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On a Random Matrix Models of Quantum Relaxation

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ABSTRACT. In paper [**7**] two of us (J.L. and L.P.) considered a matrix model for a two-level system interacting with a $n \times n$ reservoir and assuming that the interaction is modelled by a random matrix. We presented there a formula for the reduced density matrix in the limit $n \to \infty$ as well as several its properties and asymptotic forms in various regimes. In this paper we give the proofs of assertions, announced in [**7**]. We present also a new fact about the model (see Theorem 2.1) as well as additional discussions of topics of [**7**]

1. Introduction

The model considered in [**7**] can be viewed as a random matrix version of the spin-boson model, widely used in studies of open quantum systems (see e.g. review works [**8, 12**] and references therein). We mention here that one of the first models of this type, namely the model where the classical system is represented by a harmonic oscillator coupled linearly with the oscillator reservoir, was considered by N. Bogolyubov in 1945 [**3**], Chapter IV.

We recall now the model, proposed and discussed in [**7**]. Let h_n be a Hermitian $n \times n$ matrix with eigenvalues $E_j^{(n)}$, $j = 1, ..., n$. We characterize the spectrum of h_n by its normalized counting measure of eigenvalues

$$(1.1) \qquad \nu_0^{(n)}(\Delta) = n^{-1} \sum_{j=1}^{n} \chi_\Delta(E_j^{(n)}); \quad \int_{-\infty}^{\infty} \nu_0^{(n)}(dE) = 1,$$

where χ_Δ is the indicator of an interval $\Delta \subset \mathbb{R}$. We assume that $\nu_0^{(n)}$ converges weakly as $n \to \infty$ to a limiting probability measure ν_0, i.e. that for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ we have:

$$(1.2) \qquad \lim_{n \to \infty} \int_{-\infty}^{\infty} \nu_0^{(n)}(dE)\varphi(E) = \int_{-\infty}^{\infty} \nu_0(dE)\varphi(E), \quad \int_{-\infty}^{\infty} \nu_0(dE) = 1.$$

Let w_n be a Hermitian $n \times n$ random matrix, whose probability density is

$$(1.3) \qquad\qquad Q_n^{-1} \exp\left\{ -\operatorname{Tr} w_n^2/2 \right\},$$

where Q_n is the normalization constant. In other words, the entries w_{jk}, $1 \leq j \leq k \leq n$ of the matrix w_n are independent Gaussian random variables with

$$(1.4) \quad \langle w_{jk} \rangle = 0, \ \langle w_{jj}{}^2 \rangle = 1, \ j,k = 1, ..., n, \ \langle (\Re w_{jk})^2 \rangle = \langle (\Im w_{jk})^2 \rangle = \frac{1}{2}, \ j \neq k,$$

where the symbol $\langle ... \rangle$ denotes here and below the expectation with respect to the distribution (1.3). This probability distribution is known as the Gaussian Unitary Ensemble (GUE) [9].

We define the Hamiltonian of our composite system $\mathcal{S}_{2,n}$ as a random $2n \times 2n$ matrix of the form

$$(1.5) \qquad\qquad H^{(n)} = s\sigma^z \otimes \mathbf{1}_n + \mathbf{1}_2 \otimes h_n + v\sigma^x \otimes w_n/n^{1/2},$$

where $\mathbf{1}_l$ ($l = 2, n$) is the $l \times l$ unit matrix, σ^z and σ^x are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the symbol $\otimes$ denotes the tensor product, and s and v are positive parameters. The first term in (1.5) is the Hamiltonian of the two-level sytem $\mathcal{S}_2$, the second term is the Hamiltonian of the n-level system (reservoir) $\mathcal{S}_n$, and the third term is an interaction between them. Thus s determines the energy scale of the isolated small system ($2s$ is its level spacing), and v plays the role of the coupling constant between $\mathcal{S}_2$ and $\mathcal{S}_n$. We write the Hamiltonian $H^{(n)}$ in the form

$$(1.6) \qquad\qquad H^{(n)} = H_0^{(n)} + M^{(n)},$$

where

$$H_0^{(n)} = s\sigma^z \otimes \mathbf{1}_n + \mathbf{1}_2 \otimes h_n, \quad M^{(n)} = v\sigma^x \otimes w_n/n^{1/2},$$

and choose the basis in $\mathbb{C}^2 \otimes \mathbb{C}^n$ in which the matrix $H_0^{(n)}$ is diagonal:

$$(1.7) \quad (H_0^{(n)})_{\alpha j, \beta k} = \lambda_{\alpha j}^{(n)} \delta_{\alpha \beta} \delta_{jk}, \quad \lambda_{\alpha j}^{(n)} = E_j^{(n)} + \alpha s, \quad \alpha, \beta = \pm, \quad j, k = 1, .., n.$$

Assume that at $t = 0$ the density matrix of the composite system $\mathcal{S}_{2,n}$ is

$$(1.8) \qquad\qquad \mu_m^{(n)}(E_k^{(n)}, 0) = \rho(0) \otimes P_k,$$

where $\rho(0)$ is a 2×2 positive definite matrix of unit trace and P_k is the projection on the state of energy $E_k^{(n)}$ of the reservoir. Let $\mu^{(n)}(t)$ be the density matrix of the composite system $\mathcal{S}_{2,n}$ at time t, corresponding to the initial density matrix $\mu_m^{(n)}(0)$ of (1.8):

$$(1.9) \qquad \mu^{(n)}(t) = U^{(n)}(-t)\mu_m^{(n)}(E_k^{(n)}, 0)U^{(n)}(t), \quad U^{(n)}(t) = e^{itH^{(n)}}.$$

Then the reduced density matrix of the small system is defined as

$$(1.10) \qquad\qquad \widehat{\rho}_{\alpha,\delta}^{(n)}(E_k^{(n)}, t) = \sum_{j=1}^n \mu_{\alpha j, \delta j}^{(n)}(t), \quad \alpha, \delta = \pm,$$

i.e. $\widehat{\rho}^{(n)}$ is obtained from the density matrix (1.9) of the whole composite (closed) system by tracing out the reservoir degrees of freedom. It follows from Theorem 2.1 below that the variance of the reduced density matrix vanishes as $n \to \infty$, i.e.

that $\widehat{\rho}^{(n)}$ is selfaveraging. This allows us to confine ourselves to the study of the mean reduced density matrix $\rho^{(n)}(E_k^{(n)}, t)$:

$$(1.11) \qquad \rho_{\alpha,\delta}^{(n)}(E_k^{(n)}, t) = \sum_{j=1}^{n} \left\langle \mu_{\alpha j,\delta j}^{(n)}(t) \right\rangle = \sum_{\beta,\gamma=\pm} T_{\alpha\beta\gamma\delta}^{(n)}(E_k^{(n)}, t)\rho_{\beta,\gamma}(0),$$

where

$$(1.12) \qquad T_{\alpha\beta\gamma\delta}^{(n)}(E_k^{(n)}, t) = \sum_{j=1}^{n} \left\langle U_{\alpha j,\beta k}^{(n)}(-t) + U_{\gamma k,\delta j}^{(n)}(t) \right\rangle$$

is the "transfer" matrix, an analog of the influence functional by Feynman-Vernon [12].

Notice that we can equally consider the factorized initial condition $\mu_c^{(n)}(\beta, 0)$ in which the microcanonical distribution P_k of the reservoir is replaced by its canonical distribution $e^{-\beta H_n^{(0)}}/Z_0^{(n)}$. We have evidently

$$(1.13) \qquad \mu_c^{(n)}(\beta, 0) = \sum_{k=1}^{n} e^{-\beta E_k^{(n)}} \mu_m^{(n)}(E_k^{(n)}, 0)/Z_0^{(n)}.$$

It is also easy to write the corresponding reduced density matrix.

2. Selfaveraging of Reduced Density Matrix

THEOREM 2.1. Let $\widehat{\rho}^{(n)}(E_k^{(n)}, t)$ be the reduced density matrix (1.10) of the composite system $S_{2,n} = S_2 + S_n$, given by (1.1) – (1.9). Then we have

$$(2.1) \qquad \mathbf{Var}\left\{ \widehat{\rho}_{\alpha,\delta}^{(n)}(E_k^{(n)}, t) \right\} \leq \frac{8v^2 t^2}{n}, \ \alpha, \delta = \pm.$$

To prove the theorem we use the following facts.

PROPOSITION 2.2. (Poincare-Nash inequality). If a random Gaussian vector $X = \{\xi_j\}_{j=1}^{p}$ satisfies conditions $\langle \xi_j \rangle = 0$, $\langle \xi_j \bar{\xi}_k \rangle = C_{jk}$, $j, k = 1, .., p$, and functions $\Phi_{1,2} : \mathbb{R}^p \to \mathbb{C}$ have bounded partial derivatives, then

$$(2.2) \qquad \mathbf{Cov}\{\Phi_1, \Phi_2\} := \langle (\Phi_1, \Phi_2) \rangle - \langle \Phi_1 \rangle \langle \Phi_2 \rangle$$

$$\leq \langle (C\nabla\Phi_1, \nabla\Phi_1) \rangle^{\frac{1}{2}} \langle (C\nabla\Phi_2, \nabla\Phi_2) \rangle^{\frac{1}{2}},$$

where

$$(C\nabla\Phi, \nabla\Phi) = \sum_{j,k=1}^{p} C_{jk} (\nabla\Phi)_j (\nabla\bar{\Phi})_k.$$

For the proof of the inequality see e.g. [2], Theorem 1.6.4.

