On spin systems with quenched randomness: Classical and quantum

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ABSTRACT

The rounding of first-order phase transitions by quenched randomness is stated in a form which is applicable to both classical and quantum systems: The free energy, as well as the ground state energy, of a spin system on a d-dimensional lattice is continuously differentiable with respect to any parameter in the Hamiltonian to which some randomness has been added when $d \leq 2$. This implies absence of jumps in the associated order parameter, e.g., the magnetization in the case of a random magnetic field. A similar result applies in cases of continuous symmetry breaking for $d \leq 4$. Some questions concerning the behavior of related order parameters in such random systems are discussed.

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1. Introduction

The effect of quenched randomness on the equilibrium and transport properties of macroscopic systems is a subject of great theoretical and practical interest which is close to Nihat's heart. He and his collaborators [1–3] made important contributions to the study of the changes brought about by such randomness in phase transitions occurring in the pure (nonrandom) system. These effects can be profound in low dimensions. Their understanding evolved in a somewhat interesting way. As was argued by Imry, Ma and others [1,4,5], even a very small amount of randomness can change the nature of the transition in dimensions $d \leq 2$. The validity of the reasoning involved was contentious as it conflicted with the prediction of a dimensional-reduction principle, which was arrived at through perturbative calculations carried to all orders [6]. Rigorous work by Imbrie [7] and Bricmont and Kupiainen [8] showed that, in the discrete case, the rounding phenomenon does not extend beyond two dimensions, as given by the Imry–Ma argument. Aizenman and Wehr [9,10] provided a rigorous proof of the rounding effect in low dimensions, which although limited to classical spin systems included a large variety of types of randomness not previously considered. Although some questions have arisen recently about whether the effect should be equally general for quantum systems [11], we have recently shown [12] that with some modification the same proof can be extended to quantum systems. Here we present these results in a unified way, and discuss related issues which appear in specific examples of interest.

To formulate the results in a general, but not the most general way, we consider a lattice system of generalized spin variables $\underline{\sigma} = \{\sigma_x\}, x \in \mathbb{Z}^d$, where σ_x is an operator in a finite dimensional Hilbert space or, in the classical case, a function taking values in a finite set. The Hamiltonian is of the form

$$\mathcal{H}_\epsilon = \mathcal{K}(\underline{\sigma}; \underline{J}) - h \sum_x \kappa_x - \epsilon \sum_x \eta_x \kappa_x = \mathcal{H}_0 - \epsilon \sum_x \eta_x \kappa_x \tag{1}$$

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where J denotes a set of interaction parameters, $\{\kappa_x\}$ are translates of some local operator κ_0, and h and ϵ are real parameters. The quenched disorder is represented by $\{\eta_x\}$, a family of independent, identically distributed random variables with an absolutely continuous distribution with mean zero and finite third moment (weaker conditions are possible, but more awkward to state). $\mathcal{K}(\sigma; J)$ may be translation invariant and nonrandom, or it can include additional random terms, but we will not discuss the latter case. It is assumed here that the interactions decay faster than $(\text{distance})^{-\gamma}$, with $\gamma > 3d/2$ for the case described below in Theorem 1 and $\gamma > d + 2$ for Theorem 2.

We refer to the ηs as random fields, although in general they may also be associated with some other parameters, e.g. random bond strengths as in the models studied in Refs. [1,3].

An example of a system of this type (with $\kappa_x = \sigma_x^{(3)}$) is the ferromagnetic transverse-field Ising model with a random longitudinal field [13] (henceforth QRFIM), with

$$\mathcal{H}_\epsilon = - \sum J_{x-y}\sigma_x^{(3)}\sigma_y^{(3)} - \sum \left[\lambda\sigma_x^{(1)} + (h + \epsilon\eta_x)\sigma_x^{(3)} \right] \tag{2}$$

where $\sigma_x^{(i)}$ ($i = 1, 2, 3$) are single-site Pauli matrices, and $J_{x-y} \geq 0$. When $\lambda = 0$, this is the classical Ising system with a random field.

We will examine phase transitions where the order parameter is the volume average of the expectation value of κ_x with respect to an equilibrium state. These states are the infinite volume limits of sequences of Gibbs states in finite boxes $\Lambda \subset \mathbb{Z}^d$ with some boundary conditions, for which the expectation values are denoted here by $\langle - \rangle_\Lambda$.

