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Displacement Convexity and Minimal Fronts at Phase Boundaries

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Abstract

We show that certain free energy functionals that are not convex with respect to the usual convex structure on their domain of definition are strictly convex in the sense of displacement convexity under a natural change of variables. We use this to show that, in certain cases, the only critical points of these functionals are minimizers. This approach based on displacement convexity permits us to treat multicomponent systems as well as single component systems. The developments produce new examples of displacement convex functionals and, in the multi-component setting, jointly displacement convex functionals.

1. Introduction

1.1. The variational problem

We consider minimization problems for a type of functional that arises in the study of phase segregation in statistical mechanical systems. Let $F(m)$ be a function on the real line that is continuous and strictly positive except at $m = a$ and $m = b$ with $a < b$. A good example to bear in mind is the “double well potential”

$$F(m) = \frac{1}{4}(m^2 - 1)^2,$$

where of course $a = -1$ and $b = 1$.

Let $\mathcal{C}_{a,b}$ be the set of measurable functions $m(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that (for some representative)

$$\lim_{x\to-\infty} m(x) = a \quad \text{and} \quad \lim_{x\to+\infty} m(x) = b.$$

The numbers a and b represent the values of the order parameter m in two phases of a statistical mechanical system. For example, $m = a$ might correspond to a vapor phase, and $m = b$ to a liquid phase.

A function $m(x)$ in $\mathcal{C}_{a,b}$ denotes a possible one-dimensional *transition profile* across the boundary segregating the two different phases. The actual profile that one would expect to see would be one that minimizes the free energy cost of making such a transition. The free energy functional $\mathcal{F}$ to be minimized on $\mathcal{C}_{a,b}$ will in some cases of interest have the form, c.f. [10],

$$\mathcal{F}(m) = \int_{\mathbb{R}} F(m(x))\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (1.1)$$

where $J(x)$ is a non-negative integrable function on $\mathbb{R}$.

The term $\int_{\mathbb{R}} F(m(x))\,\mathrm{d}x$ is due to short range interactions and entropy effect, and is normalized so that it vanishes in the pure phases, when $m(x) = a$ or $m(x) = b$, while the term $\int_{\mathbb{R}}\int_{\mathbb{R}}(m(x)-m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y$ is due to long range interactions. This long range term in the free energy suppresses sharp transitions, as does the gradient term in the familiar but purely phenomenological Van der Waals model [13]. For more discussion of the physical context of the problem, see [8].

Much useful information can be deduced from the specific form of the minimizing profiles. In particular, the surface tension at a two dimensional phase boundary in physical three dimensional space is the minimum value of $\mathcal{F}(m)$ on $\mathcal{C}_{a,b}$; see [3] and Section 5 for more information. Hence we ask:

- *What is the minimum value of $\mathcal{F}(m)$ as m ranges over $\mathcal{C}_{a,b}$, and are the minimizing profiles, if any, unique up to translation?*

Actually, the existence of minimizers is relatively simple to prove using the rearrangement inequalities to be discussed below. However, because of the translation invariance, they are never unique: Any translate of a minimizer is again a minimizer. It is less simple to show that this is the only degeneracy.

1.2. Displacement convexity and uniqueness of fronts

For a particular choice of F in the free energy functional specified in (1.1), the minimizing profile problem has been solved in a series of papers by De Masi et al. [10,11], building on previous unpublished work of Dal Passo and de Mottoni [9]. Their solution involves the construction of a dynamics that is dissipative for the free energy functional, and then a careful analysis of limits along the time evolution for this dynamics.

Another approach that we further develop here has been introduced by Alberti and Bellettini [1,2]. They discovered an alternative convex structure which renders the variational problem for (1.1) convex, and used this to study the existence problem in [2]. Later, Alberti [1] returned to the problem and proved a uniqueness result that affirmatively answers the question raised above for this one component model.

Our goal here is to treat certain two component systems. Motivated by this problem, we were led to reconsider the single component problem from the point of view of McCann's notion of *displacement convexity* [12]. In fact, the minimization problem for (1.1) is challenging largely because the functional $\mathcal{F}$ is *not convex* on $\mathcal{C}_{a,b}$ in the usual way: For $0 < \lambda < 1$, and m_0 and m_1 in $\mathcal{C}_{a,b}$, define $m_\lambda = (1-\lambda)m_0 + \lambda m_1$ and note that $m_\lambda \in \mathcal{C}_{a,b}$. However, due to the non-convexity of the potential function F, it is *not* true in general that $\mathcal{F}(m_\lambda) \leqq (1-\lambda)\mathcal{F}(m_0) + \lambda\mathcal{F}(m_0)$.

In [12], McCann, building on groundbreaking work of Brenier [6], introduced an alternative convex structure on the space of probability densities on $\mathbb{R}^n$, and used this to prove existence and uniqueness results for minimizers of functionals that were not convex in the usual sense. We shall show here that the minimization problem for (1.1), as well as for a two component model of this type, can be handled within this framework. In the process, we provide two new examples of strictly displacement convex functionals, the second of which is jointly displacement convex. It turns out that the alternative convex structure introduced in [2] is equivalent to the displacement convexity in this one dimensional setting, although the approach is quite different. We shall see that developing the alternative convex structure explicitly in terms of displacement convexity has advantages, especially for the two component system, when one seeks to prove a uniqueness result. Moreover, as we show in the final section, our results for the two component system may be applied to the single component system in higher dimensions, yielding a new uniqueness theorem for monotone solutions of the Euler–Lagrange equation.

We now describe the alternate convex structure with respect to which $\mathcal{F}$ *is* convex. This second convex structure cannot be defined on all of $\mathcal{C}_{a,b}$, but only on the subset $\mathcal{M}_{a,b}$ consisting of *right-continuous monotone* profiles. Nothing is lost in this restriction, as rearrangement inequalities show that minimizers of $\mathcal{F}$ on $\mathcal{C}_{a,b}$ must actually be monotone, so that they have a right-continuous version belonging $\mathcal{M}_{a,b}$; see [1] and Theorem 6.1 below.

Any right-continuous profile $m(x)$ in $\mathcal{M}_{a,b}$ can be written in the form

$$m(x) = a + (b-a)\int_{(-\infty,x]} \mathrm{d}\mu(y), \tag{1.2}$$

where μ is a uniquely determined probability measure on $\mathbb{R}$. This identification of $\mathcal{M}_{a,b}$ and the set of probability measures on $\mathbb{R}$ allows us to look at $\mathcal{F}$ as a functional defined on probability measures.

This is a useful perspective since there is an alternative convex structure on the set of probability measures on $\mathbb{R}$ (or more general domains) that was introduced by McCann, and which we describe below. A functional on probability measures is said to be *displacement convex* if it is convex with respect to this alternative structure. We shall show here that $\mathcal{F}$, regarded as a functional on probability measures is, in fact, displacement convex. Using this, we shall show that any solution in $\mathcal{M}_{a,b}$ of the Euler–Lagrange equation for the variational problem concerning (1.1)

$$m(x) = \frac{1}{\widehat{J}}\left(\int_{\mathbb{R}} J(x-y)m(y)\,\mathrm{d}y - F'(m(x))\right), \tag{1.3}$$

where

$$\widehat{J} = \int_{\mathbb{R}} J(x)\,\mathrm{d}x, \tag{1.4}$$

is in fact a minimizer. Solutions to (1.3) can easily be constructed by iteration and using these surface tensions may be readily computed.

This solution to the variational problem has the advantage of applying also to free energy functionals in certain multicomponent systems, in which the determination of the minimizers has not been previously treated. Indeed, our motivation was to be able to rigorously determine the surface tension in such systems. However, we shall first present our simple solution of the minimization problem for the single component free energy functional $\mathcal{F}$ specified in (1.1), and then treat the multicomponent case.

2. The alternative convex structure

2.1. The reduction to monotone profiles

First of all, notice that if we seek to minimize $\mathcal{F}$ on $\mathcal{C}_{a,b}$, we need only consider profiles m for which $a \leqq m(x) \leqq b$ for all x. Indeed, for any $m \in \mathcal{C}_{a,b}$, define $\widehat{m}$ by

$$\widehat{m}(x) = \min\{b, \max\{a, m(x)\}\}.$$

Then $\mathcal{F}(\widehat{m}) \leqq \mathcal{F}(m)$ with equality only in case $\widehat{m} = m$, since otherwise replacing m by $\widehat{m}$ lowers both terms.

We now recall a notion of rearrangement due to Alberti [1]. For any Borel measurable set A, let $|A|$ denote its Lebesgue measure. The rearrangement is defined for Borel sets $A \subset \mathbb{R}$ such that $|A \Delta (0, \infty)| < \infty$, where $A \Delta B = A \backslash B \cup B \backslash A$ is the symmetric difference of A and B. For such a set A, define the rearranged set A^* by

$$A^* = [\alpha, \infty) \quad \text{where} \quad \alpha = |(0, \infty) \backslash A| - |A \backslash (0, \infty)|.$$

Any function m in $\mathcal{C}_{a,b}$ that takes values in $[a, b]$ can be represented in "layer-cake" form:

$$m(x) = \int_a^b 1_{\{m>z\}}(x)\, \mathrm{d}z + a.$$

For each $z \in (a, b)$, the set $\{m > z\}$ certainly has the property that $|\{m > z\} \Delta (0, \infty)| < \infty$. Hence one can define the rearrangement of m itself through $m^*(x) = \int_a^b \left(1_{\{m>z\}}\right)^* (x)\, \mathrm{d}z + a$. (Applying the rearrangement to a monotone increasing function, one simply obtains the right-continuous version.)