PROPOSITION 2.3. (Duhamel formula). If M_1, M_2 are $n \times n$-matrices, then

$$(2.3) \qquad e^{(M_1+M_2)t} = e^{M_1 t} + \int_0^t e^{M_1(t-s)} M_2 e^{(M_1+M_2)s} ds.$$

The proof is elementary. Notice, that Duhamel formula allows us to obtain the derivative of the matrix $U^{(n)}(t)$ with respect to the entry w_{lm}, $l, m = 1, ..., n$ of the matrix w_n in (1.5):

$$(2.4) \qquad \frac{\partial U_{\alpha j,\beta k}^{(n)}(t)}{\partial w_{lm}} = \frac{iv}{\sqrt{n}} \int_0^t \sum_{\kappa=\pm} U_{\alpha j,\kappa l}^{(n)}(t-s) U_{-\kappa m,\beta k}^{(n)}(s) ds.$$

PROOF. (of the Theorem 2.1). By using the Poincare-Nash inequality (2.2) with

$$\Phi_1(X) = \Phi_2(X) = \sum_{j=1}^{n} U^{(n)}_{\gamma k,\delta j}(t) U^{(n)}_{\alpha j,\beta k}(-t), \ X = \{w_{lm}\}^n_{l,m=1}, \ C = \mathbf{1}_{n^2},$$

differentiation formula (2.4), and then Schwartz inequality, we obtain

$$(2.5) \quad \mathbf{Var}\Big\{\sum_j U^{(n)}_{\gamma k,\delta j}(t) U^{(n)}_{\alpha j,\beta k}(-t)\Big\}$$

$$\leq \frac{v^2}{n}\Big\langle \sum_{l,m}\Big|\sum_j \int_0^t \sum_\kappa U^{(n)}_{\gamma k,\kappa l}(t-s) U^{(n)}_{-\kappa m,\delta j}(s)ds \, U^{(n)}_{\alpha j,\beta k}(-t)$$

$$+ \sum_j U^{(n)}_{\gamma k,\delta j}(t) \int_0^t \sum_\kappa U^{(n)}_{\alpha j,\kappa l}(-t+s) U^{(n)}_{-\kappa m,\beta k}(-s)ds\Big|^2\Big\rangle$$

$$\leq \frac{2v^2 t}{n}\Big\langle \int_0^t \sum_{\kappa,l}|U^{(n)}_{\gamma k,\kappa l}(t-s)|^2 \sum_{\kappa,m}\Big|\sum_j U^{(n)}_{-\kappa m,\delta j}(s) U^{(n)}_{\alpha j,\beta k}(-t)\Big|^2 ds$$

$$+ \int_0^t \sum_{\kappa,l}\Big|\sum_j U^{(n)}_{\alpha j,\kappa l}(-t+s) U^{(n)}_{\gamma k,\delta j}(t)\Big|^2 \sum_{\kappa,m}|U^{(n)}_{-\kappa m,\beta k}(-s)|^2 ds\Big\rangle,$$

and here and below all the sums over the Latin indices will be from 1 to n, and the sum over the Greek indices will be over $\pm$. Notice that

$$(2.6) \qquad\qquad \sum_{\kappa,m}|U^{(n)}_{-\kappa m,\beta k}(-s)|^2 = \sum_{\kappa,l}|U^{(n)}_{\gamma k,\kappa l}(t-s)|^2 = 1,$$

and

$$(2.7) \quad \sum_{\kappa,l}\Big|\sum_j U^{(n)}_{\alpha j,\kappa l}(-t+s) U^{(n)}_{\gamma k,\delta j}(t)\Big|^2 = \sum_{j_1,j_2}\Big(\sum_{\kappa,l} U^{(n)}_{\alpha j_1,\kappa l}(-t+s) U^{(n)}_{\kappa l,\alpha j_2}(t-s)\Big)$$

$$\times U^{(n)}_{\gamma k,\delta j_1}(t) U^{(n)}_{\delta j_2,\gamma k}(-t) = \sum_{j_1}|U^{(n)}_{\gamma k,\delta j_1}(t)|^2 \leq 1.$$

Hence, we have by (2.5), (2.6), and (2.7):

$$\mathbf{Var}\Big\{\sum_j U^{(n)}_{\gamma k,\delta j}(t) U^{(n)}_{\alpha j,\beta k}(-t)\Big\} \leq \frac{4v^2 t^2}{n}, \quad \alpha,\delta = \pm.$$

Now, taking into account (1.10) and the fact that $\rho(0)$ is a 2×2 positive definite matrix and of unit trace, we obtain (2.1). $\qquad\square$

3. Equilibrium Properties

We begin by considering the equilibrium (time independent) microcanonical density matrix of the composite system $\mathcal{S}_{2,n}$:

$$(3.1) \qquad\qquad \Omega(\lambda) = \delta(\lambda - H^{(n)})\Big/\mathrm{Tr}\,\delta(\lambda - H^{(n)}) .$$

Following a standard prescription of statistical mechanics, we will replace the Dirac delta-function in (3.1) by the function $(2\varepsilon)^{-1}\chi_\varepsilon$, where χ_ε is the indicator of the interval $(-\varepsilon,\varepsilon)$, and $\varepsilon \ll \lambda$. Then the reduced microcanonical density matrix, i.e.

the microcanonical density matrix of $\mathcal{S}_{2,n}$, traced with respect to the states of $\mathcal{S}_n$, is the 2×2 matrix of the form

$$(3.2) \qquad \omega^{(n)}(\lambda) = \frac{\overline{\nu}^{(n)}(\lambda)}{\sum_{\delta=\pm} \overline{\nu}_{\delta,\delta}^{(n)}(\lambda)},$$

where

$$(3.3) \qquad \overline{\nu}_{\alpha\gamma}^{(n)}(\lambda) = (2\varepsilon n)^{-1} \sum_{j=1}^{n} \chi_{\varepsilon}(\lambda - H^{(n)})_{\alpha j, \gamma j}.$$

The corresponding canonical distribution of the composite system is

$$(3.4) \qquad e^{-\beta H_n} \big/ \operatorname{Tr} e^{-\beta H_n},$$

and the reduced distribution of the small system is

$$(3.5) \qquad \frac{\int_{-\infty}^{\infty} e^{-\beta\lambda} \nu^{(n)}(d\lambda)}{\sum_{\delta=\pm} \int_{-\infty}^{\infty} e^{-\beta\lambda} \nu_{\delta,\delta}^{(n)}(d\lambda)},$$

where (cf (3.3))

$$(3.6) \qquad \nu^{(n)}(\Delta) = \{\nu_{\alpha\gamma}^{(n)}(\Delta)\}_{\alpha,\gamma=\pm}, \quad \nu_{\alpha\gamma}^{(n)}(\Delta) = n^{-1} \sum_{j=1}^{n} \chi_{\Delta}(H^{(n)})_{\alpha j, \gamma j},$$

and χ_{Δ} is the indicator of an interval Δ of the spectral axis.

THEOREM 3.1. *Consider the 2×2 matrix measure $\nu^{(n)}$ of (3.6). Then*
(i) there exists non-random diagonal 2×2 matrix measure

$$\nu = \{\nu_\alpha \delta_{\alpha\gamma}\}_{\alpha,\gamma=\pm}$$

such that the weak convergence:

$$(3.7) \qquad \lim_{n \to \infty} \nu^{(n)} = \nu$$

holds with probability 1;
(ii) if

$$f_\alpha(z) = \int_{-\infty}^{\infty} \frac{\nu_\alpha(d\lambda)}{\lambda - z}, \quad \Im z \neq 0,$$

is the Stieltjes transform of ν_α, and ν_0 is defined by (1.2), then the pair $f_\alpha(z)$, $\alpha = \pm$ is a unique solution of the system of two coupled functional equations

$$(3.8) \qquad f_\alpha(z) = \int_{-\infty}^{\infty} \frac{\nu_0(dE)}{E + s\alpha - z - v^2 f_{-\alpha}(z)}, \quad \alpha = \pm$$

in the class of functions analytic for $\Im z \neq 0$, and satisfying the condition $\Im f_\alpha(z) \cdot \Im z > 0$, $\Im z \neq 0$;
(iii) nonnegative measures ν_α, $\alpha = \pm$ have the unit total mass, $\nu_\alpha(\mathbb{R}) = 1$, and if the measure ν_0 of (1.2) is absolute continuous and $\sup_{\lambda \in \mathbb{R}} \nu_0'(\lambda) < \infty$, then ν_α, $\alpha = \pm$ are also absolute continuous, and we have

$$(3.9) \qquad \nu_\alpha'(\lambda) \leq \sup_{\mu \in \mathbb{R}} \nu_0'(\mu);$$

(iv) for any $\lambda \in \mathbb{R}$ with probability 1 there exists the limit of the reduced microcanonical distribution

$$(3.10) \qquad \lim_{n \to \infty} \omega^{(n)} = \omega,$$

where

$$(3.11) \qquad \omega(\lambda) = \frac{\overline{\nu}(\lambda)}{\sum_{\delta=\pm} \overline{\nu}_{\delta,\delta}(\lambda)},$$

and

$$(3.12) \qquad \overline{\nu}_{\alpha\gamma} = \delta_{\alpha,\gamma}\overline{\nu}_\alpha, \quad \overline{\nu}_\alpha(\lambda) = (2\varepsilon)^{-1} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \nu_\alpha(d\mu),$$

analogous formulas are also valid for the limits of the reduced canonical distribution (3.5).