As is well known, this order parameter is related to the directional derivatives ($\pm$) of the (quenched) free energy density given by

$$\mathcal{F}(T, J, h, \epsilon) = -T \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda, \tag{3}$$

where Z_Λ is the partition function of the system on a finite domain Λ of size $|\Lambda|$. Under fairly modest conditions [10,14], it is known that this limit exists, and is independent of the sequence of boundary conditions and (almost certainly) of η. Furthermore, it is a concave function of h, which means that it has directional derivatives; and these are related to the order parameter by

$$-\frac{\partial \mathcal{F}}{\partial h-} \leq \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \kappa_x \rangle_\Lambda \leq -\frac{\partial \mathcal{F}}{\partial h+} \tag{4}$$

for any sequence of sufficiently regular (e.g. rectangular) domains Λ converging to $\mathbb{Z}^d$.

More can be said, in particular in the classical case. There, convexity arguments imply that if the mean free energy density $\mathcal{F}(T, h)$ is differentiable in h, i.e.

$$\frac{\partial \mathcal{F}}{\partial h-} = \frac{\partial \mathcal{F}}{\partial h+} \equiv \frac{\partial \mathcal{F}}{\partial h}, \tag{5}$$

then the block averages of κ_x over large rectangular volumes B do not asymptotically deviate from the thermodynamic value $-\frac{\partial \mathcal{F}}{\partial h}$. More explicitly, there is a positive function $g(\delta)$ such that for every $\delta > 0$,

$$\lim_{\Lambda \to \mathbb{Z}^d} \text{Av} \left(\text{Prob}_\Lambda \left(\left| \frac{1}{|B|} \sum_{x \in B} \kappa_x + \frac{\partial \mathcal{F}}{\partial h} \right| > \delta \right) \right) \leq A e^{-g(\delta)|B|} \tag{6}$$

for some $A < \infty$, where Av is an average over the random fields, and $\text{Prob}_\Lambda (-)$ is the probability with respect to the Gibbs equilibrium measures for any sequence of Λs which converges to $\mathbb{Z}^d$. This general principle, which follows from the convexity of the free energy, can be used to show that the differentiability of the 'quenched' free energy density, i.e., (5) which is the subject of our discussion, implies the vanishing of the 'Short Long Range Order parameter' as well as of the 'Long Long Range Order Parameter' which appears in (4).

Combining the results known for some time for classical systems (where $\mathcal{H}_0$ and κ_x are functions of a local configuration) [10] with those recently proven also for quantum spin systems (where they are self-adjoint operators) [12], we obtain:

Theorem 1. *In dimensions $d \leq 2$, any system of the form of* (1) *has* $\frac{\partial \mathcal{F}}{\partial h-} = \frac{\partial \mathcal{F}}{\partial h+}$ *for all h, and $T \geq 0$, provided $\epsilon \neq 0$.*

In view of Eq. (4), or the above comment, this means that no first-order (i.e. discontinuous) transition with κ_x as the order parameter is possible in the presence of randomness in the local value of h: any such transition present in the system described by $\mathcal{H}_0$ is "rounded".

For systems where this spontaneous ordering would break a continuous symmetry, the results can be extended to $d \leq 4$. Here we must have observables which transform as vectors, so we replace (1) with

$$\mathcal{H}_\epsilon = \mathcal{K} - \sum (\vec{h} + \epsilon\vec{\eta}_x) \cdot \vec{\sigma}_x = \mathcal{H}_0 - \epsilon \sum \vec{\eta}_x \cdot \vec{\sigma}_x, \tag{7}$$

where $\vec{h}$, $\vec{\eta}_x$, and $\vec{\sigma}_x$ are quantities which transform as vectors in a space of dimension $N \geq 2$ and $\mathcal{K}$ is invariant under such rotations.

Fig. 1. Phase diagrams of (a) the simple Ising model and (b) the Ising lattice gas in two dimensions; thick lines are first-order phase transitions.

Theorem 2. *For the systems described above, with the random fields $\vec{\eta}_x$ having a rotation-invariant distribution, the free energy is continuously differentiable in $\vec{h}$ at $\vec{h} = 0$ whenever $\epsilon \neq 0$, $d \leq 4$.*

As before, the proof for classical systems is found in Ref. [10] and the proof for quantum systems in Ref. [12].

