Ergodic properties of simple model system with collisions*

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(Received 14 March 1973)

We investigate the ergodic properties of the discrete time evolution of a particle in a two-dimensional torus with velocity in the unit square. The dynamics consists of free motion for a unit time interval followed by a baker's transformation of the velocity.

1. INTRODUCTION

We are interested in the ergodic properties of dilute gas systems. These may be thought of as Hamiltonian dynamical systems in which the particles move freely except during binary "collisions". In a collision the velocities of the colliding particles undergo a transformation with "good" mixing properties (cf. Sinai's study of the billiard problem[1]). To gain an understanding of such systems we have studied the following simple discrete time model: The system consists of a single particle with coordinate $\underline{r} = (x,y)$ in a two-dimensional torus with sides of length (L_x, L_y), and "velocity" $\underline{v} = (v_x, v_y)$, in the unit square $v_x \in [0,1], v_y \in [0,1]$. The phase space Γ is thus a direct product of the torus and the unit square. The transformation T which takes the system from a dynamical state $(\underline{r}, \underline{v})$ at "time" j to a new dynamical state $T(\underline{r}, \underline{v})$ at time $j+1$ may be pictured as resulting from the particle moving freely during the unit time interval between j and $j+1$ and then undergoing a "collision" in which its velocity changes according to the baker's transformation, i.e.,

$$T(\underline{r}, \underline{v}) = (\underline{r} + \underline{v}, B\underline{v}), \tag{1.1}$$

with

$$B(v_x, v_y) = \begin{cases} (2v_x, \tfrac{1}{2}v_y), & 0 \le v_x < \tfrac{1}{2} \\ (2v_x - 1, \tfrac{1}{2}v_y + \tfrac{1}{2}), & \tfrac{1}{2} < v_x \le 1. \end{cases} \tag{1.2}$$

The normalized Lebesgue measure $d\mu = dxdydv_xdv_y/L_xL_y = d\underline{r}\,d\underline{v}/L_xL_y$ in Γ is left invariant by T. We call U_T the unitary transformation induced by T on $L^2(d\mu)$, $U_T\varphi = \varphi \circ T$. Our interest lies then in the ergodic properties of T and in the spectrum of U_T.

We note first that the transformation B on the velocities is, when taken by itself as a transformation of the unit square with measure $d\underline{v}$, well known to be isomorphic to a Bernoulli shift. It therefore has very good mixing properties. The isomorphism is obtained by setting

$$v_x = \sum_{j=1}^{\infty} 2^{-j} u_j, \qquad v_y = \sum_{j=1}^{\infty} 2^{-j} u_{1-j}, \tag{1.3}$$

with the u_j independent random variables taking the values 0 and 1 each with probability $\frac{1}{2}$. We then have

$$\begin{aligned} (B\underline{v})_x &= \sum_{j=1}^{\infty} 2^{-j} u_{j+1} = 2v_x - u_1, \\ (B\underline{v})_y &= \sum_{j=1}^{\infty} 2^{-j} u_{2-j} = \tfrac{1}{2}v_y + \tfrac{1}{2}u_1. \end{aligned} \tag{1.4}$$

2. ERGODIC PROPERTIES

The ergodic properties of our system which combines B with free motion turn out to depend on whether L_x^{-1} and L_y^{-1} satisfy the independence condition (I),

$$n_x L_x^{-1} + n_y L_y^{-1} \notin Z \text{ for } n_x \text{ and } n_y \text{ integers unless } n_x = n_y = 0. \tag{I}$$

Theorem 1: When (I) holds, the spectrum of U_T, on the complement of the one-dimensional subspace generated by the constants, is absolutely continuous with respect to Lebesgue measure and has infinite multiplicity.

It follows from Theorem 1 that when (I) holds the dynamical system (Γ, T, μ) is at least mixing. We do not know at present whether it is also a Bernoulli shift or at least a K system.

Theorem 2: When (I) does not hold the system (Γ, T, μ) is not ergodic.

The proof of Theorem 1 has two parts: a general characterization of unitary operators with Lebesgue spectrum and a set of estimates.