Alberti shows that for any two such functions m_1 and m_2,

$$\int_{\mathbb{R}} |m_1^*(x) - m_2^*(x)|^2\, \mathrm{d}x \leqq \int_{\mathbb{R}} |m_1(x) - m_2(x)|^2\, \mathrm{d}x.$$

In particular, with m being any function in $\mathcal{C}_{a,b}$ that takes values in $[a, b]$, and h any real number, let $m_1(x) = m(x)$, and $m_2(x) = m(x + h)$. Then

$$\int_{\mathbb{R}} |m^*(x) - m^*(x + h)|^2\, \mathrm{d}x \leqq \int_{\mathbb{R}} |m(x) - m(x + h)|^2\, \mathrm{d}x,$$

so that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |m^*(x) - m^*(x+h)|^2 \, \mathrm{d}x \right) J(h) \, \mathrm{d}h$$
$$\leqq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |m(x) - m(x+h)|^2 \, \mathrm{d}x \right) J(h) \, \mathrm{d}h.$$

This of course means that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (m^*(x) - m^*(y))^2 J(x-y) \, \mathrm{d}x \, \mathrm{d}y \leqq \int_{\mathbb{R}} \int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y) \, \mathrm{d}x \, \mathrm{d}y. \tag{2.1}$$

In fact, Alberti shows (see Theorem 2.11 in [1]) that there is equality in (2.1) if and only if $m = m^*$.

Of course, $\int_{\mathbb{R}} F(m^*(x)) \, \mathrm{d}x = \int_{\mathbb{R}} F(m(x)) \, \mathrm{d}x$, and so we have $\mathcal{F}(m^*) \leqq \mathcal{F}(m)$ with equality if and only if $m = m^*$. Thus, we may restrict our search for minimizers to $\mathcal{C}_{a,b}$, the subset of monotone increasing profiles in $\mathcal{C}_{a,b}$.

2.2. Displacement convexity of $m \mapsto \int_{\mathbb{R}} F(m(x)) \, \mathrm{d}x$

As noted in (1.2), if m is any profile in $\mathcal{M}_{a,b}$, then $(m(x) - a)/(b-a)$ is the cumulative distribution function of a uniquely determined probability measure μ:

$$\frac{m(x) - a}{b - a} = \int_{(-\infty, x]} \mathrm{d}\mu(y).$$

For each m in $\mathcal{M}_{a,b}$, define $x(m)$ to be the inverse function: For $m \in (a, b)$,

$$x(m) = \inf\{ \, x \ : \ m(x) > m \, \}. \tag{2.2}$$

Then of course, $m(x)$ is the inverse function of $x(m)$, so that for x in $\mathbb{R}$,

$$m(x) = \inf\{ \, m \ : \ x(m) > x \, \}. \tag{2.3}$$

Let $\mathrm{d}x(m)$ denote the Lebesgue–Stieltjes measure on $[a, b]$ induced by the monotone function $x(m)$. (In the terminology introduced below, $\mathrm{d}x(m)$ is the push-forward of Lebesgue measure on $\mathbb{R}$ under m). Then one can rewrite

$$\int_{\mathbb{R}} F(m(x)) \, \mathrm{d}x = \int_a^b F(m) \, \mathrm{d}x(m).$$

Let m_0 and m_1 be any two elements of $\mathcal{M}_{a,b}$, and let x_0 and x_1 denote their respective inverse functions. Then for any $\lambda \in (0, 1)$, define $x_\lambda(m)$ by

$$x_\lambda(m) = (1 - \lambda) x_0(m) + \lambda x_1(m). \tag{2.4}$$

Note that x_λ is also the inverse function of an element of $\mathcal{M}_{a,b}$, which we shall call m_λ. That is,

$$m_\lambda(x) = \inf\{ \, m \ : \ (1 - \lambda) x_0(m) + \lambda x_1(m) > x \, \}. \tag{2.5}$$

Note that $\mathrm{d}x_\lambda(m)$, the Lebesgue–Stieltjes measure on $[0, 1]$ induced by the monotone function $x_\lambda(m)$, satisfies $\mathrm{d}x_\lambda(m) = (1-\lambda)\,\mathrm{d}x_0 + \lambda\,\mathrm{d}x_1$. Then,

$$\begin{aligned}\int_{\mathbb{R}} F(m_\lambda(x))\,\mathrm{d}x &= \int_a^b F(m)\,\mathrm{d}x_\lambda(m)\\ &= (1-\lambda)\int_a^b F(m)\,\mathrm{d}x_0(m) + \lambda\int_a^b F(m)\,\mathrm{d}x_1(m)\\ &= (1-\lambda)\int_{\mathbb{R}} F(m_0(x))\,\mathrm{d}x + \lambda\int_{\mathbb{R}} F(m_1(x))\,\mathrm{d}x.\end{aligned} \tag{2.6}$$

This tells us that along the interpolation m_λ between m_0 and m_1 provided by (2.5), the function $\lambda \mapsto \int_{\mathbb{R}} F(m_\lambda(x))\,\mathrm{d}x$ is affine, and in particular, is convex. This is not the case for the standard interpolation given by

$$\widetilde{m}_\lambda(x) = (1-\lambda)m_0(x) + \lambda m_1(x), \tag{2.7}$$

since $\lambda \mapsto \int_{\mathbb{R}} F(\widetilde{m}_\lambda(x))\,\mathrm{d}x$ is *not*, in general, convex. That is, taking convex combinations in terms of the inverse function $x(m)$ as in (2.5), instead of $m(x)$ itself as in (2.7), has "cured" the non-convexity of the functional $m \mapsto \int_{\mathbb{R}} F(m(x))\,\mathrm{d}x$.

Of course, this will only be useful if the functional

$$m \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}} (m(x) - m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.8}$$

which was convex in the usual way, is still convex with the new convex structure. This is not at all obvious, but the main result of the next section asserts that this is the case.

The approach of ALBERTI and BELLETTINI [2], which we discovered only after our work was complete, was to rewrite the interaction directly in terms of x_m, and to show that it is convex.

However, it turns out that the convex structure in (2.4) is something that is by now well-known; it is the *displacement convexity* structure introduced by McCann. Making this connection will facilitate showing the *strict* convexity of $m \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}}(m(x)-m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y$ under this convex structure. This point was left open in [2], who explicitly asked whether one could extend the ideas to give a direct proof of uniqueness. Although ALBERTI [1] did later return to address the issue, we shall see here that the strict convexity is quite clear from the perspective of displacement convexity.

Displacement convexity is usually introduced as a convex structure in a set of probability measures. Given a probability measure μ_0 on $\mathbb{R}$, and a measurable map $T : \mathbb{R} \to \mathbb{R}$, we define the *push forward of* μ_0 *under* T, $T\#\mu_0$, by

$$\int_{\mathbb{R}} \phi(T(x))\,\mathrm{d}\mu_0(x) = \int_{\mathbb{R}} \phi(y)\,\mathrm{d}(T\#\mu_0)(y), \tag{2.9}$$

for all bounded, continuous functions ϕ.

Given two probability measures μ_0 and μ_1 on $\mathbb{R}$, there is a unique *monotone* map T such that $T\#\mu_0 = \mu_1$. To see what it must be, fix any $a \in R$, let ϕ_a be the step function $\phi_a(x) = 1_{(-\infty,a]}(x)$. Then, by definition, we must have $\int_{\mathbb{R}} \phi_a(T(x))\,\mathrm{d}\mu_0(x) = \int_{\mathbb{R}} \phi_a(y)\,\mathrm{d}\mu_1(y)$, and hence

$$\int_{-\infty}^{T^{-1}(a)} \mathrm{d}\mu_0 = \int_{-\infty}^{a} \mathrm{d}\mu_1. \tag{2.10}$$

Let m_0 and m_1 be the cumulative distribution functions of μ_0 and μ_1, respectively. Then (2.10) entails that $m_0(T^{-1}(a)) = m_1(a)$ for all a, or, what is the same thing

$$m_0(a) = m_1(T(a)) \tag{2.11}$$

for all a. As long as m_1 is free of "flat spots", so that the inverse function does the expected thing, this leads to

$$T(a) = x_1(m_0(a)). \tag{2.12}$$

As long as μ_0 and μ_1 have strictly positive densities, (2.12) does indeed define a monotone map T, and then it is very easy to see that with T defined by (2.12), $T\#\mu_0 = \mu_1$, and in fact, this is true without further technical hypotheses; see [14] for more information.

We now interpolate the map T, and hence the corresponding probability measures μ_0 and μ_1 and the corresponding cumulative distribution functions m_0 and m_1 as well. For all $\lambda \in [0, 1]$, define T_λ by

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x). \tag{2.13}$$

If we define $x_\lambda(m)$ by

$$x_\lambda(m) = T_\lambda(x_0(m)),$$

then clearly x_λ is given by (2.4).