2. Higher order odd correlations

It can be proven rigorously for the models described by (2) with purely ferromagnetic interactions ($J_{x-y} \geq 0$) and $\lambda = 0$ that in the pure ($\epsilon = 0$) system, not only the magnetization $\left\langle \sigma_x^{(3)} \right\rangle$ but also all odd correlations, e.g. $\left\langle \sigma_x^{(3)} \sigma_y^{(3)} \sigma_z^{(3)} \right\rangle$, $x \neq y \neq z$, are discontinuous at $h = 0$ for $T < T_c$ [15]. The same is presumably true quite generally, particularly for the QRFIM with $\lambda \neq 0$; a discontinuity in the magnetization brings with it also a discontinuity in all odd correlation functions. We may now ask whether the randomness in the external field, corresponding to $\epsilon \neq 0$, which removes the discontinuity in $\langle \sigma_x \rangle$ also removes the discontinuity in all the odd correlation functions when averaged with respect to the randomness.

For the cases covered by (2) when $J_{x-y} \geq 0$, this question can be answered in the affirmative, including for $\lambda \neq 0$. In this case the system satisfies the Fortuin–Kasteleyn–Ginibre (FKG) inequalities [16]. Using these, it is possible to show that whenever all Gibbs states agree on the magnetization per site $\langle \sigma_x \rangle$ (i.e. when the spontaneous magnetization M is zero) they also agree on all other local observables (the argument for random systems is spelled out in Ref. [17]). We do not have any such result for the case when J_{x-y} in (2) contains also some negative terms. We expect of course that this will be the case more generally, but we lack a proof.

For an instructive example let us consider the phase diagram of the Ising model (2) with $\lambda = 0$ and $d = 2$ with nearest neighbor interactions $J_{\hat{e}} = J > 0$ ($\hat{e} \in \{(0, 1), (1, 0)\}$), $\epsilon = 0$ (see Fig. 1a). The free energy is analytic everywhere except on the bold part of the T axis. Along that line the free energy is continuous but its derivatives are discontinuous in all directions transverse to the T axis. That is, the free energy surface has a seam, i.e. a line of 'kink singularities'.

Those singularities will be rounded through the addition of arbitrarily small randomness to the field h. However, the kink singularity would not be eliminated by small randomness in the bond strengths. That is, with J replaced by $J + \epsilon b_{x,\hat{e}}$ with the $b_{x,\hat{e}}$ i.i.d. random variables, there would still be spontaneous magnetization at low temperatures, though T_c may change, and so possibly will the nature of the singularity in $\mathcal{F}$ at T_c. The randomness is tangential to the seam, corresponding to "orthogonal randomness" in the terminology of Ref. [2].

Let us now consider this system in terms of lattice gas variables. Setting $\xi_x = \frac{1}{2}(1 + \sigma_x) \in \{0, 1\}$, the nonrandom Hamiltonian now takes the form

$$\mathcal{H}_0 = -4J \sum_{\langle x,y \rangle} \xi_x \xi_y - \mu \sum_x \xi_x + \text{constant}. \tag{8}$$

The chemical potential μ is related to the parameters of the spin model by $\mu = 2h - 8J$, and so the phase diagram is now of the form shown in Fig. 1b. At the bold line the free energy has a discontinuous derivative in both variables, and randomness in either J or μ will eliminate this first-order transition, rounding the seam in $\mathcal{F}(T, \mu, 0)$. To avoid destroying the first-order transition we have to add randomness to J and μ in a correlated manner, so as to make it 'tangential' to the seam, i.e., in the form given in (2) in the σ-variables.

Similar reasoning can be applied quite generally. In particular if we consider the free energy $\mathcal{F}(T, J, h, \epsilon = 0)$ for the system described by (2) with J containing interaction parameters multiplying both even and odd functions of the spins then, for $T < T_c$, $\mathcal{F}$ will have "multidimensional seams" in directions transverse to all the odd terms — which is why they are all discontinuous in the absence of randomness. Adding then randomness to any odd term, e.g. to the one spin term as in (2), then rounds the discontinuity in all of them. If on the contrary we add noise only to the even terms, this is tangential to the "seams" and doesn't destroy the first-order transition.

This is also the case when adding noise in the QRFIM (2) to the transverse component of the field, i.e. letting $\lambda \sum \sigma_x^{(1)} \rightarrow \sum (\lambda + \epsilon \eta_x) \sigma_x^{(1)}$. As proven in Ref. [18] this does not destroy the discontinuity in $\left\langle \sigma_x^{(3)} \right\rangle$.