Lemma: Let U by a unitary operatory on a Hilbert space h, with spectral representation $U = \int_0^{2\pi} e^{i\theta} \underline{P}(d\theta)$. Assume that there exists a total set of vectors $\{\varphi_i\}$ such that $\sum_{n=1}^{\infty} |(U^n\varphi_i|\varphi_i)| < \infty$ for all i. (A set of vectors is said to be *total* if the finite linear span of this set of vectors is dense.) Then the spectral measure $\underline{P}(d\theta)$ is absolutely continuous with respect to Lebesgue measure, i.e., if E is a Borel set of Lebesgue measure 0, then $\underline{P}(E) = 0$.

Proof: We have

$(U^n\varphi_i|\varphi_i) = \int e^{in\theta}(\underline{P}(d\theta)\varphi_i|\varphi_i)$, i.e., the function $n \to (U^n\varphi_i|\varphi_i)$ is the Fourier transform of the measure $(\underline{P}(d\theta)\varphi_i|\varphi_i)$. On the other hand, $\sum_n |(U^n\varphi_i|\varphi_i)| < \infty$, so we can compute its inverse Fourier transform in the elementary way. By the uniqueness of the Fourier transform, we get:

$$(\underline{P}(d\theta)\varphi_i|\varphi_i) = \frac{d\theta}{2\pi} \cdot \sum_{n=-\infty}^{\infty} e^{-in\theta}(U^n\varphi_i|\varphi_i),$$

so the numerical measure $(\underline{P}(d\theta)\varphi_i|\varphi_i)$ is absolutely continuous with respect to Lebesgue measure. If E is a Borel setof Lebesgue measure 0,

$$\|\underline{P}(E)\varphi_i\|^2 = (\underline{P}(E)\varphi_i|\varphi_i) = 0, \quad \text{so } \underline{P}(E)\varphi_i = 0 \text{ for all } \varphi_i.$$

But the vectors $\{\varphi_i\}$ form a total set, so $\underline{P}(E) = 0$ as desired.

Now the estimates: Let $\chi(1) = 1$, $\chi(0) = -1$. For each finite subset X of Z, we define

$$\chi_X(\underline{v}) = \prod_{j \in X} \chi(u_j).$$

The χ_X form an orthonormal basis for $L^2(d\underline{v})$. Similarly, the functions $\exp(ik \cdot r)$; $\{k = (k_x, k_y), k_x = 2\pi n_x L_x, k_y = 2\pi n_y L_y, n_x$ and n_y integers$\}$, form an orthonormal basis for $L^2(d\underline{r})$. Thus, the functions $\varphi_{X,\underline{k}} = \exp(i\underline{k} \cdot \underline{r}) \cdot \chi_X(\underline{v})$ form an orthonormal basis for $L^2(d\mu)$. We will prove that

$$\sum_{n=1}^{\infty} |(U_T^n \varphi_{X_1,\underline{k}_1} | \varphi_{X_2,\underline{k}_2})| < \infty \quad \text{unless } \underline{k}_1 = \underline{k}_2 = 0, \; X_1 = X_2 = 0.$$

By straightforward computation,

$$U_T^n \varphi_{X_1,\underline{k}_1} = \varphi_{X_1+n}(\underline{v}) \exp(i\underline{k} \cdot \underline{r}) \times \exp[i\underline{k} \cdot (\underline{v} + B\underline{v} + \cdots + B^{n-1}\underline{v})].$$

Thus

$$\int d\underline{r} (U_T^n \varphi_{X_1,\underline{k}_1}) \bar{\varphi}_{X_2,\underline{k}_2} = 0 \quad \text{unless } \underline{k}_1 = \underline{k}_2 (= \underline{k}),$$

so we assume $\underline{k}_1 = \underline{k}_2 = \underline{k}$. Also,

$$\int d\underline{v} (U_T^n \varphi_{X_1,0}) \bar{\varphi}_{X_2,0} = 0 \quad \text{unless } X_2 = X_1 + n,$$

so the result is trivally true for $\underline{k} = 0$. We therefore assume $\underline{k} \neq 0$.

