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The Sherrington–Kirkpatrick model with short range ferromagnetic interactions

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Abstract. We study a model of Ising spins with short range ferromagnetic and long range SK interactions. We generalize the results obtained for the standard SK model, computing in particular the high temperature pressure. © Académie des Sciences/Elsevier, Paris

Modèle Ising + SK

Résumé. *Nous étudions un modèle de spins binaires qui est soumis à la fois à des interactions de type Ising et de type Sherrington–Kirkpatrick. Nous généralisons des résultats obtenus dans le cas du modèle SK seul, en calculant en particulier la pression à haute température.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Dans cette Note, nous considérons un modèle d'Ising ferromagnétique perturbé par une interaction de verre de spins de champ moyen sur le réseau carré de dimension *d*. L'hamiltonien du système est la somme d'une composante déterministe à courte portée

$$H_N^I(\sigma) = -\frac{1}{2} \sum_{i,j \in \Lambda_N} K(i-j)\sigma_i\sigma_j, \tag{1}$$

et d'une composante désordonnée à longue portée

$$H_N^{\mathrm{SK}}(\sigma) = \frac{1}{\sqrt{2|\Lambda_N|}} \sum_{i,j \in \Lambda_N} J_{i,j}\sigma_i\sigma_j, \tag{2}$$

Note présentée par Michel TALAGRAND.

où $\Lambda_N = \{-N, \dots, N\}^d$, $\sigma \in \{-1,+1\}^{\Lambda_N}$, $K(\cdot) \geq 0$, $\sum_i K(i) < \infty$ et $\{J_{i,j}\}_{i,j\in\mathbb{Z}^d}$ est une famille de variables aléatoires indépendantes de loi gaussienne $\mathcal{N}(0,1)$. La fonction de partition du système est donnée par

$$\widetilde{Z}_N(\kappa,\beta) = \sum_{\sigma\in\{-1,+1\}^{\Lambda_N}} \exp\left\{-\kappa H_N^I(\sigma) - \beta H_N^{\mathrm{SK}}(\sigma)\right\}, \tag{3}$$

avec $\beta > 0$ et $\kappa > 0$. Le cas $\beta = 0$ correspond au modèle d'Ising habituel à température $1/\kappa$, tandis que pour $\kappa = 0$ on retrouve le modèle SK introduit par Kirkpatrick et Sherrington [6]. Le modèle SK a été étudié rigoureusement lorsque $\beta < 1$, d'abord par des techniques de moments et d'expansions [1], puis par calcul stochastique [2] et par des méthodes de concentration [11]. Cette dernière référence traite également le cas bien plus complexe avec champ extérieur. Notre modèle est motivé par les systèmes de verres de spins sujets à la fois à des interactions ferromagnétiques à courte portée homogènes en espace, et des interactions à longue portée fortement oscillantes représentées ici par les couplages aléatoires $J_{i,j}$. Quand la portée de ces dernières tend vers l'infini (*asymptotique de Kac)*, le système est alors décrit par le modèle (3), comme cela a été remarqué dans [4], Theorem B en l'absence d'interaction courte. Notons que les deux composantes de l'hamiltonien sont concurrents : pour des réalisations typiques des $J_{i,j}$ les configurations de minimum d'énergie seront des mélanges de spins + et −. Indépendamment des conditions frontières la pression du système est donnée par la limite quand $N \to \infty$ de

$$\widetilde{p}_N(\kappa,\beta) = \frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa,\beta). \tag{4}$$

dont l'existence, non évidente en présence de l'interaction SK, est partie de nos résultats. Le modèle d'Ising sous-jacent joue un rôle important dans la suite; dans la région d'unicité nous noterons $\mathbf{P} = \mathbf{P}_\kappa$ (resp. $\mathbf{E} = \mathbf{E}_\kappa$) la mesure de Gibbs [5] sur $\{-1,+1\}^{\mathbb{Z}^d}$ avec interaction de paires K et température inverse κ [resp., l'intégrale correspondante], en réservant la notation $\mathbb{P}$ (resp. $\mathbb{E}$) pour les variables J. Définissons $\theta_\kappa > 0$ la quantité