The displacement convex structure on probability measures on $\mathbb{R}$ is given by $\mu_\lambda = T_\lambda\#\mu_0$, and so it is nothing other than the convex structure (2.4), expressed in terms of probability measures instead of cumulative distribution functions. When μ_1 and μ_2 have strictly positive densities, so that T is given by (2.11), we denote the density of μ_λ by ρ_λ, and write

$$\rho_\lambda = T_\lambda\#\rho_1. \tag{2.14}$$

We summarize the main result of this section in a theorem:

Theorem 2.1. *Let $\lambda \mapsto m_\lambda$ be the displacement interpolation between m_0 and m_1 in $\mathcal{M}_{a,b}$. Then for $0 \leqq \lambda \leqq 1$,*

$$\int_{\mathbb{R}} F(m_\lambda(x))\,\mathrm{d}x = (1-\lambda)\int_{\mathbb{R}} F(m_0(x))\,\mathrm{d}x + \lambda\int_{\mathbb{R}} F(m_1(x))\,\mathrm{d}x.$$

3. Displacement convexity of the interaction energy

Let $\mathcal{M}$ denote the class of cumulative distribution functions on $\mathbb{R}$. Making the obvious change of variables, we will assume without loss of generality that $a = 0$ and $b = 1$ and we will set $\mathcal{M}_{0,1} = \mathcal{M}$.

Given any $m \in \mathcal{M}$, let μ be the corresponding probability measure, so that $m(x) = \int_{-\infty}^{x} \mathrm{d}\mu(y)$. The first step in the investigation of the interaction energy is to rewrite it as a functional of μ instead of m. This is done in the following lemma:

Lemma 3.1. *Assume that* $\int_{\mathbb{R}} |s| J(s)\,\mathrm{d}s < \infty$. *Define* W *in terms of* J *by setting*

$$W(u) = \int_{u}^{\infty} (s-u)(J(s)+J(-s))\,\mathrm{d}s \tag{3.1}$$

for $u \geqq 0$ *and* $W(u) = W(-u)$ *for* $u < 0$. *Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x)-m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w).$$

W *is a symmetric function, and is convex on* $(0,\infty)$ *and on* $(-\infty, 0)$, *though not on all of* $\mathbb{R}$.

Proof. Since for $x < y$, $m(y) - m(x) = \int_{x}^{y} \mathrm{d}\mu(z)$, we have from the Fubini Theorem that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x)-m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w)\right] J(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w) J(x-y)\,\mathrm{d}x\,\mathrm{d}y\right] \mathrm{d}\mu(z)\,\mathrm{d}\mu(w).
\end{aligned} \tag{3.2}$$

Thus if we define $V(z-w)$ by $V(z-w) = \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w)J(x-y)\,\mathrm{d}x\,\mathrm{d}y$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (m(x)-m(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} V(z-w)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w).$$

We next show that $V = W$. To do this, write

$$J_+(x) = \begin{cases} J(x) & \text{for } x > 0, \\ 0 & \text{for } x \leqq 0 \end{cases} \quad \text{and} \quad J_-(x) = J(x) - J_+(x).$$

We first consider $\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[x,y]}(z)1_{[x,y]}(w)J_+(x-y)\,\mathrm{d}x\,\mathrm{d}y$. Make the change of variables $s = y - x$ and $t = (x+y)/2$. Then $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}s\,\mathrm{d}t$, and

$1_{[x,y]}(z)1_{[x,y]}(w) = 1_{[t-s/2,t+s/2]}(z)1_{[t-s/2,t+s/2]}(w)$. This quantity is zero unless $|z-w| \leqq s$ and $|2t-(x+w)| \leqq s-|z-w|$, in which case it is one. Therefore

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} & 1_{[x,y]}(z)1_{[x,y]}(w)J_+(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{|z-w|}^{\infty}\left(\int_{(z+w)/2-(s-|z-w|)/2}^{(z+w)/2+(s-|z-w|)/2}\mathrm{d}t\right)J_+(s)\,\mathrm{d}s \qquad (3.3)\\ &= \int_{|z-w|}^{\infty}(s-|z-w|)J_+(s)\,\mathrm{d}s\ .\end{aligned}$$

Doing the same calculation for the part involving J_-, we obtain that $V = W$ where W is given by (3.1). Also, for $u > 0$, $W'(u) = -\int_u^\infty (J(s)+J(-s))\,\mathrm{d}s$, and so $W''(u) = J(u) + J(-u)$, which is non-negative. Thus, W is convex on $(0,\infty)$, and on $(-\infty, 0)$ by symmetry. However, it is not convex on the whole real line. Notice that $W(0) = \int_0^\infty s(J(s)+J(-s))\,\mathrm{d}s > 0$, while $\lim_{u\to\pm\infty} W(u) = 0$. □

We now prove the main result of this section:

Theorem 3.2. *Let $\lambda \mapsto m_\lambda$ be the displacement interpolation between m_0 and m_1 in $\mathcal{M}$, as defined in* (2.5)*. Then, for* $0 < \lambda < 1$*,*

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} & (m_\lambda(x)-m_\lambda(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqq (1-\lambda)\int_{\mathbb{R}}\int_{\mathbb{R}}(m_0(x)-m_0(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y \qquad (3.4)\\ &\quad +\lambda\int_{\mathbb{R}}\int_{\mathbb{R}}(m_1(x)-m_1(y))^2 J(x-y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

If J is strictly positive on some interval, and m_0 has a strictly positive derivative almost everywhere, there is equality if and only if m_1 is a translate of m_0.

Proof. If W were convex on all of $\mathbb{R}$, the displacement convexity of the interaction energy would be a classical result of McCann [12]. However, in one dimension, the partial convexity of W that was established in Lemma 3.1 suffices, as observed by Blower [5]. This is because the map T_λ is monotone for all λ. Therefore, if $z > w$, $T_\lambda(z) > T_\lambda(w)$ for all λ. Hence, as we vary λ, $T_\lambda(z) - T_\lambda(w)$ stays in a domain of convexity of W.

Hence from (2.4), if $\mathrm{d}\mu_\lambda = T_\lambda \#\,\mathrm{d}\mu_0$ is the displacement interpolation between $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\,\mathrm{d}\mu_\lambda(z)\,\mathrm{d}\mu_\lambda(w) = \int_{\mathbb{R}}\int_{\mathbb{R}} W(T_\lambda(z)-T_\lambda(w))\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w). \tag{3.5}$$

Define the map $S(x)$ by $S(x) = T(x) - x$. Then, we can rewrite (3.5) as

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} W(z-w)\,\mathrm{d}\mu_\lambda(z)\,\mathrm{d}\mu_\lambda(w) = \int_{\mathbb{R}}\int_{\mathbb{R}} & W([z-w] \\ & +\lambda[S(z)-S(w)])\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w).\end{aligned} \tag{3.6}$$

By the remarks made above, the right-hand side is clearly a convex function of λ. In fact, under mild assumptions on μ_0 or J, it is strictly convex unless T is simply a translation.

To see this *formally*, let J be symmetric for simplicity of notation, and differentiate the right-hand side of (3.6) twice in λ, finding

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 2J\left([z-w]+\lambda[S(z)-S(w)]\right)[S(z)-S(w)]^2\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w).$$

If this vanishes for all λ, then $\int_{\mathbb{R}}\int_{\mathbb{R}} 2J(z-w)[S(z)-S(w)]^2\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w)=0$. If J is strictly positive and if μ_0 has a strictly positive density, then this is possible if and only if S is constant, and that of course means that T is a translation.

To make this argument rigorous, and to relax the hypotheses, let $f(\lambda)$ denote the right-hand side of (3.4) minus the left-hand side. Then, with $g(z,w,\lambda)$ defined by

$$\begin{aligned} g(z,w,\lambda) = {} & [\lambda W(z-w)+(1-\lambda)W((z-w)+(S(z)-S(w)))] \\ & -W((z-w)+\lambda(S(z)-S(w))), \end{aligned}$$

we have $f(\lambda)=\int_{\mathbb{R}}\int_{\mathbb{R}} g(z,w,\lambda)\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w)$. Since the integrand is non-negative, we have for any measurable subsets A and B of $\mathbb{R}$,

$$f(\lambda) \geqq \int_A\int_B g(z,w,\lambda)\,\mathrm{d}\mu_0(z)\,\mathrm{d}\mu_0(w). \tag{3.7}$$

Suppose that J is strictly positive on the open interval $I=(y_0-\delta/2, y_0+\delta/2)$. Then I is an interval of strict convexity of W, so that whenever $w-z\in I$, $\lambda\mapsto g(z,w,\lambda)>0$ on $(0,1)$ unless $S(z)=S(w)$. However, if S is not constant almost everywhere, we can find an arbitrarily small interval about some z_0 on which it has strictly positive oscillation. In particular, we can find a z_0 and an $\varepsilon>0$ so that $\int_{z_0-\delta/2}^{z_0+\delta/2}(S(z)-c)^2\,\mathrm{d}z>\varepsilon$ for all c. Let $A=(z_0-\delta/2, z_0+\delta/2)$, and let $B=(y_0+x_0-\delta/2, y_0+x_0+\delta/2)$. Then for all z in A and w in B, $z-w$ belongs to I. Moreover, for every w in B, $\int_A(S(z)-S(w))^2\,\mathrm{d}w>0$, so $|S(z)-S(w)|>0$ on a subset of A of positive Lebesgue measure. Since μ_0 has a strictly positive density, this ensures that the right-hand side of (3.7) is strictly positive. □

It is clear that the conditions on J and m_0 that are invoked to ensure strict convexity can be relaxed, though they are already quite general.

We close this section with a remark. If the profile m is continuously differentiable with $m'(x)=\rho(x)$, and $\int_{\mathbb{R}} J(x)\,\mathrm{d}x=1$, then

$$\lim_{h\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(m(x)-m(y))^2}{h^2}\frac{1}{h}J\left(\frac{x-y}{h}\right)\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}}\rho^2(x)\,\mathrm{d}x.$$

It is already well known that the functional $\rho\mapsto\int_{\mathbb{R}}\rho^2(x)\,\mathrm{d}x$ is displacement convex, so the fact that Theorem 3.2 gives another proof of this is not of great interest. However, the connection between the two functionals at least gives one suggestion as to why the interaction functional might be expected to be displacement convex.