Now

$$(L_x L_y)^{-1} \int d\underline{r} d\underline{v} (U_T^n \varphi_{X_1,\underline{k}}) \bar{\varphi}_{X_2,\underline{k}} = \int d\underline{v} \chi_{X_1}(B^n \underline{v}) \chi_{X_2}(\underline{v}) \exp[i\underline{k}(\underline{v} + B\underline{v} + \cdots + B^{n-1}\underline{v})],$$

$$(B^j \underline{v})_x = \sum_{i=1}^{\infty} u_{j+i} 2^{-i},$$

$$\sum_{j=0}^{n-1} (B^j \underline{v})_x = \sum_{j=0}^{n-1} \sum_{i=1}^{\infty} u_{j+i} 2^{-i} = \sum_{l=1}^{\infty} u_l \sum_{i=1 \vee (l-n+1)}^{l} 2^{-i} = \sum_{l=1}^{\infty} u_l \alpha_l^n$$

(where this equation defines α_l^n),

$$(B^j \underline{v})_y = \sum_{i=1}^{\infty} 2^{-i} u_{j+1-i},$$

$$\sum_{j=0}^{n-1} (B^j \underline{v})_y = \sum_{j=0}^{n-1} \sum_{i=1}^{\infty} 2^{-i} u_{j+1-i} = \sum_{l=-\infty}^{n-1} u_l \sum_{i=1 \vee (-l+1)}^{n-l} 2^{-i} = \sum_{l=-\infty}^{\infty} u_l \beta_l^n.$$

Now let $l_2 = 1 \vee \max\{X_2\}$, $l_1 = \inf\{X_1\} \wedge 0$.

Then

$$U_T^n \varphi_{X_1,\underline{k}} \cdot \varphi_{X_2,\underline{k}} = \prod_{l=l_2+1}^{n+l_1-1} \exp[i(\alpha_l^n k_x + \beta_l^n k_y)^u l] \times [fn \text{ of the } u_l\text{'s for } l \notin (l_2, n + l_1)].$$

By independence, the integral of the product on the right is the product of the integrals, and the unspecified function of the u_l's, $l \notin (l_2, n + l_1)$ is no greater than one in absolute value, so

$$(L_x L_y)^{-1} \left| \int d\underline{v} d\underline{r}\, U_T^n \varphi_{X_1,\underline{k}} \cdot \varphi_{X_2,\underline{k}} \right| \le \prod_{l=l_2+1}^{n+l_1-1} \left| \tfrac{1}{2}[\exp(i\alpha_l^n k_x + \beta_l^n k_y) + 1] \right|.$$

For l's within the limits of the product, we have

$$\alpha_l^n = \sum_{i=1}^{l} 2^{-i} = 1 - 2^{-l},$$

$$\beta_l^n = \sum_{i=1}^{n-1} 2^{-i} = 1 - 2^{-(n-l)}.$$

Thus, for most of the terms in the product, $\alpha_l^n \approx \beta_l^n \approx 1$, and the number of terms is $n -$ const for large n. In particular, if we put

$$\gamma = \tfrac{1}{2} |\exp[i(k_x + k_y)] + 1| < 1$$

(by our fundamental assumption),

$$|(U_T^n \varphi_{X_1,\underline{k}} | \varphi_{X_2,\underline{k}}) < \gamma^{n/2} \quad \text{for all sufficiently large } n,$$

we have

$$\sum_{n=1}^{\infty} |(U_T^n \varphi_{X_1,\underline{k}} | \varphi_{X_2,\underline{k}})| < \infty$$

as desired.

The fact that the multiplicity is infinite is trivial. We have $L^2(d\underline{v}) \subset L^2(d\underline{r} d\underline{v})$, and we already know that the spectrum of U_T restricted to $L^2(d\underline{v})$ has infinite multiplicity.

To obtain a proof of Theorem 2, we note that ergodicity is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int d\mu\, (U_T^n \varphi) \bar{\Psi} = \left(\int d\mu\, \varphi\right)\left(\int d\mu \bar{\Psi}\right), \qquad \varphi, \Psi \in L^2(d\mu).$$

For φ or Ψ orthogonal to the constants we must then have Cesaro convergence to zero when the system is ergodic. We prove that the system is nonergodic by finding φ or Ψ orthogonal to the constants such that the above integral converges (strictly) to a nonzero number.