The situation is similar in the case when the spin at site x, S_x, assumes the values $S_x = -1, 0, 1$. A particular version of such a system is the Blume–Capel model, whose nonrandom version is defined by

$$\mathcal{H}_0 = -J \sum_{\langle x,y \rangle} (S_x - S_y)^2 - \Delta \sum S_x^2 - h \sum S_x. \tag{9}$$
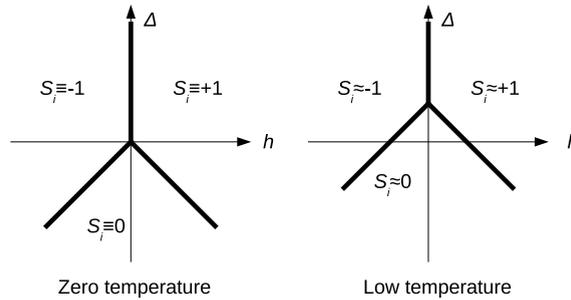
Fig. 2. Sections of the phase diagram for the Blume–Capel model.

The low temperature phase diagram of this system is known rigorously from Pirogov–Sinai theory [19,20]. A section is shown in Fig. 2 alongside the corresponding diagram at zero temperature.

In analogy with the Ising case, randomizing the magnetic field h, by using $h \mapsto h + \epsilon \eta_x$, will destroy all the first-order lines for $d \leq 2$ (for $d = 1$ this would be relevant only at $T = 0$). Randomness in J or Δ on the other hand should not destroy the discontinuity along the $h = 0$ line. This is in accord with the analysis in Ref. [1]. Similar results will hold for the quantum version of this model with S_x now interpreted as $S_x^{(3)}$, and a transverse field term $\sum_x S_x^{(1)}$ added.

3. Effects of bond randomness on the discontinuity of spontaneous magnetization

While the 'tangential disorder', in the sense discussed above, will not eliminate the discontinuity in the order parameter associated with the transverse field, e.g. the spontaneous magnetization at low enough temperatures, it *may* eliminate the possible discontinuity of the spontaneous magnetization if one was present at T_c in the non-random model. A general statement to that effect was proposed in Ref. [1].

An relevant example is provided by the two-dimensional Potts models with nearest neighbor ferromagnetic interactions. At $Q > 4$ these models exhibit a discontinuity in the spontaneous magnetization at T_c, which coincides with a jump in the energy density. While Theorem 1 assures that randomization of the bond strengths will eliminate the energy discontinuity, it does not address the persistence or lack of it for the discontinuity in the spontaneous magnetization. In Ref. [1] it was argued that also the latter discontinuity will be eliminated, but this remains an interesting open challenge from the rigorous point of view [21].

4. One-dimensional, inverse square interaction

Another class of models exhibiting a jump in the spontaneous magnetization are the one-dimensional Q-state Potts models, at $Q \geq 1$, with ferromagnetic pair interactions J_{x-y} which asymptotically behave as $J^+ / |x - y|^2$. It is known that such systems have a phase transition, and that these exhibit somewhat unusual features [22,23] (and references therein). Among those is the discontinuity of the spontaneous "magnetization" $M(T)$ at zero "magnetic" field, which jumps at T_c from zero to a value satisfying $M^2 J^+ / T_c \geq 1$.

At the moment it is not clear to us whether for the $1/r^2$ Potts models the discontinuity in $M(T)$ may coincide with a discontinuity in the energy (it does not for $Q = 2$ [24,25]). That will of course correspond to a jump at T_c in the derivative of $\mathcal{F}$ with respect to T. If that occurs then the system will have at $T = T_c$ at least $Q + 1$ phases. In the absence of energy discontinuity one may expect that at the critical temperature the systems will have Q ordered phases in analogy with the results of Ref. [15] for Ising systems.

Let us consider now the effects of the additions of a random symmetry-breaking "magnetic" field and of randomization, at suitably decaying rates, of the spin–spin interaction terms. As mentioned in the Introduction, the proof of the rounding phenomenon [10] applies also to the long range $d = 1$ models as long as the pair interaction decays faster than $|x - y|^{-3/2+\delta}$ (this ensures that the energy of interaction between an interval of size L and the rest of the system is less than $L^{1/2}$). Thus, in the presence of a random magnetic field the spontaneous magnetization $M(T)$ will vanish for all T. Randomization of the pair interaction would remove the discontinuity in the energy, if there was one, but unlike for the case discussed above we do not expect bond randomness to remove the discontinuity in $M(T)$ at the (possibly altered) critical temperature. The reason for the possible difference with the two-dimensional models is the bound $M^2 J^+ / T \geq 1$ which relates the discontinuity in the $d = 1$ models to the purely long range part of the interaction.

Acknowledgements

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