4. For the functional $\mathcal{F}$, critical points are minimizers

Theorem 4.1. *If m_0 is any critical point of $\mathcal{F}$ in $\mathcal{M}$, and m is any other profile in $\mathcal{M}$, then $\mathcal{F}(m) \geqq \mathcal{F}(m_0)$ and there is equality if and only if m is a translate of m_0.*

Proof. Let m_λ be the displacement interpolation between m_0 and m. Then $\lambda \mapsto \mathcal{F}(m_\lambda)$ is convex, and the derivative is zero at $\lambda = 0$. Hence m_0 is a minimizer of $\mathcal{F}$, so that $\mathcal{F}(m_\lambda) \geqq \mathcal{F}(m_0)$, and there is equality if and only if $\lambda \mapsto \mathcal{F}(m_\lambda)$ is constant. But in this case, the strict displacement convexity of $\mathcal{F}$ ensures that m is a translate of m_0. $\square$

5. Fronts in a binary fluid model

We now turn to the study of the analogous problem for a binary fluid model. The binary fluid model has been investigated in [7,8], and we refer to those papers for details. Although the arguments apply to that setting in full generality, we discuss here only a special case where the non-local interaction is only between particles of different species and the local term is purely entropic, for the sake of brevity. For further information and a numerical investigation of the minimizing fronts, see [4].

In what follows, $m(x)$ and $n(x)$ represent the particle number densities of two different species of particles contained in some bounded domain Ω in $\mathbb{R}^n$. Consider the functional $\mathcal{F}$ defined by

$$\begin{aligned}\mathcal{F}(m,n) = &\int_\Omega m(x)\ln m(x)\,\mathrm{d}x + \int_\Omega n(x)\ln n(x)\,\mathrm{d}x \\ &+ \beta\int_\Omega\int_\Omega J(|x-y|)m(x)n(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{5.1}$$

Here, J is a non-negative, decreasing and compactly supported function on $\mathbb{R}_+$ with range R. Notice that we must impose more conditions on J in the case of two species than we did in the single component model. The reasons for this will be made clear in Section 6.

The problem considered in [7] is to minimize $\mathcal{F}(m,n)$ subject to the constraint that

$$\frac{1}{|\Omega|}\int_\Omega m(x)\,\mathrm{d}x \quad \text{and} \quad \frac{1}{|\Omega|}\int_\Omega n(x)\,\mathrm{d}x \tag{5.2}$$

have certain prescribed values. As shown in [7], this system undergoes a segregating phase transition when β is large enough for the interaction term to overcome the entropy terms in $\mathcal{F}$. These would prefer to have m and n to be uniform and this will indeed be the minimizing state for small β, that is, high temperature β^{-1}. However, for large values of β, the advantages of segregation can dominate and the fluid separates into two phases, one rich in particles of species 1 and the other rich in particles of species 2. Our concern here is with the profiles of the densities at the interface between the two phases.

The nature of the two phases in the bulk is determined by considering the zero range model, in which the length scale R of the interaction J is negligible compared to the size of Ω. Formally, this corresponds to setting $J(x-y) = \widehat{J}\delta(x-y)$. It is also convenient to drop the constraint (5.2) and to consider the function

$$f_{\beta,\lambda_1,\lambda_2}(m,n) = m\ln m + n\ln n + \beta\hat{J}mn - \lambda_1 m - \lambda_2 n \tag{5.3}$$

as a local free energy density. Here, as in the one component case, $\widehat{J} = \int_{\mathbb{R}^n} J(|x|)\,\mathrm{d}x$, and λ_1 and λ_2 are Lagrange multipliers that ensure the constraint (5.2) on the total particle numbers. One may also think of λ_1 and λ_2 as specified chemical potentials and then determine m and n as functions of λ_1 and λ_2.

In [7] it is proved that, if $\lambda_1 \neq \lambda_2$, there is an unique couple $\bar{m}, \bar{n}$ minimizing $f_{\beta,\lambda_1,\lambda_2}(m,n)$. However, if $\lambda_1 = \lambda_2$, there is a β_c such that, if $\beta \leqq \beta_c$ the minimizer is still unique, while, if $\beta > \beta_c$ there are densities $\rho^- < \rho^+$ such that the couples (ρ^+, ρ^-) and (ρ^-, ρ^+) are both minimizers of $f_{\beta,\lambda_1,\lambda_2}(m,n)$. We focus on the last case. Thus, in what follows $\lambda_1 = \lambda_2 = \lambda$. Then $f_{\beta,\lambda_1,\lambda_2}$ is the local Gibbs free energy density.

Analysis of the zero range model suffices to determine the quantity of the fluid that is present in each phase, but not the surface tension across the boundary. We now turn to the variational problem that determines the density profiles across the interface and the surface tension. We will assume that the geometry of Ω is such that the interface is perpendicular to the first coordinate axis; for example, we take Ω to be a very long cylinder along the x_1-axis with periodic boundary conditions along the other coordinate axes.

First, we need to introduce the one dimensional version of J. Choose coordinates (s,t) on $\mathbb{R}^n$ with $s \in \mathbb{R}$ and $t \in \mathbb{R}^{n-1}$, and define $\bar{J}$ on $\mathbb{R}$ by

$$\bar{J}(s) = \int_{\mathbb{R}^{n-1}} J(\sqrt{s^2 + |t|^2})\,\mathrm{d}t$$

and then $\widehat{J} = \int_{\mathbb{R}} \bar{J}(s)\,\mathrm{d}s$. Let $g_{\beta,\lambda} = \inf_{m,n \geqq 0} f_{\beta,\lambda,\lambda}(m,n)$. By what has been noted above,

$$g_{\beta,\lambda} = f_{\beta,\lambda,\lambda}(\rho^-, \rho^+) = f_{\beta,\lambda,\lambda}(\rho^+, \rho^-).$$

The functional $\mathcal{G}$ defined by

$$\mathcal{G}(m,n) = \int_{\mathbb{R}} \left[m(x)\ln m(x) + n(x)\ln n(x) + \beta \int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\,\mathrm{d}y - g_{\beta,\lambda} \right] \mathrm{d}x$$

is the *excess free energy* of a front. We look for the minimizers of this functional for $\beta > \beta_c$. The minimum value gives the surface tension across the planar phase boundary. Note that we have reduced Ω to $\mathbb{R}$, the line along the axis of Ω and that minimizing $\mathcal{G}$ yields the free energy per unit $(d-1)$-dimensional area.

Our goal in the next sections is to prove a strict displacement convexity property of this excess free energy functional and to show, as a consequence, the uniqueness of the minimizing fronts up to translation. As in the one component case, a rearrangement inequality will enable us to restrict our attention to monotone profiles.

Let $\mathcal{M}_{\rho^-,\rho^+}$ be the subset of $\mathcal{C}_{\rho^-,\rho^+}$ consisting of monotone increasing profiles, let $\mathcal{M}_{\rho^+,\rho^-}$ be the subset of $\mathcal{C}_{\rho^+,\rho^-}$ consisting of monotone decreasing profiles.

Our main goal mathematically in what follows is to show that the functional

$$(m,n) \mapsto \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\,\mathrm{d}y - \hat{J}\rho^+\rho^- \right] \mathrm{d}x$$

is displacement convex on $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, where now we have both an increasing and a decreasing density profile.

We shall prove the displacement convexity results in the next section. This time, we shall require certain moment conditions to obtain the displacement convexity. Hence, before we can apply these results, we need to show a priori that all minimizers have good localization properties. We do this by an analysis of the Euler Lagrange equation.

5.1. Convexity of the interaction energy for $\mathcal{G}$

Define the functional $\mathcal{I}$ on $\mathcal{M}_{a,b} \times \mathcal{M}_{c,d}$ by

$$\mathcal{I}(m,n) = \int_{\mathbb{R}} \mathrm{d}x \left[\int_{R} \bar{J}(x-y)m(x)m(y)\,\mathrm{d}y - \hat{J}\hat{m}(x)\hat{n}(x) \right]. \tag{5.4}$$

We assume J to be non-negative, even and compactly supported on $\mathbb{R}$ with range R. We define $\widehat{J}$ to be the total mass of J, and we define

$$\widehat{m}(x) = \begin{cases} b & \text{for } x \geqq 0, \\ a & \text{for } x < 0 \end{cases} \quad \text{and} \quad \widehat{n}(x) = \begin{cases} d & \text{for } x \geqq 0, \\ c & \text{for } x < 0. \end{cases}$$

Note that in the special case $a = d = \rho^-$ and $b = c = \rho^+$,

$$\mathcal{I}(m,n) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \bar{J}(x-y)m(x)n(y)\,\mathrm{d}y - \hat{J}\rho^+\rho^- \right] \mathrm{d}x.$$

Although this special case is all that is needed for our applications here, we treat the general case because the small extra effort yields a broad new class of jointly displacement convex functionals.

The first step in our analysis is to rewrite $\mathcal{I}$ as a functional on probability densities. Let the probability densities ρ_1 and ρ_2 be defined by

$$m(x) = a + (b-a)\int_{-\infty}^{x} \rho_1(t)\,\mathrm{d}t; \quad n(x) = c + (d-c)\int_{-\infty}^{x} \rho_2(t)\,\mathrm{d}t. \tag{5.5}$$

We the rewrite the functional in terms of ρ_1 and ρ_2, and integrate by parts. Formally, one moves an antiderivative from each of ρ_1 and ρ_2 over to $\bar{J}$. Since $\bar{J}$ is positive, integrating it twice produces a convex function, W, different from the one constructed in the one-component case. This is indeed what happens, but one must be careful about the boundary terms. The boundary terms do not vanish but, as we shall see, they depend on the densities in a very nice way and altogether, one obtains the desired displacement convexity.