Let n_x, n_y be such that $n_x/L_x + n_y/L_y \in Z$ and n_x and n_y are not both 0, and let $k_x = 2\pi n_x/L_x$, $k_y = 2\pi n_y/L_y$. We set $\varphi = \Psi = \varphi_{0,\underline{k}}$ and compute as before the relevant integrals:

$$I_n = \int d\mu (U_T^n \varphi_{0,\underline{k}}) \bar{\varphi}_{0,\underline{k}} = \int d\underline{v} \exp\left[i\underline{k} \cdot \left(\sum_{j=0}^{n-1} B^j \underline{v} \right) \right]$$
$$= \int d\underline{v} \prod_{l=\infty}^{\infty} \exp[i(k_x \alpha_l^n + k_y \beta_l^n) u_l]$$
$$= \prod_{l=-\infty}^{\infty} \tfrac{1}{2}[1 + \exp(i\alpha_l^n k_x + \beta_l^n k_y)].$$

Here

$$\alpha_l^n = \sum_{i=1 \vee (l-n+1)}^{l} 2^{-i} = 2^{-l} \sum_{m=0}^{(n-1)\wedge(l-1)} 2^m = 2^{-l}(2^{n \wedge l} - 1)$$

for $l > 0$ and vanishes for $l \le 0$, and

$$\beta_l^n = \sum_{i=1 \vee (-l+1)}^{n-l} 2^{-i} = 2^{l-1} \sum_{m=0 \vee l}^{n-1} 2^{-m} = 2^{0 \wedge l} - 2^{l-n}$$

for $l < n$ and vanishes for $l \ge n$.

We thus have found that

$$I_n = \prod_{l=-\infty}^{0} \tfrac{1}{2}\{1 + \exp[i(2^l - 2^{l-n})k_y]\}$$
$$\times \prod_{l=1}^{n-1} \tfrac{1}{2}(1 + \exp\{i[(1-2^{-l})k_x + (1-2^{-(n-l)})k_y]\})$$
$$\times \prod_{l-n}^{\infty} \tfrac{1}{2}\{1 + \exp[ik_x(2^{-(l-n)} - 2^{-l})]\}$$
$$= F_n^1(\underline{k})F_n^2(\underline{k})F_n^3(\underline{k})$$

with

$$F_n^1(\underline{k}) = F_n^1(k_y) = \prod_{m=0}^{\infty} \tfrac{1}{2}\{1 + \exp[ik_y(2^{-m} - 2^{-(m+n)})]\},$$
$$F_n^3(\underline{k}) = F_n^3(k_x) = F_n^1(k_x),$$
$$F_n^2(\underline{k}) = \prod_{l=1}^{n-1} \tfrac{1}{2}(1 + \exp\{i[(1-2^{-l})k_x + (1-2^{-(n-l)})k_y]\}).$$

Since $k_x + k_y \in 2\pi Z$, we have

$$F_n^2(\underline{k}) = \prod_{l=1}^{n-1} \tfrac{1}{2}\{1 + \exp[-i(k_x 2^{-l} + k_y 2^{-(n-l)})]\}.$$

We now assert that (for $k_x + k_y \in \pi Z$)

$$\lim_{n\to\infty} F_n^i(\underline{k}) = \alpha^i \neq 0, \quad i = 1, 2, 3.$$

This is verified by observing that the log $F_n^i(\underline{k})$ coverage to a finite limit, thus completing the proof.

(If k_x and k_y are such that some of the terms at the beginning of the series which one obtains from the log $F_n^i(\underline{k})$ are singular, one easily removes the difficulty by an appropriate change in the functions φ and Ψ introduced at the beginning of the proof of Theorem 2. We also note that for the case where L_x/L_y is rational we can find explicitly a nonconstant function f which is left invariant by U_T. From the fact that $U_B(v_x + 2v_y) = 2v_x + v_y$ it follows that $f(x - y - v_x - 2v_y)$ is invariant if f is doubly periodic with periods L_x and L_y, so that we can construct an infinite family of orthonormal invariant functions $f_n : f_n = \exp\{(i2\pi n/L)(x - y - v_x - 2v_y)\}$ with $L_x/r = L_y/s = L$, r and s integers.)

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*Supported in part by USAFOSR Grant No. 73-2430.
†National Science Foundation Fellow.
‡Alfred P. Sloan Foundation Fellow; also supported in part by NSF Grant GP-15735.

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