REMARK 3.2. The limiting measures ν_α, $\alpha = \pm$ can be found from their Stieltjes transforms f_α, $\alpha = \pm$ via the inversion formula [**1**]:

$$(3.13) \qquad \nu_\alpha(\triangle) = \pi^{-1} \lim_{\tau \to 0} \int_\triangle \Im f_\alpha(\lambda + i\tau)d\lambda.$$

To prove the theorem we need the following auxiliary fact.

PROPOSITION 3.3. Let Φ be a $\mathbf{C}^1$ function of $n \times n$ hermitian matrix, bounded together with its derivatives. Then we have for the GUE matrix w_n of (1.3):

$$(3.14) \qquad \left\langle \frac{\partial \Phi(w_n)}{\partial w_{jk}} \right\rangle = \left\langle \Phi(w_n)w_{kj} \right\rangle.$$

The proof of proposition follows from (1.3)–(1.4) and the integration by parts formula.

PROOF. (of the Theorem 3.1). Denote

$$(3.15) \qquad G^{(n)}(z) = (H^{(n)} - z)^{-1}, \ \Im z \neq 0$$

the resolvent of (1.5) and set

$$(3.16) \qquad g_{\alpha\gamma}^{(n)}(z) = n^{-1} \sum_{j=1}^n G_{\alpha j, \gamma j}^{(n)}(z).$$

It follows from the spectral theorem for Hermitian matrices that $g_{\alpha\gamma}^{(n)}$ is the Stieltjes transform of $\nu_{\alpha\gamma}^{(n)}$ and in view of the one-to-one correspondence between measures and their Stieltjes transforms (see [**1**], Section 59) to prove the weak convergence (3.7) with probability 1 it suffices to prove that with probability 1 $g_{\alpha\gamma}^{(n)}$ converges to $\delta_{\alpha\gamma}f_\alpha$ uniformly on a compact set of $\mathbb{C}\backslash\mathbb{R}$. Denote

$$(3.17) \qquad f_{\alpha\gamma}^{(n)}(z) := \left\langle g_{\alpha\gamma}^{(n)}(z) \right\rangle = n^{-1} \sum_{j=1}^n \left\langle G_{\alpha j, \gamma j}^{(n)}(z) \right\rangle.$$

For further purposes it is convenient to start by considering the functions

$$(3.18) \qquad u_{\alpha\gamma}^{(n)}(t) = n^{-1} \sum_{j=1}^n \left\langle U_{\alpha j, \gamma j}^{(n)}(t) \right\rangle,$$

where the matrix $U^{(n)}(t)$ is defined in (1.9). By the spectral theorem for Hermitian matrices $u_{\alpha\gamma}^{(n)}$ is the Fourier transfrom of $\nu_{\alpha\gamma}^{(n)}$ and $f_{\alpha\gamma}^{(n)}(z)$ is the generalized Fourier transform (see e.g. [**11**]) of $u_{\alpha\gamma}^{(n)}(t)$:

$$(3.19) \qquad f_{\alpha\gamma}^{(n)}(z) = i^{-1} \int_0^\infty e^{-izt} u_{\alpha\gamma}^{(n)}(t)dt, \quad \Im z < 0.$$

Notice that the matrix $\langle U^{(n)}(t) \rangle$ is diagonal with respect to the Latin indices. Indeed, since w_n in (1.5) is the GUE random matrix whose probability law (1.3) is unitary invariant, we have for any unitary $n \times n$-matrix U:

$$\langle \exp\{itH^{(n)}\} \rangle = \langle \exp\{it\big(H_0^{(n)} + vn^{-1/2}\sigma^x \otimes Uw^{(n)}U^*\big)\} \rangle.$$

In particularly, for any diagonal unitary matrix $U = \{e^{i\varphi_j}\delta_{jk}\}_{j,k=1}^n$ with distinct $\varphi_j \in [0, 2\pi)$, $j = 1, ..., n$ we obtain

$$\langle (\exp\{itH^{(n)}\})_{\alpha j, \beta k} \rangle = e^{i(\varphi_j - \varphi_k)}\langle (\exp\{itH^{(n)}\})_{\alpha j, \beta k} \rangle.$$

This implies

$$(3.20) \qquad \langle U_{\alpha j, \beta k}^{(n)}(t) \rangle = U_{\alpha\beta,j}^{(n)}(t)\delta_{jk}, \quad U_{\alpha\beta,j}^{(n)}(t) = \langle \exp\{itH^{(n)}\}_{\alpha j, \beta j} \rangle.$$

Hence we can write (3.18) as

$$(3.21) \qquad u_{\alpha\gamma}^{(n)}(t) = \frac{1}{n} \sum_{j=1}^n U_{\alpha\gamma,j}^{(n)}(t).$$

It follows now from (2.3), (2.4), (1.6), and (3.14) that

$$(3.22) \quad \langle e^{itH^{(n)}} \rangle_{\alpha j, \beta j} = \big(e^{itH_0^{(n)}}\big)_{\alpha j, \beta j} + i \int_0^t ds \, \langle e^{i(t-s)H_0^{(n)}} M^{(n)} e^{itH^{(n)}} \rangle_{\alpha j, \beta j}$$

$$= e^{it\lambda_{\alpha j}^{(n)}}\delta_{\alpha\beta} + \frac{iv}{\sqrt{n}} \int_0^t ds \, e^{i(t-s)\lambda_{\alpha j}^{(n)}} \sum_m \langle w_{jm} U_{-\alpha m, \beta j}^{(n)}(s) \rangle$$

$$= e^{it\lambda_{\alpha j}^{(n)}}\delta_{\alpha\beta} - \frac{v^2}{n} \int_0^t ds \, e^{i(t-s)\lambda_{\alpha j}^{(n)}} \int_0^s d\tau \sum_m \sum_\nu \langle U_{-\alpha m, \nu m}^{(n)}(s-\tau) U_{-\nu j, \beta j}^{(n)}(\tau) \rangle.$$

Hence taking into account (3.20) and (3.21), we obtain:

$$(3.23) \qquad U_{\alpha\beta,j}^{(n)}(t) = \int_0^t ds \, e^{i(t-s)\lambda_{\alpha j}^{(n)}} \, r_{\alpha\beta,j}^{(n)}(s)$$

$$+ e^{it\lambda_{\alpha j}^{(n)}}\delta_{\alpha\beta} - v^2 \int_0^t ds \, e^{i(t-s)\lambda_{\alpha j}^{(n)}} \int_0^s d\tau \sum_\nu u_{-\alpha\nu}^{(n)}(s-\tau) U_{-\nu\beta,j}^{(n)}(\tau),$$

where

$$r_{\alpha\beta,j}^{(n)}(s) = -\frac{v^2}{n} \int_0^s d\tau \sum_m \sum_\nu \langle \overset{\circ}{U}_{-\alpha m, \nu m}^{(n)}(s-\tau) U_{-\nu j, \beta j}^{(n)}(\tau) \rangle, \quad (\overset{\circ}{U} = U - \langle U \rangle).$$

By using Schwartz inequality and inequality (3.34) below we have that

$$(3.24) \qquad \big| r_{\alpha\beta,j}^{(n)}(s) \big| \leq \frac{Cs^3}{n^{3/2}}.$$

Here and below we use the notation C for all positive quantities that do not depend on n, t, z and indexes.

We will use the notations $G_j^{(n)}(z)$, $f^{(n)}(z)$ and $R_j^{(n)}(z)$ for the generalized Fourier transforms (see (3.19)) of the 2×2-matrices

$$(3.25) \quad U_j^{(n)} = \{U_{\alpha\beta,j}^{(n)}(t)\}_{\alpha\beta=\pm}, \ u^{(n)} = \{u_{\alpha\beta}^{(n)}(t)\}_{\alpha\beta=\pm}, \ r_j^{(n)}(t) = \{r_{\alpha\beta,j}^{(n)}(t)\}_{\alpha\beta=\pm}.$$

We have from the spectral theorem, (3.18), (3.20), and (3.15) (cf (3.19))

$$(3.26) \qquad G_j^{(n)}(z) = i^{-1} \int_0^\infty e^{-izt} U_j^{(n)}(t)dt = \big\{ \langle G_{\alpha j, \beta j(z)} \rangle \big\}_{\alpha,\beta=\pm}, \quad \Im z < 0.$$

This, (3.17), and (3.23) lead to the matrix relation:
$$(E_j^{(n)} + s\sigma^z - z - v^2\sigma^x f^{(n)}(z)\sigma^x)G_j^{(n)}(z) = 1 + R_j^{(n)}(z).$$

Since the resolvent $G^{(n)}(z)$ possesses the property $\Im z \Im(G^{(n)}(z)x, x) \geq 0, \forall x \in \mathbb{C}^{2n}$, the matrix $f^{(n)}(z)$ possesses the same property $\forall \xi \in \mathbb{C}^2$, and
$$\Im((E_j^{(n)} + s\sigma^z - z - v^2\sigma^x f^{(n)}(z)\sigma^x)\xi, \xi) = -\Im z ||\xi||^2 - v^2(f^{(n)}(z)\sigma^x\xi, \sigma^x\xi) \geq -\Im z ||\xi||^2.$$

Thus the matrix $E_j^{(n)} + s\sigma^z - z - v^2\sigma^x f^{(n)}(z)\sigma^x$ is invertible, its inverse

(3.27) $$f^{(n)}(E_j^{(n)}, z) = (E_j^{(n)} + s\sigma^z - z - v^2\sigma^x f^{(n)}(z)\sigma^x)^{-1}$$

admits the bound

(3.28) $$||f^{(n)}(E_j^{(n)}, z)|| \leq |\Im z|^{-1},$$

and equation for $G_j^{(n)}(z)$ takes the form

(3.29) $$G_j^{(n)}(z) = f^{(n)}(E_j^{(n)}, z) + f^{(n)}(E_j^{(n)}, z)R_j^{(n)}(z).$$