To carry out this analysis, define

$$W(x) = \begin{cases} \int_0^x \left(\int_0^t \bar{J}(s)\,\mathrm{d}s \right) \mathrm{d}t & \text{for } x > 0, \\ W(-x) & \text{for } x < 0. \end{cases} \tag{5.6}$$

Then $W''(x) = \bar{J}(x)$, $W(0) = 0$, and W is an even convex function. Furthermore,

$$\lim_{x\to\infty} W'(x) = \frac{\hat{J}}{2}, \quad W(x) = \alpha + \frac{\hat{J}}{2}|x| \text{ for } |x| \geqq R. \tag{5.7}$$

Lemma 5.1. *Let $m \in \mathcal{M}_{a,b}$ and $n \in \mathcal{M}_{c,d}$. Let ρ_1 and ρ_2 be the corresponding probability densities defined in (5.5). Then, provided ρ_1 and ρ_2 have finite first moments, and with W and α defined as above,*

$$\begin{aligned} \mathcal{I}(m_1, m_2) = (a-b)(d-c) \int_{\mathbb{R}} \int_{\mathbb{R}} W(x-y)\rho_1(x)\rho_2(y)\,\mathrm{d}x\,\mathrm{d}y \\ + [2(b-a)(d-c) + bc + ad]\alpha \\ - \frac{\hat{J}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} x \ [(b+a)(d-c)\rho_2(x) + (b-a)(c+d)\rho_1(x)]\,\mathrm{d}x. \end{aligned} \tag{5.8}$$

Note that $(a-b)(d-c) > 0$ for $b > a$ and $c > d$, which is the case when $a = d = \rho^-$ and $c = b = \rho^+$. Thus, $(a-b)(d-c)W(z)$ is a convex function of z on all of $\mathbb{R}$. It follows in the usual way that the first term on the right is displacement convex. Since W is strictly convex on the support of J, it follows as in the proof of Theorem 3.2 that this part of the functional (5.8) is in fact strictly convex apart from translation. The second term on the right of (5.8) is a constant. The third term is a linear combination of the first moments of ρ_1 and ρ_2. Since these first moments are displacement affine, we see that altogether, $\mathcal{I}(m, n)$ is strictly displacement convex, apart from translation.

The fact that Lemma 5.1 requires a conditions on first moments, while Theorem 3.2 does not, means that it will be a little more work to apply Lemma 5.1: We shall need an a priori estimate guaranteeing that for any critical point (m, n) of $\mathcal{G}$, the corresponding densities have finite first moments. We shall return to this after first proving the theorem.

Proof. We start by considering the integral in x first, on a bounded interval $[-L, L]$. Since $\bar{J}(x-y) = -\frac{\partial^2}{\partial x \partial y} W(x-y)$ we have that

$$\begin{aligned} -\int_{-L}^{L} \frac{\partial^2}{\partial x \partial y} W(x-y)m(x)\,\mathrm{d}x = \int_{-L}^{L} \frac{\partial}{\partial y} W(x-y)(b-a)\rho_1(x)\,\mathrm{d}x \\ - \frac{\partial}{\partial y} W(L-y)m(L) + \frac{\partial}{\partial y} W(-L-y)m(-L). \end{aligned} \tag{5.9}$$

Moreover,

$$
\begin{aligned}
&\int_{-L}^{L}\int_{-L}^{L} \bar{J}(x-y)m(x)n(y)\,\mathrm{d}y\\
&= \int_{-L}^{L}\int_{-L}^{L} \frac{\partial}{\partial y} W(x-y)(b-a)\rho_1(x)n(y)\,\mathrm{d}x\,\mathrm{d}y \qquad (5.10)\\
&\quad + \int_{-L}^{L}\left[-\frac{\partial}{\partial y}W(L-y)m(L) + \frac{\partial}{\partial y}W(-L-y)m(-L)\right]n(y)\,\mathrm{d}y.
\end{aligned}
$$

Now we integrate by parts once more, this time in y:

$$
\begin{aligned}
\int_{-L}^{L}\frac{\partial}{\partial y}W(x-y)n(y)\,\mathrm{d}y = -\int_{-L}^{L} W(x-y)(d-c)\rho_2(y)\,\mathrm{d}y&\\
+W(x-L)n(L) - W(x+L)n(-L).& \qquad (5.11)
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
&\int_{-L}^{L}\int_{-L}^{L} \bar{J}(x-y)m(x)n(y)\,\mathrm{d}y\\
&\quad = -(b-a)(d-c)\int_{-L}^{L}\int_{-L}^{L} W(x-y)\rho_1(x)\rho_2(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad + \int_{-L}^{L}\left[-\frac{\partial}{\partial y}W(L-y)m(L) + \frac{\partial}{\partial y}W(-L-y)m(-L)\right]n(y)\,\mathrm{d}y\\
&\qquad + (b-a)\int_{-L}^{L}\left[W(x-L)n(L) - W(x+L)n(-L)\right]\rho_1(x)\,\mathrm{d}x.
\end{aligned}
\qquad (5.12)
$$

Let us examine the boundary terms

$$
\begin{aligned}
B_1 &:= \int_{-L}^{L}\left[-\frac{\partial}{\partial y}W(L-y)m(L) + \frac{\partial}{\partial y}W(-L-y)m(-L)\right]n(y)\,\mathrm{d}y,\\
B_2 &:= (b-a)\int_{-L}^{L}\left[W(x-L)n(L) - W(x+L)n(-L)\right]\rho_1(x)\,\mathrm{d}x.
\end{aligned}
$$

We have

$$
\begin{aligned}
B_1 &= m(L)\int_{-L}^{L} W(L-y)(d-c)\rho_2(y)\,\mathrm{d}y - m(-L)\\
&\qquad \times \int_{-L}^{L} W(-L-y)(d-c)\rho_2(y)\,\mathrm{d}y\\
&\qquad + m(L)\left[-W(L-y)n(y)\right]_{-L}^{+L} + m(-L)\left[W(-L-y)n(y)\right]_{-L}^{+L}\\
&= (d-c)\int_{-L}^{L}\left[m(L)W(L-y) - m(-L)W(-L-y)\right]\rho_2(y)\,\mathrm{d}y\\
&\qquad + m(L)\left[-W(0)n(L) + W(2L)n(-L)\right] + m(-L)\\
&\qquad \times \left[W(2L)n(L) - W(0)n(-L)\right].
\end{aligned}
$$

For $2L > R$, where R is the range of the interaction $\bar{J}$, the last two terms give

$$(bc + ad)(\hat{J}L + \alpha) + \mathcal{O}(1).$$

To compute the other term, we consider, for a function f rapidly decaying, $\int_{-L}^{L} f(x)W(x+L)\,\mathrm{d}x$ and $\int_{-L}^{L} f(x)W(x-L)\,\mathrm{d}x$. We have

$$\begin{aligned}\int_{-L}^{L} f(x)W(x+L)\,\mathrm{d}x &= \int_{0}^{2L} f(z-L)W(z)\,\mathrm{d}z\\ &= \int_0^R f(z-L)W(z)\,\mathrm{d}z + \int_R^{2L} f(z-L)\left(\frac{\hat{J}}{2}z+\alpha\right)\mathrm{d}z.\end{aligned}$$

The first term vanishes in the limit $L \to \infty$ because of the decay of f and of the boundedness of $W(z)$ for $z \in [0, R]$. The second term becomes, if $\int_{\mathbb{R}} |x|f(x)\,\mathrm{d}x < \infty$,

$$\int_{R-L}^{L} f(x)\left(\frac{\hat{J}}{2}(x+L)+\alpha\right)\mathrm{d}x = \frac{\hat{J}}{2}\int_{\mathbb{R}} xf(x)\,\mathrm{d}x + \left(\alpha+\frac{\hat{J}}{2}L\right)\int_{\mathbb{R}} f(x)\,\mathrm{d}x + \mathcal{O}(1).$$

In conclusion,

$$\int_{-L}^{L} f(x)W(x\pm L)\,\mathrm{d}x = \pm\frac{\hat{J}}{2}\int_{\mathbb{R}} xf(x)\,\mathrm{d}x + \left(\alpha + L\frac{\hat{J}}{2}\right)\int_{\mathbb{R}} f(x)\,\mathrm{d}x + \mathcal{O}(1).$$

Now we apply this result to B_2, where the decaying function is ρ_1, to get

$$\begin{aligned}B_2 = (b-a)\Bigg[&-(c+d)\frac{\hat{J}}{2}\int_{\mathbb{R}} x\rho_1(x)\,\mathrm{d}x + \alpha(d-c)\int_{\mathbb{R}}\rho_1(x)\,\mathrm{d}x\\ &+ \frac{\hat{J}}{2}L(d-c)\int_{\mathbb{R}}\rho_1(x)\,\mathrm{d}x\Bigg] + \mathcal{O}(1).\end{aligned}$$

Now we apply it to B_1:

$$\begin{aligned}B_1 = (d-c)\Bigg[&-(b+a)\frac{\hat{J}}{2}\int_{\mathbb{R}} x\rho_2(x)\,\mathrm{d}x\\ &+ \alpha(b-a)\int_{\mathbb{R}}\rho_2(x)\,\mathrm{d}x + \frac{\hat{J}}{2}L(b-a)\int_{\mathbb{R}}\rho_2(x)\,\mathrm{d}x\Bigg]\\ &+ (bc+ad)(\hat{J}L+\alpha) + \mathcal{O}(1).\end{aligned} \tag{5.13}$$

