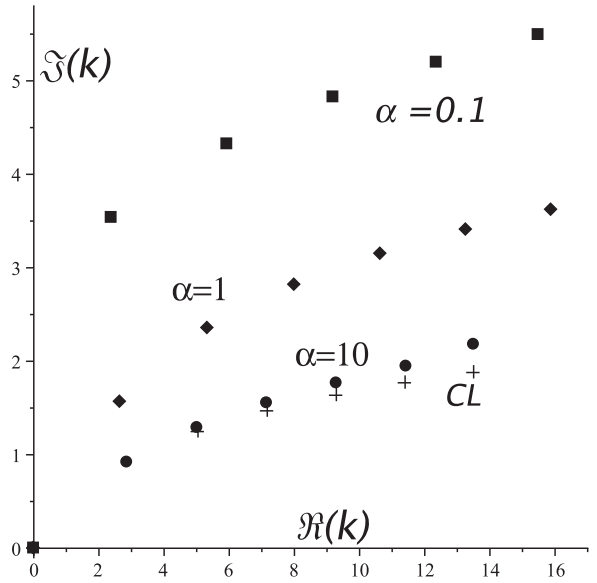

FIG. 3. First seven roots of k for $F(f_0) = \alpha f_0$ when $A = 0.19$. Crosses are for CL flow.

symmetric about the vertical axis in Fig. 3 and all figures below). All higher roots have larger imaginary parts and, therefore, faster decay rates. The flow is always stable and when α increases, i.e., a cathode more responsive to the field emission, these rates approach the ones of the CL flow corresponding to $\alpha = \infty$ whose roots[4] are indicated by crosses in Fig. 3.

2. Fixed electron density ρ_0 at cathode

The dispersion relation (23) with $q = 1$ now reads as

$$(2+s)e^{-s} = 2 - s + As^3 + \frac{s}{pj_0T}\left[f_0(1+s)e^{-s} - f_0 + \frac{2s\omega_0}{T}(As^3 - s)\right]. \quad (24)$$

This clearly will not change our conclusion for the case of a very weak emission dependence on the applied electric field, $|p| \gg T^2$.

The straightforward analysis of Eq. (24) near $s = 0$, similar to one made in Ref. 2, shows that up to the forth order in s Eq. (24) can be reduced to the form

$$2 - s + \frac{s^3}{6} - \frac{s^4}{12} + O(s^5) = 2 - s + \frac{s^3}{6} + s^4\left(\frac{A - 1/6}{F'T} + \frac{f_0}{3j_0T}\right), \quad (25)$$

which implies that for $|s| \ll 1$ and $f_0 > 0$, the right side can be smaller than the left one (this would produce the existence of a positive solution and, therefore, an unstable flow) only in exotic situations[8] when $F' < 0$ which we do not consider here. Thus, near $s = 0$, the left side of Eq. (24) is smaller than the right one and decreases when real $s > 0$ increases. The right side behaves differently but stays larger when s grows. In the case of negative s, a straightforward analysis shows that real solutions of Eq. (25) with $s < 0$ can be realized. This occurs when $|s|$ is rather small because the negative term $2s^5\omega_0 A/j_0T^2 = \omega_0 Ts^5$ on the right can be stronger than the terms proportional to s^4. Another possibility is a much larger $-s$, which makes f_0e^{-s} large but comparable with $2As^3\omega_0/T$. In Fig. 4 when $A = 2$, these quantities differ by about ~25% and a small root $k \sim 0.1$ exists too.

For fixed ρ_0, our numerical computations using Eq. (24) confirmed the flow stability for the linear and quadratic current-field emission laws. In addition to the usual complex solutions of Eq. (23), similar to ones in Fig. 3, we see in Fig. 4 the presence of negative real roots, which describe non-oscillating decaying flow perturbations. These real solutions appear at relatively large A, which correspond to significant initial velocities. The decay rate of terms, which are described by them, can be small and decreasing when A grows: for the linear emission model with $\alpha = 2$ and $A > 8$, we have $\Im k < 0.2$ and $\Im k < 0.07$ for $\alpha = 10$, $A > 10$. Such decay rates are almost an order smaller then the ones shown in Fig. 3, but they might be relevant only for low anode voltages, i.e., when $\omega_0^2 \sim 1$ or larger, see below the case of a more realistic model.

In Fig. 4, we plot a few the smallest decay exponents for the linear emission model with fixed ρ_0 at the cathode. Compared with Fig. 3 and the results of Ref. 4, the first decay exponents in this regime can be much smaller and the decay can be without oscillations.

Fig. 5 exhibits locations of the roots of $k_1 \ldots k_6$ for the quadratic current-field law $j = 25f^2$ when $A = 0.24$ in two cases of $q = 0$ (ω_0 fixed) and $q = 1$, (ρ_0 fixed).