$$\theta_\kappa^2 = \sum_{i\in\mathbb{Z}^d} [\mathbf{E}_\kappa(\sigma_0\sigma_i)]^2, \tag{5}$$

connue dans la littérature sous le nom de *bubble diagram* [3], ainsi que

$$c_\kappa(i) = c(i) = \mathbf{E}_\kappa(\sigma_0\sigma_i)\ , \qquad \chi_\kappa = \sum_{i\in\mathbb{Z}^d} c(i), \tag{6}$$

et κ_c la plus petite valeur de κ telle que $\chi_\kappa = \infty$; il est bien connu que $c(i) \geq 0$ est une fonction croissante de κ. Soit D le domaine de l'espace des paramètres

$$D = \left\{(\kappa,\beta)\,;\ \kappa < \kappa_c, \exists\varepsilon > 0\ :\ \sup_N \mathbf{E}_\kappa^{\otimes 2} \exp \frac{\beta^2+\varepsilon}{2} X_N(\sigma,\sigma')^2 < \infty\right\},$$

$$X_N(\sigma,\sigma') = \frac{1}{\sqrt{|\Lambda_N|}} \sum_{i\in\Lambda_N} \sigma_i\sigma_i' \tag{7}$$

Sur D la variable $\widetilde{Z}_N/\mathbb{E}\widetilde{Z}_N$ est bornée dans L^2 uniformément en N. Résumons nos résultats principaux, qui sont énoncés dans le théorème 1 ci-dessous.

i) Le domaine D contient $\{\kappa < \kappa_c, \beta^2\chi_\kappa < 1\}$.

ii) Sur le domaine D, $\mathbf{E}_\kappa[\exp(-\beta H_N^{SK})]/\mathbb{E}\mathbf{E}_\kappa[\exp(-\beta H_N^{\mathrm{SK}})]$ converge vers une loi log-normale.

iii) Si $(\kappa,\beta)\in\overline{D}$, la limite $\widetilde{p}(\kappa,\beta)$ quand $N\to\infty$ de $\widetilde{p}_N(\kappa,\beta)$ existe $\mathbb{P}$-p.s., et elle est donnée par $\widetilde{p}(\kappa,\beta) = p^I(\kappa) + \frac{\beta^2}{4}$, où $p^I(\kappa) = \lim_N(|\Lambda_N|)^{-1}\log \widetilde{Z}_N(\kappa,0)$ désigne la pression du modèle d'Ising.

Notre preuve suit l'approche par le calcul calcul stochastique introduite dans [2]. Nous utilisons aussi de façon cruciale les propriétés du modèle d'Ising (avec l'interaction ferromagnétique seule) ; le champ aléatoire $(\sigma_i)_i$ est positivement associé (inégalité FKG), et ses moments sont majorés par ceux du champ gaussien possédant les mêmes covariances, d'après un résultat de C. Newman [7].

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1. Introduction

We consider the Ising ferromagnetic d-dimensional model, perturbed by a mean field interaction. A spin configuration is an element σ of $\{-1,+1\}^{\Lambda_N}$, with $\Lambda_N = \{-N,\dots,N\}^d$, $N \in \mathbb{Z}^+$, and the Hamiltonian of the system is given by $H_N^I(\sigma) + H_N^{\mathrm{SK}}(\sigma)$ (*cf.* (1) and (2)). The partition function is given by (3), in which there are two positive parameters β and κ, which play the role of two *independent* inverse temperatures. If $\beta = 0$ the model reduces to the Ising model with temperature $1/\kappa$, and if $\kappa = 0$ we are back to the standard SK model [6]. The SK model has been rigorously studied for $\beta < 1$, first with expansion techniques [1], and then via a stochastic calculus technique [2] and finally by using concentration inequalities [11]. In [11] the (much more complex) model with an external magnetic field is treated. Our model is motivated by spin glass systems in which there are two different types of interactions: one of short range, ferromagnetic and translation invariant, and the other one of long range type with strongly oscillating (random) couplings $J_{i,j}$. When the range of the J-interaction tends to infinity (Kac asymptotics), the system is described by a model with partition function (3), as remarked in [4], Theorem B for the case *without* short range interactions. Observe that the two terms in the Hamiltonian are competing: for typical realizations of $J_{i,j}$, the minimal energy configurations are mixtures of ± 1 spins. Independently of the boundary conditions, the pressure of the system is given by the limit $N \to \infty$ of (4), the existence of which is part of our result.