Applying to the equation the operation $n^{-1}\sum_j$ we obtain in view of (1.1)

(3.30) $$f^{(n)}(z) = \int_{-\infty}^{\infty} \nu_0^{(n)}(dE)f^{(n)}(E, z) + n^{-1}\sum_{j=1}^{n} f^{(n)}(E_j^{(n)}, z)R_j^{(n)}(z).$$

Since the resolvent $G^{(n)}(z)$ is analytic if $\Im z \neq 0$ and bounded from above by $|\Im z|^{-1}$, we have the bound

(3.31) $$||f^{(n)}(z)|| \leq |\Im z|^{-1},$$

implying that the sequence $\{f^{(n)}(z)\}_{n\geq 1}$ consists of functions, analytic and uniformly bounded in n and in z by η_0^{-1} if $|\Im z| \geq \eta_0 > 0$. Hence there exists analytic 2×2 matrix function $f(z)$, $\Im z \neq 0$, such that $||f(z)|| \leq |\Im z|^{-1}$, and an infinite subsequence $\{f^{(n_k)}(z)\}_{k\geq 1}$ that converges to $f(z)$ uniformly on any compact set of $\mathbb{C} \setminus \mathbb{R}$. This and estimates (3.24), (3.28) allow us to pass to the limit $n_k \to \infty$ in (3.30) and obtain that the limit of any converging subsequence of the sequence $\{f^{(n)}(z)\}_{n\geq 1}$ satisfies the matrix functional equation

(3.32) $$f(z) = \int_{-\infty}^{\infty} \nu_0(dE)f(E, z),$$

where

(3.33) $$f(E, z) = (E + s\sigma^z - z - v^2\sigma^x f(z)\sigma^x)^{-1}, \quad ||f(E, z)|| \leq |\Im z|^{-1}.$$

The equation is uniquely soluble in the class of 2×2 matrix functions, analytic for $\Im z \neq 0$, and such that $\Im z \Im(f(z)\xi, \xi) \geq 0, \forall \xi \in \mathbb{C}^2$. Indeed, for any two solutions f_1, f_2 of the class, and $g = f_1 - f_2$ we have
$$g(z) = \int_{-\infty}^{\infty} \nu_0(dE)\left[E + s\sigma^z - z - v^2\sigma^x f_1(z)\sigma^x\right]^{-1} v^2\sigma^x g(z)\sigma^x$$
$$\times \left[E + s\sigma^z - z - v^2\sigma^x f_2(z)\sigma^x\right]^{-1},$$

and by (1.2), (3.33) we obtain inequality $||g(z)|| \leq v^2|\Im z|^{-2}||g(z)||$ from which it follows that $g(z) = 0$ for $v\Im z < 1$, hence for any $\Im z \neq 0$ by analyticity. The solution of (3.32) is diagonal, $f_{\alpha\beta} = f_\alpha \delta_{\alpha\beta}$, and pair f_α, $\alpha = \pm$ satisfies system (3.8). This follows from the unique solvability of (3.8). We can rewrite (3.8) in

the form $f_\alpha(z) = f_\alpha^0(z + v^2 f_{-\alpha}(z))$, where $f_\alpha^0(z)$ is the Stieltjes transform of the unit non-negative measure $\nu_0^\alpha(E) = \nu_0(E - \alpha s)$. Since $f_\alpha^0(z)$ possesses the property $\lim_{\eta \to \infty} \eta |f_\alpha^0(i\eta)| = 1$ and $|\Im(z + v^2 f_{-\alpha}(z))| \geq |\Im z|$, then $\lim_{\eta \to \infty} \eta |f_\alpha(i\eta)| = 1$ and $f_\alpha(z)$, $\alpha = \pm$ are Stieltjes transforms of the unit non-negative measures $\nu_\alpha(\lambda)$ (3.13) (see [**1**], Section 59).

In addition, the Tchebyshev inequality and bound (3.35) below imply that for any $\varepsilon > 0$

$$\mathbf{P}\{|f_{\alpha\gamma}^{(n)}(z) - g_{\alpha\gamma}^{(n)}(z)| > \varepsilon\} \leq \frac{1}{\varepsilon^2}\mathbf{Var}\{g_{\alpha\gamma}^{(n)}(z)\} \leq \frac{2v^2}{\varepsilon^2 n^2 |\Im z|^4}.$$

Hence the series

$$\sum_{n=1}^\infty \mathbf{P}\{|f_{\alpha\gamma}^{(n)}(z) - g_{\alpha\gamma}^{(n)}(z)| > \varepsilon\}$$

converges for any $\varepsilon > 0$, $|\Im z| \geq \eta_0 > 0$, and by the Borel-Cantelli lemma we have for any fixed z, $|\Im z| \geq \eta_0 > 0$, $\lim_{n \to \infty} g_{\alpha\gamma}^{(n)}(z) = f_{\alpha\gamma}(z)$ with probability 1. With the same probability this limiting relation is valid for all points of an infinite countable sequence $\{z_j\}_{j \geq 1}$, $|\Im z_j| \geq \eta_0 > 0$, possessing an accumulation point. Hence on any compact of $\mathbb{C} \setminus \mathbb{R}$ with probability 1 $\lim_{n \to \infty} g_{\alpha\gamma}^{(n)}(z) = f_{\alpha\gamma}(z)$, and we have the weak convergence (3.7) with the formulas (3.1)-(3.8).

Let us prove assertion (ii) of theorem. It follows from the (3.8)

$$\Im f_\alpha(z) \leq \sup_{\mu \in \mathbb{R}} \nu_0'(\mu) \int_{-\infty}^\infty \frac{\Im(z + v^2 f_{-\alpha}(z))dE}{(E + s\alpha - \Re(z + v^2 f_{-\alpha}(z)))^2 + (\Im(z + v^2 f_{-\alpha}(z)))^2}$$
$$= \pi \sup_{\mu \in \mathbb{R}} \nu_0'(\mu).$$

We have now by (3.13)

$$\nu_\alpha(\triangle) \leq |\triangle| \sup_{\mu \in \mathbb{R}} \nu_0'(\mu).$$

This implies (ii). To prove (iii) we notice that by (ii) measures ν_α, $\alpha = \pm$ are continuous. Thus we can pass to the limit $n \to \infty$ in (3.3), written as

$$\overline{\nu}_\alpha^{(n)}(\lambda) = \nu_\alpha^{(n)}([\lambda + \varepsilon, \lambda - \varepsilon)).$$

This and (3.2) imply (3.10)-(3.11). $\square$

LEMMA 3.4. *Under the conditions of Theorem 3.1*

$$(3.34) \qquad \mathbf{Var}\{U_{\alpha j, \gamma k}^{(n)}(t)\} \leq \frac{v^2 t^2}{n}, \qquad \mathbf{Var}\{u_{\alpha\gamma}^{(n)}(t)\} \leq \frac{v^2 t^2}{n^2},$$

$$(3.35) \qquad \mathbf{Var}\{G_{\alpha j, \gamma k}^{(n)}(z)\} \leq \frac{v^2}{n|\Im z|^4}, \qquad \mathbf{Var}\{g_{\alpha\gamma}^{(n)}(z)\} \leq \frac{2v^2}{n^2 |\Im z|^4}, \quad \Im z \neq 0.$$

PROOF. Acting as in the case of Theorem 2.1 we obtain (3.34). The differentiation formula for the resolvent

$$\frac{\partial G_{\alpha j, \beta k}^{(n)}(z)}{\partial w_{lm}^{(n)}} = \frac{iv}{\sqrt{n}} \sum_\kappa G_{\alpha j, \kappa l}^{(n)}(z) G_{-\kappa m, \beta k}^{(n)}(z),$$

following from the resolvent identity, together with Poincare-Nash inequality (2.2) imply the first inequality in (3.35):

$$\mathbf{Var}\{G^{(n)}_{\alpha j,\gamma k}(z)\} \leq \frac{v^2}{n}\langle\sum_{l,m}|\sum_{\kappa}G^{(n)}_{\alpha j,\kappa l}(z)G^{(n)}_{-\kappa m,\beta k}(z)|^2\rangle$$

$$\leq \frac{v^2}{n}\langle\sum_{l,\kappa}|G^{(n)}_{\alpha j,\kappa l}(z)|^2\sum_{m,\kappa}|G^{(n)}_{-\kappa m,\beta k}(z)|^2\rangle \leq \frac{v^2}{n|\Im z|^4}, \quad \Im z > 0.$$

The second inequality in (3.35) can be proved by a similar argument.   □

We will return now to functions of variable t and find the $n \to \infty$ limits of the sequences $\{U^{(n)}_j(t)\}_{n\geq 1}$ and $\{u^{(n)}(t) = n^{-1}\sum_{j=1}^n U^{(n)}_j(t)\}_{n\geq 1}$.