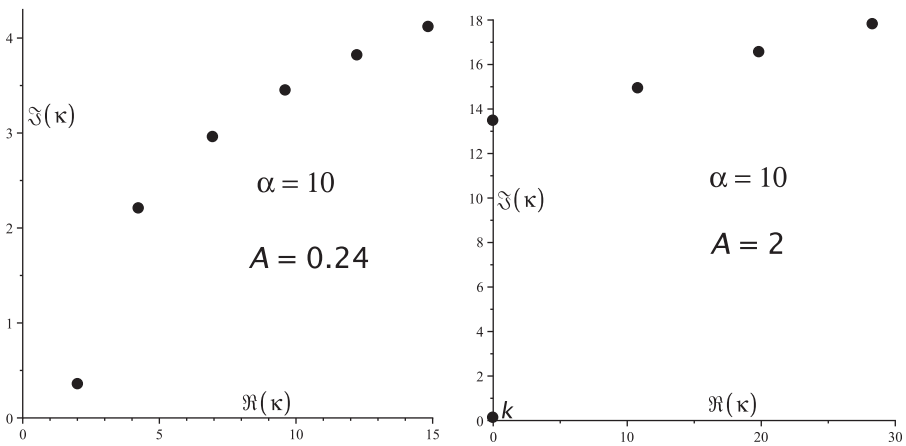

FIG. 4. The first roots of k for $F(f_0) = 10f_0$ when $\rho(t) = \rho_0$. In right plot for $A = 2$, the root indicated by k is located at (0, 0.124).

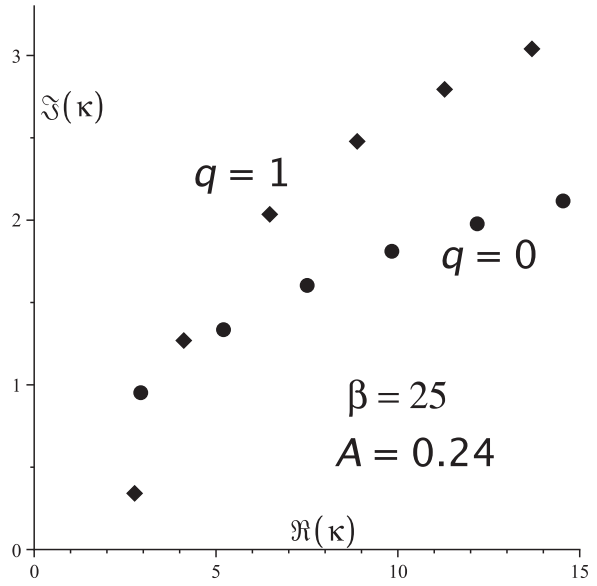

FIG. 5. Decay exponent when $F(f_0) = \beta f_0^2$. Diamonds for $q = 1$, circles for $q = 0$.

For the same values of A and β, the rate of decay of the first root for this model is much slower when ρ_0 is fixed while higher roots do not differ significantly. One can say now that for a given current density in the case of fixed ρ_0 at the cathode, the flow stays stable but its perturbations decay much slower than when the initial velocity is fixed. The period of these oscillations are always of the order of the electron transit time or shorter. When ω is much larger and $A = 4$, there are strictly imaginary solutions $k = 0.064i$ and $k = 19.50i$. The first of them means a very slow decay, but nevertheless the flow is stable, the second imaginary root describes a very rapid decay like the one in Fig. 4 for $A = 2$. The existence of imaginary roots of k always requires significant initial velocities of emitted electrons.

C. Fowler-Nordheim emission model

The linear flow stability is determined by the dispersion relation (23) whose general properties were studied above and thus we cannot expect to see a drastic difference in the case of the realistic emission FN model[5]

$$F(f_0) = Q\sqrt{V} f_0^2 e^{-PL/Vf_0}, \quad F' = j_0(2f_0 + PL/V)/f_0^2.$$

Here, V is the anode voltage in kilovolts, L is the inter-electrode distance in micrometers, and parameters Q, P depend on properties of emitter material, mainly the work function W, see modification in Ref. 9. The typical values using kilovolts, micrometers, and electron-volts are approximately (Copper) $W = 4.5$, $Q = 2.06$, and $P = 65.2$, while for Cesium $W = 2$, $Q = 4.63$, and $P = 19.3$. In the case when the anode potential $V = 10$, a micro-diode size $L = 1\mu$M, considered in Ref. 6, taking dimensionless $f \sim 0.7$, one has $F' \sim 0.005$, and for $L = 10\,\mu$M, $f \sim 1$, it would be $\sim 2 \cdot 10^{-26}$. This means that in such practical situations, p is very large, the second term in Eq. (23) can be neglected, and the results of Refs. 1–4 should be valid here, see Sec. III A. When the work function is smaller (Cesium), then F' is not small for $L = 1\,\mu$M, but emitters with low W are unstable in such strong fields. For $L = 10\,\mu$M, the field is weaker and $F' \sim 3 \cdot 10^{-10}$.