Finally,

$$\begin{aligned}&B_1 + B_2 - \hat{J}\int_{\mathbb{R}} \hat{m}(x)\hat{n}(x)\,\mathrm{d}x\\ &= [2(b-a)(d-c) + bc + ad]\alpha - \frac{\hat{J}}{2}(b+a)(d-c)\int_{\mathbb{R}} y\rho_2(y)\,\mathrm{d}y\\ &\quad -\frac{\hat{J}}{2}(b-a)(c+d)\int_{\mathbb{R}} x\rho_1(x)\,\mathrm{d}x + \mathcal{O}(1).\end{aligned}$$

□

Lemma 5.1 is the key ingredient to prove the analog of Theorem 3.2 for the two-component model introduced in the beginning of this section. We now return to this model, and shall apply the lemma with $a = d = \rho^-$ and $b = c = \rho^+$. Let (w_1, w_2) and (v_1, v_2) be in $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, with corresponding probability densities (η_1, η_2) and (ζ_1, ζ_2), and let T_1, T_2 be the monotone maps such that $\zeta_i = T_i \# \eta_i$, $i = 1, 2$. Moreover, let $\lambda \mapsto (m_\lambda, n_\lambda)$ be the displacement interpolations between (w_1, w_2) and (v_1, v_2) and $T_i^\lambda(x) = \lambda x + (1 - \lambda) T_i(x)$.

Theorem 5.2. *Suppose that the probability densities η_i and ζ_i, $i = 1, 2$ have finite first moments. Then for $0 < \lambda < 1$,*

$$\mathcal{G}(m_\lambda, n_\lambda) \leqq (1 - \lambda)\mathcal{G}(w_1, w_2) + \lambda \mathcal{G}(v_1, v_2). \tag{5.14}$$

If J is strictly positive on some interval, and (w_1, w_2) have strictly positive derivatives almost everywhere, there is equality if and only if (v_1, v_2) is a translate of (w_1, w_2).

Proof. Lemma 5.1 is applicable by the assumption that the probability densities have finite first moments. We set $S_i(x) = T_i(x) - x$, so that

$$\begin{aligned}\mathcal{I}(m_\lambda, n_\lambda) = (\rho^+ - \rho^-)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} W[(x - y) + \lambda(S_1(x) - S_2(y))] d\eta(x) d\eta(y) \\ + \mathcal{A}(m_\lambda, n_\lambda),\end{aligned}$$

with $\mathcal{A}$ affine. The function W is convex on all $\mathbb{R}$, thus the interaction part of the $\mathcal{G}$ is strictly displacement convex. The remaining term is simply a linear combination of functions of m and n to which we can apply Theorem 2.1. The strict convexity up to translation follows as in the proof of Theorem 3.2. □

Remark. In the two component case we need to use two monotone maps instead of one as in Theorem 3.2. Therefore it is crucial that W is convex on all of $\mathbb{R}$ and not just on $(0, +\infty)$ and $(-\infty, 0)$ as in the one component case.

We close this section with a corollary showing that one could also use Lemma 5.1 to prove displacement convexity of the interaction energy in the one component model. In fact, in this application, the first moment condition drops out.

Corollary 5.3. *Let $\bar{J}$ satisfy the conditions below (5.4), and W defined as in the (5.6). Let m be a function that increases monotonically from $-m_\beta$ to m_β. Let ρ denote m', the derivative of m. Consider the functional $\Phi(m)$ given by $\Phi(m) = \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{J}(x - y) \left[m(x)m(y) - m_\beta^2 \right] \mathrm{d}x\, \mathrm{d}y$. Then*

$$\Phi(m) = -4m_\beta^2 \int_{\mathbb{R}} \int_{\mathbb{R}} W(x - y)\rho(x)\rho(y)\, \mathrm{d}x\, \mathrm{d}y - 6\alpha m_\beta^2.$$

Proof. The functional $\Phi(m)$ is equal to $-\mathcal{G}(m_1, m_2)$ by putting $m_1(x) = m(x)$ and $m_2(x) = -m(x)$. This shows that $-\Phi$ is strictly displacement convex, up to translation. □

6. Properties of the minimizers of $\mathcal{G}$

We restrict our attention to the case $a = d$, $b = c$. We need two results on the minimizers for $\mathcal{G}$, the first of which allows us to restrict our attention to monotone profiles when seeking to minimize $\mathcal{G}$. The second guarantees the existence of moments for the two densities corresponding to any minimizing pair (m, n). These theorems are:

Theorem 6.1. *Suppose that $J(x)$ is even, non-negative and decreasing. Then any minimizer (m_1, m_2) of $\mathcal{G}(m_1, m_2)$ in $\mathcal{C}_{\rho^-,\rho^+} \times \mathcal{C}_{\rho^+,\rho^-}$ is monotone in the sense that m_1 is increasing and m_2 is decreasing.*

This theorem makes it easy to establish the existence of minimizers for $\mathcal{G}$. The minimizers satisfy an Euler–Lagrange equation from which we can deduce a priori estimates needed to apply Lemma 5.1.

Theorem 6.2. *Suppose that $J(x)$ is even non-negative and decreasing on $\mathbb{R}_+$. Any minimizer $w = (w_1, w_2)$ of $\mathcal{G}$ in $\mathcal{C}_{\rho^-,\rho^+} \times \mathcal{C}_{\rho^+,\rho^-}$ satisfies $\rho^- < w_i(x) < \rho^+$ for any $x \in \mathbb{R}$. It has a derivative almost everywhere which is strictly positive and with $\|w_i'\|_{L^1(\mathbb{R})}$ bounded. Furthermore, it satisfies the Euler–Lagrange equations*

$$\ln m(x) + \beta(J * n)(x) = \mu, \quad \ln n(x) + \beta(J * m)(x) = \mu, \tag{6.1}$$

where $\mu = \mu_1 - 1$ and $*$ denotes convolution. Its derivative w satisfies almost everywhere the equations

$$\frac{w_1'(x)}{w_1(x)} + \beta(J * w_2')(x) = 0, \quad \frac{w_2'(x)}{w_2(x)} + \beta(J * w_1')(x) = 0. \tag{6.2}$$

Finally, it converges to its asymptotic values exponentially fast, in the sense that there is $\alpha > 0$ such that $(w_1(x) - \rho_{\mp})e^{\alpha|x|} \to 0$ as $x \to \mp\infty$ and $(w_2(x) - \rho_{\pm})$ $e^{\alpha|x|} \to 0$ as $x \to \mp\infty$.

The proof of Theorem 6.1 is adapted from a related result in [7] for functions on the d dimensional torus. One could instead adapt the proof of Alberti's rearrangement inequality in [1] and remove the requirement that J be decreasing. But the present approach has the advantage of also working on the torus, and not only the line. The proof of the final part of Theorem 6.2, which is important for our application here since it provides the existence of moments, is adapted from the proof of a similar result for the one component system in [11]. In the rest of this section, we present these proofs.

Proof of Theorem 6.1. To show this, we use a rearrangement inequality similar to those introduced in [7] for the analogous problem in the d-dimensional torus. For any $x_0 \in \mathbb{R}$, let T_{x_0} denote the reflection about x_0:

$$T_{x_0}(x) = 2x_0 - x.$$

Then define $\mathcal{D}$, as the set of functions on $\mathbb{R}$ having finite limits at $\pm\infty$ and the operators $R^{\pm}_{x_0}$ on $\mathcal{D}$ by

$$R^{+}_{x_0} g(x) = \begin{cases} \max\{g(x), g(T_{x_0})\} & \text{if } x \geqq x_0, \\ \min\{g(x), g(T_{x_0})\} & \text{if } x \leqq x_0. \end{cases} \tag{6.3}$$

$$R^{-}_{x_0} h(x) = \begin{cases} \max\{h(x), h(T_{x_0})\} & \text{if } x \leqq x_0, \\ \min\{h(x), h(T_{x_0})\} & \text{if } x \geqq x_0. \end{cases} \tag{6.4}$$

Let us also define $\hat{g}(x) = \begin{cases} \lim_{x\to-\infty} g(x) & \text{if } x < 0 \\ \lim_{x\to+\infty} g(x) & \text{if } x \geqq 0 \end{cases}$ and $\hat{h}$ similarly.

For any fixed x_0 and $g, h \in \mathcal{D}$, let $g^\star$ denote $R^{+}_{x_0} g$ and $h_\star = R^{-}_{x_0} h$. We now wish to show that

$$\begin{aligned} &\int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(x) J(x-y) h(y)\, \mathrm{d}y - \hat{J}\hat{g}(x)\hat{h}(x) \right] \mathrm{d}x \\ &\quad \geqq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g^\star(x) J(x-y) h_\star(y)\, \mathrm{d}y - \hat{J}\hat{g}(x)\hat{h}(x) \right] \mathrm{d}x \end{aligned}$$

with equality if and only if $g = g^\star$ and $h = h^\star$.