THEOREM 3.5.   *Consider the 2×2 matrices $\{U^{(n)}_j(t)\}_{n\geq 1}$ and $\{u^{(n)}(t)\}_{n\geq 1}$, defined in (3.25) and (3.21) and choose a subsequence $\{E^{(n)}_{j_n}\}$ that converges to a given E of the support of ν_0 of (1.2). Then there exist the limits*

$$(3.36) \qquad\qquad U_E(t) = \lim_{n\to\infty} U^{(n)}_{j_n}(t), \quad u(t) = \lim_{n\to\infty} u^{(n)}(t),$$

where

$$(3.37) \quad U_E(t) = \frac{i}{2\pi}\int_L e^{izt}f(E,z)dz := \frac{i}{2\pi}\lim_{N\to\infty}\int_{-N-i\eta}^{N-i\eta} e^{izt}f(E,z)dz, \quad \forall\eta > 0,$$

$$(3.38) \qquad\qquad\qquad ||U_E(t)|| = 1 \quad \forall t \geq 0,$$

and

$$(3.39) \qquad\qquad u(t) = \frac{i}{2\pi}\int_L dz\, e^{izt}\int_{-\infty}^{\infty}\nu_0(dE)f(E,z)$$

with $f(E,z)$ defined in (3.33):

$$(3.40) \quad f_{\alpha\beta}(E,z) = f_\alpha(E,z)\delta_{\alpha\beta}, \quad f_\alpha(E,z) = (E_\alpha - z - v^2 f_{-\alpha}(z))^{-1}, \; E_\alpha = E + \alpha s.$$

PROOF.   It follows from (3.26), (3.29), and the inversion formula for the genererelized Fourier transform [11] that

$$(3.41) \qquad U^{(n)}_{j_n}(t) = Q^{(n)}(E^{(n)}_{j_n}, t) + i^{-1}\int_0^t Q^{(n)}(E^{(n)}_{j_n}, t-s)r^{(n)}_{j_n}(s)ds,$$

where

$$(3.42) \quad Q^{(n)}(E^{(n)}_{j_n}, t) = \frac{i}{2\pi}\int_L e^{izt}f^{(n)}(E^{(n)}_{j_n}, z)dz = U_E(t)$$

$$+ \frac{i}{2\pi}\int_L e^{izt}f^{(n)}(E^{(n)}_{j_n}, z)\big[E - E^{(n)}_{j_n} + v^2\sigma^x(f^{(n)}(z) - f(z))\sigma^x\big]f(E,z)dz.$$

The resolvent identity yields

$$f^{(n)}(E^{(n)}_{j_n}, z) = -\frac{1}{z-E} + \frac{1}{z-E}\big[s\sigma^z + (E - E^{(n)}_{j_n}) + v^2\sigma^x f^{(n)}(z)\sigma^x\big]f^{(n)}(E^{(n)}_{j_n}, z),$$

and we have for sufficiently big $\eta = |\Im z|$:

$$(3.43) \qquad ||f^{(n)}(E^{(n)}_{j_n}, z)|| \leq \frac{2}{|z-E|}, \quad ||f(E,z)|| \leq \frac{2}{|z-E|}.$$

This together with (3.31) allow us to pass to the limit under the integral in the r.h.s. of (3.42) and to show that it vanishes as $n \to \infty$. Moreover, we conclude that integral in the r.h.s. of (3.42) is bounded uniformly in n and $\forall t \geq 0$. The uniform boundedness of the matrix $U_E(t)$ follows from the equalities $(U_E)_{\alpha\beta}(t) = (U_E)_{\alpha\alpha}(t)\delta_{\alpha\beta}$ and

$$(U_E)_{\alpha\alpha}(t) = e^{iE_\alpha t} + \frac{i}{2\pi}\int_L dz \, e^{itz} \frac{v^2 f_{-\alpha}(z)}{(E_\alpha - z - v^2 f_{-\alpha}(z))(E_\alpha - z)}.$$

Hence $Q^{(n)}(E_{j_n}^{(n)}, t)$ converges to $U_E(t)$ as $n \to \infty$ and is uniformly bounded in n and t. This together with (3.41), (3.24) and equality $\|U_j^{(n)}(t)\| = 1$, $\forall t \geq 0$ give us (3.37) and (3.38).

To prove (3.39) notice first that we have from (3.30)

$$(3.44) \qquad u^{(n)}(t) = \frac{i}{2\pi}\int_L dz \, e^{izt}\int_{-\infty}^{\infty} \nu_0^{(n)}(dE)f(E, z)$$

$$+ \frac{i}{2\pi}\int_L dz \, e^{izt}\int_{-\infty}^{\infty} \nu_0^{(n)}(dE)\left[f^{(n)}(E, z) - f(E, z)\right]$$

$$+ i^{-1}\int_0^t n^{-1}\sum_{j=1}^n Q^{(n)}(E_j^{(n)}, t-s)r_j^{(n)}(s)ds.$$

We integrate by parts with respect to z in the first integral to obtain in view of (3.40)

$$-\frac{1}{2\pi t}\int_{-\infty}^{\infty} \nu_0^{(n)}(dE)\int_L dz \, e^{izt}\frac{1 + v^2 f'_{-\alpha}(z)}{(E_\alpha - z - v^2 f_{-\alpha}(z))^2}.$$

It follows from (3.32)-(3.33) and (1.2) that $\|f_\alpha(z)\| = |z|^{-1}(1 + o(1))$, $|z| \to \infty$ and $\|f'_\alpha(z)\| \leq |\Im z|^{-2}$. Thus the integral with respect to z is bounded and continuous function of E. This and the weak convergence $\nu_0^{(n)}$ to ν_0 (see (1.2)) yield the convergence of the first term of the r.h.s. of (3.44) to the r.h.s. of (3.39).

Furthermore, by using (3.43), (3.31) and (1.1) we obtain

$$\int_{-\infty}^{\infty} \nu_0^{(n)}(dE)\int_L |dz|\frac{\|f^{(n)}(z+E) - f(z+E)\|}{|z|^2}$$

$$\leq C\left\{\int_{|E|\geq T} \nu_0^{(n)}(dE) + \int_{|x|\geq A}\frac{dx}{x^2 + \eta^2} + \max_{|y|\leq A+T}\|f^{(n)}(y - i\eta) - f(y - i\eta)\|\right\}.$$

For any $\varepsilon > 0$ choosing consequently $A = A(\varepsilon)$, $T = T(\varepsilon, A)$, $N_0 = N_0(\varepsilon, A, T)$, and taking in account (1.2), and convergence $f^{(n)}(z)$ to $f(z)$ on any compact set in $\mathbb{C}\setminus\mathbb{R}$, we obtain that the second term of the r.h.s. of (3.44) vanishes as $n \to \infty$.

At last (3.24) yields for the third term of the r.h.s. of (3.44):

$$\int_0^t ds\frac{Cs^3}{n}\frac{1}{n}\sum_j \|\int_L dz \, e^{iz(t-s)}f^{(n)}(E_j, z)\|$$

$$\leq \int_0^t ds\frac{Cs^3}{n}\int_{-\infty}^{\infty} \nu_0^{(n)}(dE)\left\{\|U_E(t-s)\| + \int_L dz\|f^{(n)}(E, z) - f(E, z)\|\right\},$$

and taking into account (3.38), (3.43), and (3.31) we conclude that the term also vanishes as $n \to \infty$ uniformly in t, varying on a compact interval. $\qquad\square$

4. Time Evolution

We will prove now the main general result of [**7**], a formula for the limit as $n \to \infty$ of the expectation (1.11) of the reduced density matrix (1.10) of our model formula (4.7) of [**7**].

THEOREM 4.1. *Consider the model of composite system, defined by (1.1)-(1.9). Choose a subsequence $\{E_{k_n}^{(n)}\}$ of eigenvalues of h_n of (1.5) that converges to a certain $E \in supp\ \nu_0$. Then we have for the limit as $n \to \infty$ of the expectation (1.11) of the reduced density matrix (1.10) uniformly in t varying on a finite interval:*

$$(4.1)\quad \rho_{\alpha,\delta}(E,t) := \lim_{n\to\infty} \rho_{\alpha,\delta}^{(n)}(E_{k_n}^{(n)}, t) = \frac{1}{(2\pi)^2} \int_{L_2} dz_2 \int_{L_1} dz_1 e^{it(z_2-z_1)}$$

$$\times \frac{f_\alpha(E,z_1)f_\delta(E,z_2)\rho_{\alpha,\delta}(0) + v^2 f_{-\alpha}(E,z_1)f_{-\delta}(E,z_2)f_{\alpha,\delta}(z_1,z_2)\rho_{-\alpha,-\delta}(0)}{1 - v^4 f_{\alpha,\delta}(z_1,z_2)f_{-\alpha,-\delta}(z_1,z_2)},$$

where $L_1 = (-\infty + i\eta_1, \infty + i\eta_1)$, $L_2 = (-\infty - i\eta_2, \infty - i\eta_2)$, $\eta_1 > 0$, $\eta_2 > 0$;

$$(4.2)\qquad\qquad f_{\beta,\gamma}(z_1,z_2) = \int_{-\infty}^{\infty} \nu_0(dE)f_\beta(E,z_1)f_\gamma(E,z_2)$$

with $f_\alpha(E,z)$ defined in (3.40).