Note that the above version of the FN emission law is chosen mainly for the illustration, a more advanced equation of $F(f_0)$, used in Ref. 10, would not change significantly the following consideration and its results:

(i) We consider now the FN emission and start from the case with ω_0 fixed. Using first the parameters mentioned above, we constructed Fig. 6 where the solid circles exhibit the solutions for $k_1 \ldots k_6$, i.e., represent possible harmonics of the flow perturbation.

The trajectory of k_1 starts at $A = 28.6$ and goes up to $A = 150$. The emission is quite low in all cases in Fig. 6. This makes the cathode electric field close to 1, i.e., like in a vacuum, the current density stays within $0.0087 < j_0 < 0.0091$, while ω_0 runs from 0 to 0.6 when $A = 150$. The decay parameters are rather high, >2.6, the minimal value of the parameter A is large, close to 28.6 in this model.

The current density is much larger when a material with parameters close to Cesium is used for the emitter. This case is shown in Fig. 7 in the same way as in Fig. 6. Circles in Fig. 7 show the set of poles $k_1..k_6$ for $A = 0.73$, which is close to its minimal value. The current density is an increasing function of A, it grows from $j \approx 0.25$ and reaches 0.81 when $A = 20$, i.e., exceeds the CL limit 4/9 by more than 80%. This should not be surprising because such a large A is compatible only with a big initial electron velocity $\omega_0 \approx 1.9$, which can be realized only in exceptional situations (the initial electron energy ω_0^2 is almost 4 times higher than the gained energy $\varphi_0^2 = 1$ at the anode).

(ii) Now we consider the limiting case of $q = 1$ when the current perturbations are caused by the electron initial velocity only. To make the effects visible, we focus on the case of "small" work function and use the Cesium

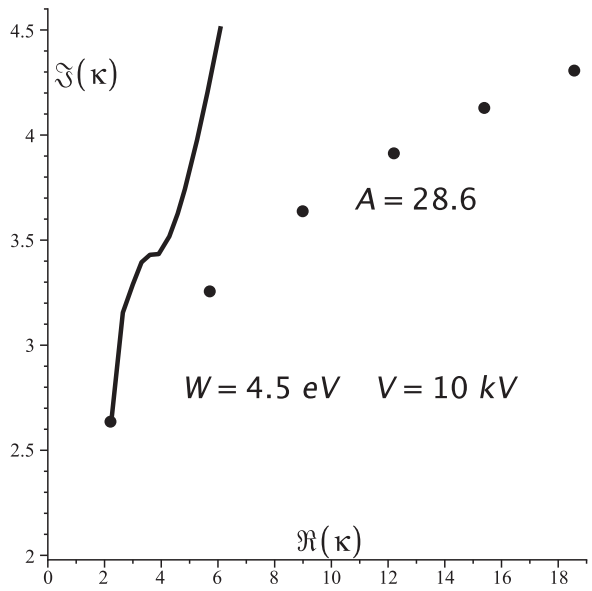

FIG. 6. The decay exponent for FN emission (Cu). Circles for $A = 28.6$, solid curve is trajectory of $k_1(A)$ when $28.6 \le A \le 150$.

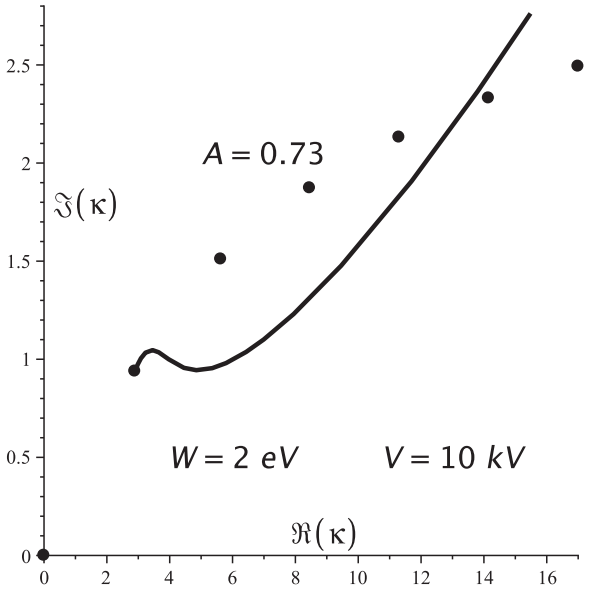

FIG. 7. The decay exponent for FN emission (Cs). Trajectory of $k_1(A)$ for $0.73 \le A \le 20$.