The underlying Ising model plays a crucial role: in the uniqueness region we denote by $\mathbf{P} = \mathbf{P}_\kappa$ (resp. $\mathbf{E} = \mathbf{E}_\kappa$) the Gibbs measure [5] on $\{-1,+1\}^{\mathbb{Z}^d}$ with 2-body interaction K and inverse temperature κ (resp. the corresponding expectation), and we will use the notation $\mathbb{P}$ (resp. $\mathbb{E}$) for the random interactions J. An important role will be played by the so called *bubble diagram* (*see* e.g. [3]), i.e. the positive quantity $\theta_\kappa > 0$ defined by (5), as well as the quantities c_κ (the correlations), χ_κ (the susceptibility), both defined in (6), and κ_c, which is the smallest value of κ such that $\chi_\kappa = \infty$. It is well known that $c_\kappa(i)$ is non-negative and increasing in κ. If $(\kappa,\beta) \in D$ (*cf.* (7)) the random variable $\widetilde{Z}_N/\mathbb{E}\widetilde{Z}_N$ is uniformly bounded in L^2 with respect to N. We have the following:

THEOREM 1.1. – i) *The domain D contains the parameter values $\{\kappa < \kappa_c, \beta^2\chi_\kappa < 1\}$.*

ii) *In the domain D,*

$$\lim_{N\to\infty} \frac{\mathbf{E}_\kappa\left[\exp\left(-\beta H_N^{SK}\right)\right]}{\mathbb{E}\mathbf{E}_\kappa\left[\exp\left(-\beta H_N^{SK}\right)\right]} = \exp\left(Y - \frac{1}{2}\phi(\beta^2\theta_\kappa^2)\right) \tag{1.1}$$

in distribution, with $Y \overset{\mathcal{D}}{=} \mathcal{N}(0, \phi(\beta^2\theta_\kappa^2))$ and $\phi(t) = (1/2)\log(1/1-t)$.

iii) *If $(\kappa,\beta) \in \overline{D}$, the limit $\widetilde{p}(\kappa,\beta)$ of $\widetilde{p}_N(\kappa,\beta)$ as $N \to \infty$ exists $\mathbb{P}$-a.s., and it is given by*

$$\widetilde{p}(\kappa,\beta) = p^I(\kappa) + \frac{\beta^2}{4}, \tag{1.2}$$

where $p^I(\kappa) = \lim_{N\to\infty}(|\Lambda_N|)^{-1}\log\widetilde{Z}_N(\kappa,0)$ is the pressure of the Ising model.

Our proof follows the stochastic calculus method introduced in [2]. A crucial element in our proofs are the properties of the Ising model with short range ferromagnetic interactions: besides the FKG inequality, we will use the fact that the moments of such an Ising field are bounded by those of a Gaussian field with the same covariances [7].