PROOF. In view of (1.11) it suffices to prove the following expression for the average transfer matrix (1.12):

$$(4.3)\qquad T_{\alpha\beta\gamma\delta}(E,t) := \lim_{n\to\infty} T_{\alpha\beta\gamma\delta}^{(n)}(E_{k_n}^{(n)}, t)$$

$$= \frac{1}{(2\pi)^2} \int_{L_2} dz_2 \int_{L_1} dz_1 e^{it(z_2-z_1)}\widetilde{T}_{\alpha\beta\gamma\delta}(E,z_1,z_2),$$

where the "two-point" functions $\widetilde{T}_{\alpha\beta\gamma\delta}(E,z_1,z_2)$ are analytic in z_1 and in z_2 outside the real axis and have the form

$$(4.4)\quad \widetilde{T}_{\alpha\beta\gamma\delta}(E,z_1,z_2) = f_\beta(E,z_1)f_\gamma(E,z_2)$$

$$\times (\delta_{\alpha,\beta}\delta_{\gamma,\delta} + v^2\delta_{-\alpha,\beta}\delta_{-\gamma,\delta}f_{-\beta,-\gamma}(z_1,z_2))[1 - v^4 f_{\beta,\gamma}(z_1,z_2)f_{-\beta,-\gamma}(z_1,z_2)]^{-1}.$$

Acting as in the proof of the Theorem 3.1, i.e., by using the Duhamel formula (2.3), (3.14) and differentiation formula (2.4) we obtain the relation (cf (3.22))

$$(4.5)\quad T_{\alpha\beta\gamma\delta}^{(n)}(E_{k_n}^{(n)}, t_1, t_2) := \sum_{j=1}^{n} \left\langle U_{\alpha j,\beta k}^{(n)}(-t_1)U_{\gamma k,\delta j}^{(n)}(t_2) \right\rangle = e^{it_2\lambda_{\gamma k_n}^{(n)}}\delta_{\gamma\delta}U_{\alpha\beta,k_n}^{(n)}(-t_1)$$

$$- v^2 \int_0^{t_2} ds\, e^{i(t_2-s)\lambda_{\gamma k_n}^{(n)}} \int_0^s d\tau \sum_\kappa T_{\alpha,\beta,-\kappa,\delta}^{(n)}(E_{k_n}^{(n)}, t_1, \tau)u_{-\gamma\kappa}^{(n)}(s-\tau)$$

$$+ v^2 \int_0^{t_2} ds\, e^{i(t_2-s)\lambda_{\gamma k_n}^{(n)}} \int_0^{t_1} d\tau \sum_\kappa U_{-\kappa\beta,k_n}^{(n)}(-\tau)K_{\alpha,\kappa,-\gamma,\delta}^{(n)}(t_1-\tau,s)$$

$$+ \int_0^{t_2} ds\, e^{i(t_2-s)\lambda_{\gamma k_n}^{(n)}} r_{\alpha\beta\gamma\delta}^{(n)}(t_1,s),$$

where

$$(4.6)\quad K_{\alpha\beta\gamma\delta}^{(n)}(t_1,t_2) = \frac{1}{n}\sum_{m=1}^{n}\sum_{j=1}^{n}\left\langle U_{\alpha j,\beta m}^{(n)}(-t_1)U_{\gamma m,\delta j}^{(n)}(t_2)\right\rangle, \quad |K_{\alpha\beta\gamma\delta}^{(n)}(t_1,t_2)| \le 1,$$

and

$$(4.7) \quad r^{(n)}_{\alpha\beta\gamma\delta}(t_1, t_2)$$

$$= \frac{v^2}{n} \sum_j \left[-\int_0^{t_2} d\tau \sum_m \sum_\kappa \langle \overset{\circ}{U}^{(n)}_{-\gamma m, \kappa m}(t_2 - \tau) U^{(n)}_{-\kappa k_n, \delta j}(\tau) U^{(n)}_{\alpha j, \beta k_n}(-t_1) \rangle \right.$$

$$\left. + \int_0^{t_1} d\tau \sum_m \sum_\kappa \langle \overset{\circ}{U}^{(n)}_{-\kappa k_n, \beta k_n}(-\tau) U^{(n)}_{-\gamma m, \delta j}(t_2) U^{(n)}_{\alpha j, \kappa m}(\tau - t_1) \rangle \right].$$

It follows from Schwartz and Poincare-Nash inequalities and estimate (3.34) that (cf (3.24))

$$(4.8) \qquad\qquad |r^{(n)}_{\alpha\beta\gamma\delta}(t_1, t_2)| = O(n^{-1/2}).$$

Let $\widetilde{T}^{(n)}_{\alpha\beta\gamma\delta}(E, z_1, z_2), \Im z_1 > 0, \Im z_2 < 0$ be generalized Fourier transform of $T^{(n)}_{\alpha\beta\gamma\delta}(E, t_1, t_2)$:

$$\widetilde{T}^{(n)}_{\alpha\beta\gamma\delta}(E, z_1, z_2) = i \int_0^\infty dt_1 \, e^{iz_1 t_1} \left(\frac{1}{i} \int_0^\infty dt_2 \, e^{-iz_2 t_2} T^{(n)}_{\alpha\beta\gamma\delta}(E, t_1, t_2) \right),$$

so that

$$T^{(n)}_{\alpha\beta\gamma\delta}(E, t_1, t_2) = \frac{1}{(2\pi)^2} \int_{L_2} dz_2 \int_{L_1} dz_1 e^{i(t_2 z_2 - t_1 z_1)} \widetilde{T}^{(n)}_{\alpha\beta\gamma\delta}(E, z_1, z_2),$$

where $L_1 = (-\infty + i\eta_1, \infty + i\eta_1)$, $L_2 = (-\infty - i\eta_2, \infty - i\eta_2)$, $\eta_1 > 0$, $\eta_2 > 0$. In view of relations

$$U^{(n)}_{\alpha j, \beta k}(-t_1) = \overline{U^{(n)}_{\alpha k, \beta j}(t_1)}, \quad G^{(n)}_{\alpha j, \beta k}(z_1) = \overline{G^{(n)}_{\alpha k, \beta j}(\overline{z_1})}$$

we have

$$i \int_0^\infty dt_1 \, e^{iz_1 t_1} U^{(n)}_{\alpha j, \beta k}(-t_1) = G^{(n)}_{\alpha j, \beta k}(z_1)$$

and (4.5) yields

$$(4.9) \quad \widetilde{T}^{(n)}_{\alpha\beta\gamma\delta}(E^{(n)}_{k_n}, z_1, z_2)$$

$$= \frac{1}{\lambda^{(n)}_{\gamma k_n} - z_2} \left[G^{(n)}_{\alpha\beta, k_n}(z_1) \delta_{\gamma\delta} + v^2 \sum_\kappa f^{(n)}_{-\gamma\kappa}(z_2) \widetilde{T}^{(n)}_{\alpha\beta-\kappa\delta}(E^{(n)}_{k_n}, z_1, z_2) \right.$$

$$\left. + v^2 \sum_\kappa G^{(n)}_{-\kappa\beta, k_n}(z_1) \widetilde{K}^{(n)}_{\alpha, \kappa, -\gamma, \delta}(z_1, z_2) + \widetilde{r}^{(n)}_{\alpha\beta\gamma\delta}(z_1, z_2) \right].$$

Here $\widetilde{K}^{(n)}$ and $\widetilde{r}^{(n)}$ are generalized Fourier transforms of $K^{(n)}$ and $r^{(n)}$ of (4.6) and (4.7) respectively, and as it follows from (4.6) the absolute values of $\widetilde{K}^{(n)}_{\alpha\beta\gamma\delta}(z_1, z_2)$ are bounded uniformly in n by $|\Im z_1|^{-1} |\Im z_2|^{-1}$.

To write (4.9) in the matrix form for any fixed pair α, β we denote $\widetilde{K}^{(n)}_{\alpha\beta}$, $\widetilde{S}^{(n)}_{\alpha\beta}$, $\widetilde{r}^{(n)}_{\alpha\beta}$ the 2×2-matrices, which entries are $(\widetilde{K}^{(n)}_{\alpha\beta})_{\gamma\delta} = \widetilde{K}^{(n)}_{\alpha\beta\gamma\delta}$ etc., and $\widetilde{K}^{-(n)}_{\alpha\beta}$ are 2×2-matrices with the entries $(\widetilde{K}^{-(n)}_{\alpha\beta})_{\gamma\delta} = \widetilde{K}^{(n)}_{\alpha,\beta,-\gamma,\delta}$, $\gamma, \delta = \pm$, so

$$\widetilde{T}^{(n)}_{\alpha\beta}(E^{(n)}_{k_n}, z_1, z_2) = f^{(n)}(E^{(n)}_{k_n}, z_2) \widetilde{r}^{(n)}_{\alpha\beta}(z_1, z_2) + f^{(n)}(E^{(n)}_{k_n}, z_2)$$

$$\times \left[G^{(n)}_{\alpha\beta, k_n}(z_1) \mathbf{1}_2 + v^2 \sum_\kappa G^{(n)}_{-\kappa\beta, k_n}(z_1) \widetilde{K}^{-(n)}_{\alpha\kappa}(z_1, z_2) \right].$$

Plugging expression (3.29) for $G_{k_n}^{(n)}(z_2)$ we obtain

$$(4.10) \quad \widetilde{T}_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1, z_2) = \widetilde{R}_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1, z_2)$$
$$+ f^{(n)}(E_{k_n}^{(n)}, z_2)\left[f_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1)\mathbf{1}_2 + v^2 \sum_\kappa f_{-\kappa\beta}^{(n)}(E_{k_n}^{(n)}, z_1)\widetilde{K}_{\alpha\kappa}^{-(n)}(z_1, z_2)\right],$$

where reminder $\widetilde{R}_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1, z_2)$ is a 2×2-matrix, and according to (3.24), (4.8), and uniform boundedness of $\widetilde{K}^{(n)}(z_1, z_2)$ and $f^{(n)}(E_m^{(n)}, z)$, we have

$$(4.11) \quad \lim_{n\to\infty} ||\widetilde{R}_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1, z_2)|| = 0, \quad \lim_{n\to\infty} ||n^{-1}\sum_{m=1}^n \widetilde{R}_{\alpha\beta}^{(n)}(E_m^{(n)}, z_1, z_2)|| = 0.$$