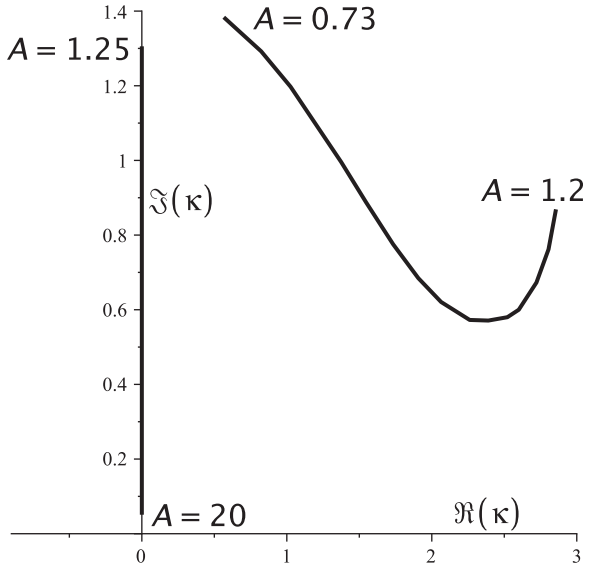

FIG. 9. Location of root k_1 for different A for FN model when $\rho(t) = \rho_0$.

parameters for illustration disregarding question of the material stability.

Like in the idealized models above, there are both real and complex solutions of the dispersion relation (24), but the real s, which correspond to imaginary k, appear only when $A \ge 1.25$. They represent the first 2 roots for $A = 1.5$ shown by circles in Fig. 8. For smaller A, the solutions are similar to the case with fixed ω_0 in Fig. 7.

The location of the root k_1 as a function of A is plotted in Fig. 9. The smallest possible A is close to 0.72 in our case. When A grows from 0.73 to 1.2 k_1 can be only complex and its trajectory in the complex plane is presented by the curve with a minimum of the decay rate at $k \approx 2.39 + 0.571i$ when $A \approx 0.8$. For $A > 1.2$, the root k_1 jumps to the imaginary axis and its trajectory goes down approaching to the origin. When $A = 20$, it reaches $k_1 = 0.05$ and gets smaller with increasing A. This regime ($A = 20$) corresponds to $\omega_0 \approx 1.9$ and $T \approx 0.5$.

We studied above only two extreme situations $q = 0$ and $q = 1$, but in practice changes of current density are caused often by simultaneous variations of the charge density and initial velocity. This translates in our terms to considering some $0 < q < 1$, which can be used in the dispersion relation (23).

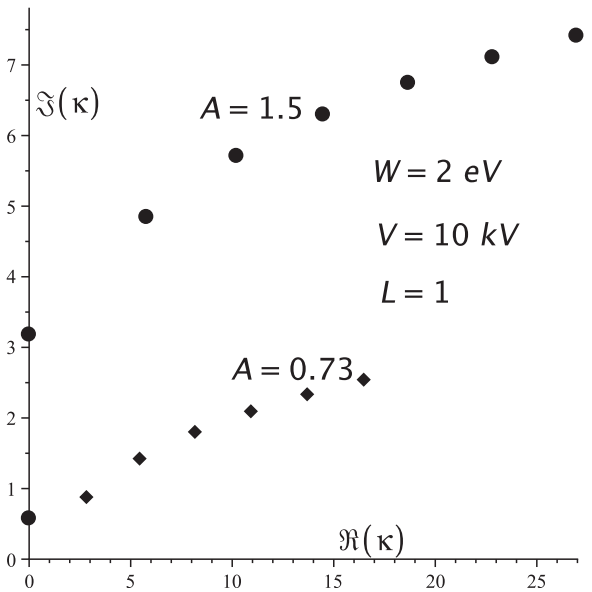

FIG. 8. The first roots of k for FN model when $\rho(t) = \rho_0$.

IV. CONCLUSION

The linear analysis shows that the field emitted electron flows with non-negative cathode electric field are stable. The dispersion relation, which describes the normal modes of the flow decay, gives a richer picture of possible states than in earlier works devoted to externally injected and Child-Langmuir flows. Especially, interesting is a situation when variations of the emitted current are due mainly to the variations of the flow velocity while the density of emitted particles is almost fixed. In this case, the decay of flow perturbations can be very slow and proceed without oscillations. The physical explanation of this is quite obvious: the increase/decrease of the current is not accompanied by the corresponding increase/decrease of the charge density and, therefore, opposite change of screening the electric field. Thus, the usual negative feedback is replaced with a positive one. This effect turns out to be insufficient to create flow instability.

Our approach to analyze the flow stability is based on the one-to-one correspondence between the Eulerian and Lagrangian variables. This is incorrect if electrons pass each other in the flow when their initial velocities are spread in some interval (the electrons in time t at a point x might be emitted at different τ). In this case, there are serious limitations for such a method, see comments by Lomax in Ref. 1.

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