We observe also that the concentration method [11] gives the statement iii), but it does not identify the limit fluctuation behavior in ii). On the other hand the method in [1], succesfully applied to the case $\kappa = 0$ where the underlying measure is of the product form, seems to be hard to adapt to our case. The computations in Section 4 and Section 5 of [2] still hold and, for example, we have convergence in law of the average energy

$$\mathcal{D}\text{-}\lim_{N\to\infty}\left[\frac{\mathbf{E}_\kappa(-H_N^{SK}\exp(-\beta H_N^{SK}))}{\mathbf{E}_\kappa(\exp(-\beta H_N^{SK}))} - \frac{\beta}{2}|\Lambda_N|\right] = \mathcal{N}(\beta v_\kappa, v_\kappa)$$

under the conditions stated above,with $v_\kappa = \theta_\kappa^2/2(1-\beta^2\theta_\kappa^2)^2$. We have also a similar result on the entropy of the Gibbs measure with random interactions with respect to $\mathbf{P}_\kappa$, namely the entropy can be written as $|\Lambda_N|\beta^2/4 + U_N$ and U_N converges in law to a (non-centered) Gaussian variable.

2. Preliminary results

We discuss in this Section some results about products of independent realizations of Ising spins with $\kappa < \kappa_c$, which are needed to study the random model later on. We fix such a κ from now on, so we will forget the subscripts κ in our notations.

Since $\chi = \sum_i c(i) < \infty$ in D it follows from Newman's Central Limit Theorem for stationary fields of positively associated variables (*see* Proposition 4 together with Theorem 3 in [8]) that:

Lemma 2.1. – *If η, η', η'' are independent Gaussian variables with common variance θ, we have for finite χ*

$$\mathcal{D}\text{-}\lim_{N\to\infty}(X_N(\sigma,\sigma'), X_N(\sigma,\sigma''), X_N(\sigma',\sigma'')) = (\eta,\eta',\eta''). \tag{2.1}$$

Recall now the Gaussian inequality for the Ising model with ferromagnetic two-body interaction, proven by C. Newman [7], Th. 3:

Proposition 2.2. – *Let $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ be a collection of centered random variables which are jointly Gaussian under the measure $\mathcal{P}$ ($\mathcal{E}$ denotes the expectation), with covariance $\mathcal{E}(\xi_i\xi_j) = c(i-j)$ for all $i,j \in \mathbb{Z}^d$. Then, for any integer $k > 0$ and any $\{i_q\}_{q=1,\dots,2k}$, $i_q \in \mathbb{Z}^d$, we have that*

$$\mathbf{E}\{\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2k}}\} \le \mathcal{E}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2k}}\}. \tag{2.2}$$

Proof of Theorem 1.1. – i): We will prove that $\mathbf{E}^{\otimes 2}\exp(tX_N^2/2)) \le \exp\phi(t\chi)$ for $t\chi < 1$. For any positive integer k we have

$$\mathbf{E}^{\otimes 2}\{X_N^{2k}\} = \frac{1}{|\Lambda_N|^k}\sum_{i_1\in\Lambda_N}\cdots\sum_{i_{2k}\in\Lambda_N}\mathbf{E}\{\sigma_{i_1}\cdots\sigma_{i_{2k}}\}\mathbf{E}\{\sigma'_{i_1}\cdots\sigma'_{i_{2k}}\} \le \mathbf{E}\mathcal{E}\left\{\left[\frac{1}{\sqrt{|\Lambda_N|}}\sum_{i\in\Lambda_N}\sigma_i\xi'_i\right]^{2k}\right\} \tag{2.3}$$

where we have used the first GKS inequality [9] $\mathbf{E}\{\sigma_{i_1}\cdots\sigma_{i_{2k}}\} \ge 0$, applied Proposition 2.2 to the σ'-variables on the right-hand side of (2.3) and re-summed the expression. Therefore, expanding the exponential in power series we get, for $t\chi < 1$,

$$\begin{aligned}\mathbf{E}^{\otimes 2}\left(\exp\left(\frac{t}{2}X_N^2\right)\right) &\le \mathbf{E}\mathcal{E}\left(\exp(t/2)\left[\frac{1}{\sqrt{|\Lambda_N|}}\sum_{i\in\Lambda_N}\sigma_i\xi'_i\right]^2\right)\\ &= \mathbf{E}\left(\exp\phi\left(t|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\sigma_i\sigma_j\right)\right) \le \exp\phi(t\chi),\end{aligned} \tag{2.4}$$