Applying the operation $n^{-1}\sum_{m=1}^n$ to (4.10) with $k_n = m$ we obtain:

$$(4.12) \quad \widetilde{K}_{\alpha\beta}^{(n)}(z_1, z_2) = v^2 \sum_\kappa \int_{-\infty}^\infty \nu_0^{(n)}(dE)f_{-\kappa\beta}^{(n)}(E, z_1)f^{(n)}(E, z_2)\widetilde{K}_{\alpha\kappa}^{-(n)}(z_1, z_2)$$
$$+ \int_{-\infty}^\infty \nu_0^{(n)}(dE)f_{\alpha\beta}^{(n)}(E, z_1)f^{(n)}(E, z_2) + n^{-1}\sum_m \widetilde{R}_{\alpha\beta}^{(n)}(E_m^{(n)}, z_1, z_2).$$

This implies that for any fixed α, β, γ, δ the limiting values $\widetilde{K}_{\alpha\beta\gamma\delta}(z_1, z_2) = \lim_{n\to0}\widetilde{K}_{\alpha\beta\gamma\delta}^{(n)}(z_1, z_2)$ and $\widetilde{K}_{\alpha,-\beta,-\gamma,\delta}(z_1, z_2)$ satisfy the system of linear equations

$$\widetilde{K}_{\alpha\beta\gamma\delta}(z_1, z_2) = f_{\beta\gamma}(z_1, z_2)\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + v^2\widetilde{K}_{\alpha,-\beta,-\gamma,\delta}(z_1, z_2)\right],$$

where $f_{\beta\gamma}(z_1, z_2)$ are defined in (4.2). Solving this system we obtain

$$(4.13) \quad \widetilde{K}_{\alpha,-\beta,-\gamma,\delta}(z_1, z_2) = \frac{f_{-\beta,-\gamma}(z_1, z_2)\left[\delta_{\alpha,-\beta}\delta_{-\gamma,\delta} + v^2 f_{\beta,\gamma}(z_1, z_2)\delta_{\alpha,\beta}\delta_{\gamma,\delta}\right]}{1 - v^4 f_{\beta,\gamma}(z_1, z_2)f_{-\beta,-\gamma}(z_1, z_2)}.$$

Now we return to the variables t_1, t_2. It follows from (4.10) and (4.11) that

$$(4.14) \quad T_{\alpha\beta\gamma\delta}(E, t_1, t_2) = \lim_{n\to\infty}\frac{1}{(2\pi)^2}\int_{L_2} dz_2 \int_{L_1} dz_1\, e^{i(t_2 z_2 - t_1 z_1)}$$
$$\times \left[f_{\alpha\beta}^{(n)}(E_{k_n}^{(n)}, z_1)f_{\gamma\delta}^{(n)}(E_{k_n}^{(n)}, z_2)\right.$$
$$\left. + v^2\sum_{\kappa,\nu} f_{-\kappa\beta}^{(n)}(E_{k_n}^{(n)}, z_1)f_{\gamma\nu}^{(n)}(E_{k_n}^{(n)}, z_2)\widetilde{K}_{\alpha,\kappa,-\nu,\delta}^{(n)}(z_1, z_2)\right],$$

and we have to prove the equality:

$$(4.15) \quad T_{\alpha\beta\gamma\delta}(E, t_1, t_2) = \frac{1}{(2\pi)^2}\int_{L_2} dz_2 \int_{L_1} dz_1 e^{i(t_2 z_2 - t_1 z_1)} f_\beta(E, z_1)f_\gamma(E, z_2)$$
$$\times \left[\delta_{\alpha\beta}\delta_{\gamma\delta} + v^2\widetilde{K}_{\alpha,-\beta,-\gamma,\delta}(z_1, z_2)\right],$$

which together with (4.13) yields (4.3). Notice that for any fixed non-real z_1, z_2 the integrand of (4.14) tends to integrand of (4.15), but it has no an integrable majorant. Because of this fact we replace $\widetilde{K}_{\alpha,\kappa,-\nu,\delta}^{(n)}(z_1, z_2)$ in (4.14) by the corresponding entry

of the r.h.s. matrix of (4.12) to obtain

$$
T_{\alpha\beta\gamma\delta}(E, t_1, t_2) = (U_E)_{\beta\beta}(-t_1)(U_E)_{\gamma\gamma}(t_2)\delta_{\alpha\beta}\delta_{\gamma\delta}
$$
$$
+ \lim_{n\to\infty} \frac{v^2}{(2\pi)^2} \sum_{\kappa,\nu} \int \nu_0^{(n)}(d\mu) dz_2 dz_1 e^{i(t_2 z_2 - t_1 z_1)} f_{-\kappa\beta}^{(n)}(E_{k_n}^{(n)}, z_1) f_{\gamma\nu}^{(n)}(E_{k_n}^{(n)}, z_2)
$$
$$
\times \left[f_{\alpha\kappa}^{(n)}(\mu, z_1) f_{-\nu\delta}^{(n)}(\mu, z_2) + v^2 \sum_{\kappa_1,\nu_1} f_{-\kappa_1\kappa}^{(n)}(\mu, z_1) f_{-\nu\nu_1}^{(n)}(\mu, z_2) \widetilde{K}_{\alpha,\kappa_1,-\nu_1,\delta}(z_1, z_2) \right].
$$

Here we denote $\int \nu_0^{(n)}(d\mu) dz_2 dz_1 = \int_{-\infty}^{\infty} \nu_0^{(n)}(d\mu) \int_{L_2} dz_2 \int_{L_1} dz_1$. Now it remains to prove that the following expressions

$$
\int \nu_0^{(n)}(d\mu) dz_2 dz_1 \left[f^{(n)}(E_{k_n}^{(n)}, z_1) - f(E, z_1) \right] f^{(n)}(E_{k_n}^{(n)}, z_2) f^{(n)}(\mu, z_1) f^{(n)}(\mu, z_2),
$$

$$
\int \nu_0^{(n)}(d\mu) dz_2 dz_1 f(E, z_1) f(E, z_2) \left[f^{(n)}(\mu, z_1) - f(\mu, z_1) \right] f^{(n)}(\mu, z_2),
$$

$$
\int \nu_0^{(n)}(d\mu) dz_2 dz_1 f(E, z_1) f(E, z_2) f(\mu, z_1) f(\mu, z_2) \left[K^{(n)}(z_1, z_2) - K(z_1, z_2) \right],
$$

$$
\int_{-\infty}^{\infty} \left[\nu_0^{(n)} - \nu_0 \right](d\mu) \int_{L_2} dz_2 \int_{L_1} dz_1 f(E, z_1) f(E, z_2) f(\mu, z_1) f(\mu, z_2) K(z_1, z_2),
$$

where $K^{(n)}(z_1, z_2) = K_{\alpha\beta\gamma\delta}^{(n)}(z_1, z_2)$, vanish as $n \to \infty$.

Since $||f^{(n)}(\mu, z_1)|| \leq \eta_1^{-1}$ and there exist η_i, $i = 1, 2$ such that $||f^{(n)}(\mu, z_2)|| \leq 2|z_2 - \mu|^{-1}$, $||f^{(n)}(E_{k_n}^{(n)}, z_i)|| \leq 2|z_i|^{-1}$, $|\Im z_i| \geq \eta_i$, then the norm of the first expression is bounded by

$$
\frac{1}{\eta_1} \int_{L_1} |dz_1| \frac{1}{|z_1|^2} \left(v^2 ||f^{(n)}(z_1) - f(z_1)|| + |E_{k_n}^{(n)} - E| \right) \int_{-\infty}^{\infty} \nu_0^{(n)}(d\mu) \int_{L_2} \frac{|dz_2|}{|z_2||z_2 - \mu|}.
$$

We have by Schwartz inequality

$$
\int_{L_2} \frac{|dz_2|}{|z_2||z_2 - \mu|} \leq \left(\int_{-\infty}^{\infty} \frac{dx}{x^2 + \eta_2^2} \int_{-\infty}^{\infty} \frac{dx}{(x - \mu)^2 + \eta_2^2} \right)^{1/2} = \frac{\pi}{\eta_2}.
$$

This and the uniform convergence of $f^{(n)}$ to f on a compact set of $\mathbb{C} \setminus \mathbb{R}$ imply that the first expression vanishes as $n \to \infty$. Treating similarly the remaining three expressions we prove that they tends to zero as $n \to \infty$. $\qquad \square$

5. Van-Hove Limit

In this section we study the limiting case, where the coupling constant v of the system-reservoir interaction tends to zero, the time t tends to infinity while the transition rate, given by first order perturbation in the interaction, is kept fixed [4, 5, 6, 10]. In terms of (4.3) this corresponds to making simultaneously the limits

$$
(5.1) \qquad\qquad v \to 0, \quad t \to \infty, \quad \tau = tv^2 \quad \text{fixed}
$$

after the limit $n \to \infty$, i.e., in formula (4.1).

We note that this limit as well as several other important topics of the small system-reservoir dynamics were considered by N.N. Bogolubov in 1945 [3] in the context of classical oscillator interacting linearly with the oscillator reservoir.