To do this, let $\mathbb{H}_+$ denote the half line $\{x \mid x > x_0\}$, and $\mathbb{H}_-$ denote the half line $\{x \mid x < x_0\}$ and observe that

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\mathbb{H}_+}\int_{\mathbb{H}_+} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{H}_-}\int_{\mathbb{H}_-} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \int_{\mathbb{H}_-}\int_{\mathbb{H}_+} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{H}_+}\int_{\mathbb{H}_-} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\mathbb{H}_+}\int_{\mathbb{H}_+} g(x)J(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{H}_+}\int_{\mathbb{H}_+} g(T_{x_0}x)J(x-y)h(T_{x_0}y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \int_{\mathbb{H}_+}\int_{\mathbb{H}_+} g(T_{x_0}x)J(T_{x_0}x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \int_{\mathbb{H}_+}\int_{\mathbb{H}_+} g(x)J(x-T_{x_0}y)h(T_{x_0}y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{6.5}$$

The desired inequality is then a consequence of the following inequality for pairs of real numbers: Let a_1 and a_2 and b_1 and b_2 be any four positive real numbers. Rearrange a_1 and a_2 to decrease, and b_1 and b_2 to increase; that is, let $a_1^\star = \max\{a_1, a_2\}$, $a_2^\star = \min\{a_1, a_2\}$, $b_1^\star = \min\{b_1, b_2\}$ and $b_2^\star = \max\{b_1, b_2\}$. Then

$$a_1^\star b_1^\star + a_2^\star b_2^\star - a_1 b_1 - a_2 b_2 = \Delta \leqq 0, \tag{6.6}$$

$$a_1^\star b_2^\star + a_2^\star b_1^\star - a_1 b_2 - a_2 b_1 = -\Delta \geqq 0, \tag{6.7}$$

and there is equality if and only if $a_1 = a_1^\star$ and $b_1 = b_1^\star$ or $a_1^\star = a_2$ and $b_1^\star = b_2$.

We now apply the above inequalities with

$$a_1 = g(x), \quad a_2 = g(T_{x_0}x), \quad b_1 = h(y) \quad \text{and} \quad b_2 = h(T_{x_0}y). \tag{6.8}$$

Then

$$a_1^\star = R_{x_0}^+ g(x), \quad a_2^\star = R_{x_0}^+ g(T_{x_0}x), \quad b_1^\star = R_{x_0}^- h(y) \quad \text{and} \quad b_2^\star = R_{x_0}^- h(T_{x_0}y). \tag{6.9}$$

Since $J(T_{x_0}x - y) = J(x - T_{x_0}y) < J(x - y)$, we get

$$\begin{aligned}
&g(x)J(x-y)h(y) + g(T_{x_0}x)J(x-y)h(T_{x_0}y)\\
&\quad + g(T_{x_0}x)J(T_{x_0}x - y)h(y) + g(x)J(T_{x_0}x - y)h(T_{x_0}y)\\
&\quad - R_{x_0}^+ g(x)J(x-y)R_{x_0}^- h(T_{x_0}y) + R_{x_0}^+ g(T_{x_0}x)J(x-y)R_{x_0}^- h(T_{x_0}y)\\
&\quad - R_{x_0}^+ g(T_{x_0}x)J(T_{x_0}x - y)R_{x_0}^- h(y) + R_{x_0}^+ g(x)J(T_{x_0}x - y)R_{x_0}^- h(T_{x_0}y)\\
&= -\Delta\left[J(x-y) - J(x - T_{x_0}y)\right] \geqq 0
\end{aligned} \tag{6.10}$$

for almost every x and y in $\mathbb{H}_+$, with equality if and only if

$$g(T_{x_0}x) \leqq g(x) \quad \text{and} \quad h(T_{x_0}y) \geqq h(y) \tag{6.11}$$

or

$$g(T_{x_0}x) \geqq g(x) \quad \text{and} \quad h(T_{x_0}y) \leqq h(y) \tag{6.12}$$

for almost every x and y in $\mathbb{H}_+$. Now unless g is constant, we can find x and x_0 so that either $g(T_{x_0}x) < g(x)$ or $g(T_{x_0}x) > g(x)$. Suppose it is the first case. Then (6.11) holds, and for almost every y, we must have $h(T_{x_0}y) \geqq g(y)$. Making a similar argument for h, we see that one of (6.11) or (6.12) must hold for almost every x and y. The only way that this can happen is if g and h are monotone. Now, by integrating (6.10) on $\mathbb{H}_+$ we conclude the proof. □

Proof of Theorem 6.2. Everything but the exponential decay is standard, and details of the proofs of similar results can be found in [7]. To prove the exponential decay, we once again take advantage of the finite range R of J.

Define a transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(m, n) = (e^{\mu - \beta\widehat{J}n}, e^{\mu-\beta\widehat{J}m})$. Then (ρ^+, ρ^-) and (ρ^-, ρ^+) are two stable fixed points of Φ; the Jacobian of Φ, $D\Phi$, is a strict contraction at either of them. Thus, there is a $\delta > 0$ and an $\varepsilon > 0$ so that if

$$|m - \rho^+| + |n - \rho^-| < \delta \quad \Rightarrow \quad \|D\phi(m, n)\| < 1 - \varepsilon.$$

Now, consider any minimizer $w = (w_1, w_2)$ with $\lim_{x\to\infty} w_1(x) = \rho^+$ and $\lim_{x\to\infty} w_2(-x) = \rho^-$. Then there is an $L < \infty$ so that $x \geqq L \quad \Rightarrow \quad |w_1(x) - \rho^+| + |w_2(-x) - \rho^-| < \delta$. Now for $x > L + R$,

$$\frac{J}{\widehat{J}} * w_1(x) \geqq \rho^+ - \delta \quad \text{and} \quad \frac{J}{\widehat{J}} * w_2(x) \leqq \rho^- + \delta.$$

Since $(w_1(x), w_2(x)) = \Phi\left(\frac{J}{\widehat{J}} * w_1(x), \frac{J}{\widehat{J}} * w_2(x)\right)$, it follows that for $x > L+R$, $|w_1(x) - \rho^+| + |w_2(-x) - \rho^-| < (1-\varepsilon)\delta$. Iterating this argument leads to the conclusion that for $x > L + kR$, $|w_1(x) - \rho^+| + |w_2(-x) - \rho^-| < (1-\varepsilon)^k\delta$. A similar argument applies as x tends to $-\infty$. □

7. For the functional $\mathcal{G}$, critical points are minimizers

We are now ready to prove the main theorem for $\mathcal{G}$:

Theorem 7.1. *If (w_1, w_2) and (v_1, v_2) are any two critical points of $\mathcal{G}$ in $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, then there is an $a \in R$ so that*

$$(v_1(x), v_2(x)) = (w_1(x-a), w_2(x-a)). \tag{7.1}$$

Thus, there is exactly one critical point (w_1, w_2) such that $w_1(0) = w_2(0)$. It is symmetric in the sense that $w_1(x) = w_2(-x)$ for all x.

Proof. We keep the notation of Section 5. Theorem 5.2 is applicable since by Theorem 6.2, the probability densities η_i and ζ_i, $i = 1, m$, have finite moments of every order. Now, if (m_λ, n_λ) is the displacement convex interpolation between (w_1, w_2) and (v_1, v_2), $\mathcal{G}(m_\lambda, n_\lambda)$ is constant since both endpoints are critical points. By the strict convexity up to translation, we see that (7.1) is true.

Since $(w_1, w_2) \in \mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, and both functions are strictly monotonic, there is some b such that $w_1(b) = w_2(b)$. Because of the strict monotonicity of w_1, that is, the strict positivity of its derivative which was proved in Theorem 6.2, this value of b is unique.

Next, by the symmetries of the functional, since $(w_1(x), w_2(x))$ is any minimizer of $\mathcal{G}$ in $\mathcal{M}_{\rho^-,\rho^+} \times \mathcal{M}_{\rho^+,\rho^-}$, then so is $(w_2(-x), w_1(-x))$. Hence, by the first part of the Theorem, there is an $a \in R$ so that

$$(w_2(-x), w_1(-x)) = (w_1(x-a), w_2(x-a)). \tag{7.2}$$

Evaluating both sides at $x = 0$, we see that since $w_1(0) = w_2(0)$, $w_1(-a) = w_2(-a)$. By the uniqueness of the crossing point established above, $a = 0$, so that $(w_2(-x), w_1(-x)) = (w_1(x), w_2(x))$ for all x. $\square$

8. Stationary monotone profiles in several dimensions

We close the paper by pointing out that our analysis of the two component case can be adapted to yield a uniqueness theorem for the one component case in higher dimensions.

Let Ω be a $(d-1)$-dimensional cube of size L spanned by the orthogonal vectors $e_1, \dots, e_{d-1}$ and $\mathcal{C}_{a,b,\Omega}$ be the set of continuous functions $m(x, y)$ from $\mathbb{R} \times \mathbb{R}^{d-1}$ to $\mathbb{R}$ such that for all $y \in \mathbb{R}^{d-1}$

$$\lim_{x\to-\infty} m(x, y) = a \quad \text{and} \quad \lim_{x\to+\infty} m(x, y) = b,$$

and such that m is L-periodic on $\mathbb{R}^{d-1}$ in the sense that $m(x, y + Le_k) = m(x, y)$ for each $k = 1, \dots, d-1$ and for each $y \in \mathbb{R}^{d-1}$.

Consider the following d-dimensional free energy on $\mathcal{C}_{-a,a,\Omega}$

$$\begin{aligned}\mathcal{F}(m) = \int_{\mathbb{R}\times\Omega} F(m(x, y))\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}\times\Omega}\int_{\mathbb{R}\times\Omega} &(m(x_1, y_1) - m(x_2, y_2))^2 \\ &\times J(x_1 - x_2, y_1 - y_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2,\end{aligned} \tag{8.1}$$

$J(x, y) = U(\sqrt{x^2 + |y|^2})$, with U monotone decreasing, finite range smooth function on $[0, +\infty)$ and F an even double well potential with minima in $-a$ and a and $F(\pm a) = 0$. (These specific conditions on F enable us to be brief, and can easily be relaxed.)