In (2.4) we used the fact that, for all fixed σ, $|\Lambda_N|^{-1/2}\sum_{i\in\Lambda_N}\sigma_i\xi_i'$ has a normal distribution with mean 0 and variance $|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\sigma_i\sigma_j$ to get the first equality and then that the maximum value of the variance is achieved when $\sigma_i=1\ \forall i$, and that $|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\leq\chi$. □

3. Martingale embedding and convergence

Following [2] we introduce a collection of independent, standard Brownian motions $\{B_{i,j}(t)\}_{i,j\in\mathbb{Z}^d}$, $t\geq 0$, defined on some probability space $(\Omega, A, \mathbb{P})$, and we consider the following positive, mean 1 martingales

$$e_N(t,\sigma)=\exp\left\{\frac{1}{\sqrt{2|\Lambda_N|}}\sum_{i,j\in\Lambda_N}B_{i,j}(t)\sigma_i\sigma_j-\frac{t}{4}|\Lambda_N|\right\},\quad Z_N(t)=\mathbf{E}[e_N(t,\sigma)]. \tag{3.1}$$

Note that

$$Z_N(\beta^2)\overset{\mathcal{D}}{=}\frac{\mathbf{E}\left[\exp\left(-\beta H_N^{SK}\right)\right]}{\mathbb{E}\mathbf{E}\left[\exp\left(-\beta H_N^{SK}\right)\right]}, \tag{3.2}$$

and observe that by definition we have $\sup_N\mathbb{E}Z_N(t)^2<\infty$ if $(\kappa,t^{1/2})\in D$. Set $\mathbf{E}_{N,t}[\,\cdot\,]=\mathbf{E}_{N,t,\kappa}[\,\cdot\,]$ $=\mathbf{E}[\,\cdot\,e_N(t,\sigma)]/Z_N(t)$: this is the expectation for our system (3). A key lemma (analogue to Lemma 3.1 in [2]) is

Lemma 3.1. – *For $(\kappa,t^{1/2})\in D,\varepsilon>0$ with $\sup_N\mathbf{E}_\kappa^{\otimes 2}\exp\frac{\beta^2+\varepsilon}{2}X_N(\sigma,\sigma')^2<\infty$, we have as $N\to\infty$*

$$Z_N(t)^2\left[\mathbf{E}_{N,t}^{\otimes 2}F(X_N(\sigma,\sigma'))-\mathbb{E}_\eta F(\eta)\right]\to 0\quad\text{in } \mathrm{L}^1(\Omega) \tag{3.3}$$

for any continuous function F such that $F(x)=o(\exp(\delta x^2/2))$ at $x=\pm\infty$ with some $0<\delta<\varepsilon$. Here η denotes a mean 0 Gaussian variable with variance $\theta^2/(1-t\theta^2)$ and $\mathbb{E}_\eta$ the expectation in η.

Proof. – The proof follows the line of the proof of Lemma 3.1 in [2], and it reduces to check (3.3) for $F(x)=G(x)\exp\{-tx^2/2\}$ with G bounded and such that $\mathbb{E}_\eta G(\eta)=0$. This in turn is implied by the convergence to 0 of $w_N=\mathbf{E}^{\otimes 3}\{e^{\frac{t}{2}X_N(\sigma',\sigma'')^2}G(X_N(\sigma,\sigma'))G(X_N(\sigma,\sigma''))\}$. However, the central limit (Lemma 2.1) and uniform integrability for $(\kappa,\sqrt{t})\in D$ – by definition of D – imply that $\lim_N w_N=\mathbb{E}_\eta^{\otimes 3}[G(\eta'')G(\eta\prime)\exp(t\eta^2/2)]=0$. □