Obviously, if $\bar{m}(x)$ is a minimizer for the corresponding one dimensional problem, then

$$\bar{m}(x, y) := \bar{m}(x)$$

is a critical point of $\mathcal{F}$ on $\mathcal{C}_{-a,a,\Omega}$, and is an obvious candidate to be the unique minimizer. We shall show here that not only is it the minimizer—this fact has been proved by Alberti [1]—but that, up to translation in x, $\bar{m}(x, y)$ is the *only* solution of the Euler–Lagrange equation for minimization of $\mathcal{F}$ that is monotone in x for all y. A related question as to whether all monotone solutions of the Euler–Lagrange equation have this special form has been extensively investigated for the local variant of the free energy (Allen-Cahn or van der Waals) with $\int |\nabla m(x, y)|^2$ in place of the non-local interaction integral above. It turns out that the non-local case may be easily treated by regarding the one dimensional profiles $x \mapsto m(x, y)$ for different y as profiles for different components, and applying our previous results.

Define $\mathcal{M}_{a,b,\Omega}$ to be the subset of $\mathcal{C}_{a,b,\Omega}$ for which $m(x, y)$ is monotone in x for each $y \in \mathbb{R}^{d-1}$. As before, there is a rearrangement inequality that allows one to reduce the minimization problem over $\mathcal{C}_{-a,a,\Omega}$ to minimization over $\mathcal{M}_{-a,a,\Omega}$: Given $m \in \mathcal{C}_{-a,a,\Omega}$ we define $m^* \in \mathcal{M}_{-a,a,\Omega}$ as follows, by separately rearranging $m(\,\cdot\,, y)$ for each $y \in \mathbb{R}^{d-1}$, using the one dimensional rearrangement procedure. By the rearrangement results cited above, $\mathcal{F}(m^*) \leqq \mathcal{F}(m)$. In [1], Alberti proceeds with a careful study of the cases of equality here. Instead, we henceforth restrict our attention to $m \in \mathcal{M}_{-a,a,\Omega}$, and shall show that up to translation in x, there is just one solution of the Euler–Lagrange equation in this set.

First, rewrite the functional in (8.1) as

$$\begin{aligned}\mathcal{F}(m) = &\int_{\mathbb{R}\times\Omega} [F(m(x, y)) - \hat{J}(m^2(x, y) - a^2)]\,\mathrm{d}x\,\mathrm{d}y\\ &+ \int_{\mathbb{R}\times\Omega}\left[\int_{\mathbb{R}\times\Omega} m(x_1, y_1)m(x_2, y_2)J(x_1-x_2, y_1-y_2)\,\mathrm{d}x_1\,\mathrm{d}y_1 - \widehat{J}a^2\right]\mathrm{d}x_2\,\mathrm{d}y_2,\end{aligned} \tag{8.2}$$

where $\widehat{J} = \int_{\Omega\times\mathbb{R}} J(x, y)\,\mathrm{d}x\,\mathrm{d}y$. We now observe that the second term on the right can be written in terms of the $\mathcal{I}$ functional that has been studied in Section 5. Indeed, this term can be written as

$$-\int_{\Omega\times\Omega} \mathcal{I}(m(\,\cdot, y_1), -m(\,\cdot, y_2))\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

This identity relates the multidimensional problem to the two species problem: here $m(\,\cdot, y_1)$ plays the role of the profile for one species, and $-m(\,\cdot, y_2)$ plays the role of the profile for the other species.

Now notice that for $a + b = 0$ (or $c + d = 0$), the statement of Lemma 5.1 simplifies in a significant way: The first moments drop out as in Corollary 5.3, and we have (using the notation from the lemma)

$$-\mathcal{I}(m_1, -m_2) = 4a^2 \int_{\mathbb{R}} \int_{\mathbb{R}} W(x_1 - x_2)\rho_1(x_1, y_1)\rho_2(x_2, y_2))\, dx_1\, dx_2 + 6a^2\alpha.$$

(First moments could be dealt with as before, but we avoid doing so in order to focus on how one may regard the multidimensional problem as a multi-component problem, which is the main point of this section.)

Given two profiles m_0 and m_1 in $\mathcal{M}_{-a,a,\Omega}$, let m_λ be the interpolation defined by interpolating between $m_0(\cdot, y)$ and $m_1(\cdot, y)$ separately in each y. Let $x \mapsto T(x, y)$ be the corresponding optimal transportation plan, and let $S(x, y) = T(x, y) - x$. Let $m_\lambda(x, y)$ be the induced interpolation between $m_0(x, y)$ and $m_1(x, y)$. Then

$$\begin{aligned} -\mathcal{I}(m_\lambda(\cdot, y_1), -m_\lambda(\cdot, y_2)) = 4a^2 \int_{\mathbb{R}} \int_{\mathbb{R}} & W[x_1 - x_2 + \lambda(S(x_1, y_1) - S(x_2, y_2)] \\ & \times \rho_1(x_1, y_1)\rho_2(x_2, y_2)\, dx_1\, dx_2 + 6a^2\alpha. \end{aligned}$$

Since W is strictly convex near the origin if J is strictly positive near the origin, it follows that if y_2 and y_1 are sufficiently close to one another, $\rho_1(x_1, y_1)\, dx_1$ and $\rho_2(x_2, y_2)\, dx_2$ both assign positive mass to some small interval around some x_0. Therefore, for such y_1 and y_2, we see that $\lambda \mapsto -\mathcal{I}(m_\lambda(\cdot, y_1), -m_\lambda(\cdot, y_2))$ is strictly convex, and for *any* y_1 and y_2 it is convex. Clearly, the set of points (y_1, y_2) for which we have strict convexity is a set of positive measure (containing the diagonal) with respect to $dy_1\, dy_2$, and so

$$\lambda \mapsto -\int_{\Omega\times\Omega} \mathcal{I}(m_\lambda(\cdot, y_1), -m_\lambda(\cdot, y_2))\, dy_1\, dy_2$$

is strictly convex, apart from translation in x. This strict convexity proves that, up to translation in x, there is just one critical point of $\mathcal{F}$ in $\mathcal{M}_{-a,a,\Omega}$. Since clearly $\bar{m}(x, y)$ is a critical point, we have the following:

Theorem 8.1. *Assume that J is bounded below by a strictly positive number on some neighborhood of the origin. Let $m(x, y)$ be any solution of the Euler–Lagrange equation for the minimization of $\mathcal{F}$ that belongs to $\mathcal{M}_{-a,a,\Omega}$. Then for some $x_0 \in \mathbb{R}$, $m(x, y) = \bar{m}(x - x_0)$ for all x and y, where $\bar{m}$ the antisymmetric minimizer for one dimension.*

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References

1. Alberti, G.: Some remarks about a notion of rearrangement. *Ann. Scuola Norm. Sup. Pisa Cl. Sci* **XXIX**, 457–472 (2000)
2. Alberti, G., Bellettini, G.: A nonlocal anisotropic model for phase transitions Part i: the optimal profile problem. *Math. Annalen.* **310**, 527–560 (1998)

3. Alberti, G., Bellettini, G., Cassandro, M., Presutti, E.: Surface Tension in Ising Systems with Kac Potentials. *J. Stat. Phys.* **82**, 743–796 (1996)
4. Bastea, S., Esposito, R., Lebowitz, J.L., Marra, R.: Sharp interface motion of a binary fluid mixture. *J. Stat. Phys.* **124**, 445–484 (2006)
5. Blower, G.: Displacement convexity for the generalized orthogonal ensemble. *J. Stat. Phys.* **116**, 1359–1387 (2004)
6. Brenier, Y.: Polar factorization and monotone rearrangement of vector valued functions. *Comm. Pure Appl. Math.* **64**, 375–417 (1991)
7. Carlen, E.A., Carvalho, M., Esposito, R., Lebowitz, J.L., Marra, R.: Free energy minimizers for a two-species model with segregation and liquid-vapor transition. *Nonlinearity* **16**, 1075–1105 (2003)
8. Carlen, E.A., Carvalho, M., Esposito, R., Lebowitz, J.L., Marra, R.: Phase transitions in equilibrium systems: microscopic models and mesoscopic free energies *J. Molecular Phys.* **103**, 3141–3151 (2005)
9. Dal Passo, R., De Mottoni, P.: *The heat equation with a non local density dependent advection term* (preprint) (1991)
10. De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Uniqueness and global stability of the instanton in non local evolution equations. *Rendiconti di Matematica* **14**, 693–723 (1994)
11. De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Stability of the interface in a model of phase separation. *Proc. R. Soc. Edinb.* **124A**, 1013–1022 (1994)
12. McCann, R.: A convexity principle for interacting gases. *Adv. Math.* **128**(1), 153–179 (1997)
13. van der Waals, J.D.: Théorie moléculaire d'une substance composée de deux matières différentes, I. *Verhandelingen, Kon. Akad. Wet. Amsterdam* **20** (1880) and Théorie moléculaire d'une substance composée de deux matières différentes, II. *Arch. Néerl.* **24**(1) (1891)
14. Villani, C.: *Topics in Optimal Transportation, Graduate Studies in Mathematics*, vol. **58**. American Mathematical Socity, Providence, 2003

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Displacement Convexity and Minimal Fronts

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