Define the logarithmic martingale of Z_N as the stochastic integral $M_N(t)=\int_0^t Z_N(s)^{-1}\mathrm{d}Z_N(s)$ This new process is itself a mean zero, L^2 martingale with quadratic variation

$$\langle\, M_N\,\rangle(t)=\frac{1}{2}\int_0^t\mathbf{E}_{N,s}^{\otimes 2}\left(X_N(\sigma,\sigma')^2\right)\mathrm{d}s \tag{3.4}$$

Lemma 3.1 with $F(x)=x^2$ implies that $V_N(t)=Z_N(t)^2\frac{\mathrm{d}}{\mathrm{d}t}[\langle\, M_N\,\rangle(t)-\phi(t\theta^2)]$ tends to 0 in L^1 and is bounded in the L^1 norm uniformly for $N\geq 1$ and t in compact subsets of $[0,t_\kappa)$, where $t_\kappa:=\sup\{t:(\kappa,\sqrt{t})\in D\}$. Using the arguments in the proof of Proposition 3.2 in [2] it follows that

$$\sup_{s\in[0,t]}|\langle\, M_N\,\rangle(s)-\phi(s\theta^2)|\to 0\qquad\text{in probability as } N\to\infty\,. \tag{3.5}$$

for $t<t_\kappa$. Since ϕ is deterministic in this limit, it follows that the sequence M_N converges to a Gaussian process M_∞ on the time interval $[0,t_\kappa)$. We get finally a process convergence as stated below, which implies ii) in Theorem 1.1.

THEOREM 3.2. – *When $\kappa < \kappa_c$ the sequence of martingales M_N converges as $N \to \infty$ in distribution to the independent increments Gaussian process M_∞ on $[0, t_\kappa)$, with $M_\infty(0) = 0$, mean 0 and variance $\mathbb{E}[M_\infty(t)^2] = \phi(t\theta^2)$. Also, the random process $(Z_N(t), t \in [0, t_\kappa))$ converges in law to*

$$\exp\{M_\infty(t) - \phi(t\theta^2)/2\}.$$

Proof of Theorem 1.1, iii). – Using the assumption $\sum_i K(i) < \infty$, standard estimates yields

$$\frac{1}{|\Lambda_N|} \log \mathbf{E}\left(\mathrm{e}^{-\beta H_N^{\mathrm{SK}}}\right) = \frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa, \beta) - \frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa, 0) + \mathrm{o}(1), \tag{3.6}$$

with $\mathrm{o}(1) \to 0$ uniformly on Ω as $N \to \infty$. Note that $\lim_{N\to\infty}(1/|\Lambda_N|) \log \widetilde{Z}_N(\kappa, 0) = p^I(\kappa)$ and that by Theorem 1.1, point ii), and a straightforward annealed computation

$$\mathcal{D}\text{-}\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa, \beta) = \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}\left[\widetilde{Z}_N(\kappa, \beta)\right] = \frac{\beta^2}{4} \tag{3.7}$$

when $(\kappa, \beta) \in D$. This ensures convergence in probability of the pressure. Almost sure convergence follows from the standard Gaussian concentration inequality (*see* e.g. [11]):

$$\mathbb{P}\left(\left|\frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa, \beta) - \mathbb{E}\left(\frac{1}{|\Lambda_N|} \log \widetilde{Z}_N(\kappa, \beta)\right)\right| \geq \delta\right) \leq 2\exp\left(-\frac{|\Lambda_N|\delta^2}{\beta^2}\right) \tag{3.8}$$

for $\delta \geq 0$. Notice now that $\{\mathbb{E}((1/|\Lambda_N|) \log \widetilde{Z}_N(\cdot, \cdot))\}_{N\in\mathbb{N}}$ is a family of convex functions, uniformly bounded over compact sets. It is hence a compact family, in the topology of uniform convergence over compact sets and from this we conclude that the result obtained for $(\kappa, \beta) \in D$ can be extended by continuity to $\overline{D}$. □

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