THEOREM 5.1. *Let the Fourier transform $\widehat{\nu_0}(u)$ of the density ν_0' of the measure ν_0 in (1.2) be absolutely integrable function:*

$$(5.2) \qquad \int_{-\infty}^{\infty} |\widehat{\nu_0}(u)| du = c_0 < \infty,$$

$$(5.3) \qquad \widehat{\nu_0}(u) = \int_{-\infty}^{\infty} e^{-iuE} \nu_0'(E) dE.$$

Then the diagonal entries of the limiting reduced density matrix in (4.1) in the van Hove limit are

$$(5.4) \qquad \rho_{\alpha,\alpha}^{vH}(E,\tau) = 2\pi \left[\frac{\nu_0'(E)}{\Gamma_\alpha(E)} \rho_{\alpha,\alpha}(0) + \frac{\nu_0'(E - 2\alpha s)}{\Gamma_{-\alpha}(E)} \rho_{-\alpha,-\alpha}(0) \right.$$
$$\left. + e^{-\tau\Gamma_\alpha(E)} \frac{\nu_0'(E + 2\alpha s)}{\Gamma_\alpha(E)} \rho_{\alpha,\alpha}(0) - e^{-\tau\Gamma_{-\alpha}(E)} \frac{\nu_0'(E - 2\alpha s)}{\Gamma_{-\alpha}(E)} \rho_{-\alpha,-\alpha}(0) \right]$$

where

$$(5.5) \qquad \Gamma_\alpha(E) = 2\pi \left[\nu_0'(E) + \nu_0'(E + 2\alpha s) \right];$$

and the off-diagonal entries are

$$(5.6) \qquad \rho_{\alpha,-\alpha}^{vH}(E,\tau) = \rho_{\alpha,-\alpha}(0) e^{-2\alpha sit} e^{i\tau(f_0(E+2\alpha s+i0) - f_0(E-2\alpha s-i0))},$$

where f_0 is the Stiltjes transform of ν_0':

$$f_0(z) = \int_{-\infty}^{\infty} \frac{\nu_0'(E) dE}{E - z}, \quad \Im z \neq 0.$$

LEMMA 5.2. *In conditions (5.2), (5.3) of the Theorem 5.1 next statements for the functions $f_\alpha(z)$, $\alpha = \pm$ are valid:*

   (i) $\sup_{\Im z \geq 0} |f_\alpha(z)| \leq c_0$,
   (ii) $\lim_{v \to 0} \frac{1}{\pi} \Im f_\alpha(\lambda + i0) = \nu_0'(\lambda - \alpha s)$, $\lambda \in \mathbb{R}$.

PROOF. Estimate (i) follows from the representation of the functions $f_\alpha(z)$, $\Im z > 0$ in the form

$$(5.7) \qquad f_\alpha(z) = i \int_0^\infty e^{iu(-\alpha s + z + v^2 f_{-\alpha}(z))} \widehat{\nu_0}(u) du$$

and condition $\Im z \Im f_\alpha(z) \geq 0$. It also follows from (5.2), (5.7) that

$$(5.8) \qquad \lim_{v \to 0} f_\alpha(z) = f_0(z - \alpha s) = i \int_0^\infty e^{iu(z - \alpha s)} \widehat{\nu_0}(u) du, \quad \Im z \geq 0.$$

Hence

$$(5.9) \qquad \lim_{v \to 0} \frac{1}{\pi} \Im f_\alpha(\lambda + i0) = \frac{1}{\pi} \Re \int_0^\infty e^{iu(-\alpha s + \lambda)} \widehat{\nu_0}(u) du = \nu_0'(\lambda - \alpha s).$$

$\square$

PROOF. (of the Theorem 5.1). By using equalities (see (4.2))

$$f_{\alpha,\alpha}(z_1, z_2) = \frac{\delta f_\alpha}{\delta z + v^2 \delta f_{-\alpha}}, \quad \delta z = z_1 - z_2, \quad \delta f_\alpha = f_\alpha(z_1) - f_\alpha(z_2),$$

and by using analyticity of the integrand of (4.1) in z_1 and in z_2, we can write the following representation for the diagonal entries of the limiting reduced density matrix

(5.10)

$$
\rho_{\alpha,\alpha}(E,t) = \frac{\rho_{\alpha,\alpha}(0)}{(2\pi)^2} \int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} f_\alpha(E,z_1) f_\alpha(E,z_2)
$$
$$
+ \frac{\rho_{\alpha,\alpha}(0)}{(2\pi)^2} \int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} \frac{v^4 \delta f_\alpha \delta f_{-\alpha} f_\alpha(E,z_1) f_\alpha(E,z_2)}{\delta z(\delta z + v^2 \delta f_\alpha + v^2 \delta f_{-\alpha})}
$$
$$
+ \frac{\rho_{-\alpha,-\alpha}(0)}{(2\pi)^2} \int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} \frac{v^2 \delta f_\alpha(\delta z + v^2 \delta f_\alpha) f_{-\alpha}(E,z_1) f_{-\alpha}(E,z_2)}{\delta z(\delta z + v^2 \delta f_\alpha + v^2 \delta f_{-\alpha})}
$$
$$
= I_1^v(E,t) + I_2^v(E,t) + I_3^v(E,t),
$$

where $L_1^v = \{z_1 : \Im z_1 = v^2 \eta_1\}$, $L_2^v = \{z_2 : \Im z_2 = -v^2 \eta_2\}$, η_1 and η_2 are arbitrarily chosen positive constants.

To compute the limit (5.1) of $I_1^v(E,t)$ we change variables to $\zeta_j = v^{-2}(z_j - E_\alpha)$, $j = 1, 2$, and by Lemma 5.2 we have

$$
I_1^v(E,t) = \frac{\rho_{\alpha,\alpha}(0)}{(2\pi)^2} \int_{L_1} d\zeta_1 \frac{e^{-i\tau\zeta_1}}{\zeta_1 + f_{-\alpha}(E_\alpha + v^2\zeta_1)} \int_{L_2} d\zeta_2 \frac{e^{i\tau\zeta_2}}{\zeta_2 + f_{-\alpha}(E_\alpha + v^2\zeta_2)}
$$
$$
= \frac{\rho_{\alpha,\alpha}(0)}{(2\pi)^2} \int_{L_1} d\zeta_1 \frac{e^{-i\tau\zeta_1}}{\zeta_1 + f_0(E + 2\alpha s + i0)} \int_{L_2} d\zeta_2 \frac{e^{i\tau\zeta_2}}{\zeta_2 + f_0(E + 2\alpha s - i0)} + o(1).
$$

Computing last integrals by residues and applying equality

(5.11)
$$
f_0(\lambda + i0) - f_0(\lambda - i0) = 2\pi i \nu_0'(\lambda)
$$

we obtain

(5.12)
$$
\mathrm{vH-}\lim I_1^v(E,t) = \rho_{\alpha,\alpha}(0) e^{-2\pi\tau\nu_0'(E+2\alpha s)},
$$

where the symbol "vH-lim" denotes the double limit (5.1).

Changing variables in $I_3^v(E,t)$ to $\zeta_2 = v^{-2}(z_2 - E_{-\alpha}) \in L_2 = \{\zeta : \Im\zeta = -\eta_2\}$, $\zeta_1 = v^{-2}(z_1 - z_2) \in L_1 = \{\zeta : \Im\zeta = \eta_1 + \eta_2\}$ yields

(5.13)
$$
I_3^v(E,t) = \frac{\rho_{-\alpha,-\alpha}(0)}{(2\pi)^2} \int_{L_2} d\zeta_2 \int_{L_1} d\zeta_1 e^{-i\tau\zeta_1} \frac{\delta f_\alpha(\zeta_1 + \delta f_\alpha)}{\zeta_1(\zeta_1 + \delta f_\alpha + \delta_{-\alpha})}
$$
$$
\times \frac{1}{\zeta_2 + f_\alpha(E_{-\alpha} + v^2\zeta_2)} \cdot \frac{1}{\zeta_1 + \zeta_2 + f_\alpha(E_{-\alpha} + v^2(\zeta_1 + \zeta_2))}.
$$

It follows from (5.2) that the absolute value of integrand of (5.14) is bounded from above by

$$
\frac{c}{|\zeta_1||\zeta_2||\zeta_1 + \zeta_2|} = \frac{c}{\sqrt{\lambda_1^2 + (\eta_1 + \eta_2)^2}\sqrt{\lambda_2^2 + \eta_2^2}\sqrt{(\lambda_1 + \lambda_2)^2 + \eta_1^2}},
$$

where $c > 0$ does not depend on v, $\lambda_j = \Re\zeta_j$, $j = 1, 2$. Now Schwartz inequality yields for any $B > 0$

$$\int_B^\infty \frac{d\lambda_1}{\lambda_1} \int_{-\infty}^\infty \frac{d\lambda_2}{\sqrt{\lambda_2^2 + 1}\sqrt{(\lambda_1 + \lambda_2)^2 + 1}}$$

$$= 2\int_B^\infty \frac{d\lambda_1}{\lambda_1} \int_0^\infty \frac{d\lambda_2}{\sqrt{(\lambda_2 - \frac{\lambda_1}{2})^2 + 1}\sqrt{(\lambda_2 + \frac{\lambda_1}{2})^2 + 1}}$$

$$\leq 2\left(\int_B^\infty \frac{d\lambda_1}{\lambda_1} \int_0^\infty \frac{d\lambda_2}{((\lambda_2 - \frac{\lambda_1}{2})^2 + 1)((\lambda_2 + \frac{\lambda_1}{2})^2 + 1)^{\frac{1}{4}}}\right)^{\frac{1}{2}}$$

$$\times \left(\int_B^\infty \frac{d\lambda_1}{\lambda_1} \int_0^\infty \frac{d\lambda_2}{((\lambda_2 + \frac{\lambda_1}{2})^2 + 1)^{\frac{3}{4}}}\right)^{\frac{1}{2}} < \infty.$$

This allows us to pass to limit in integral (5.13) by using (5.8) and (5.11):

$$\text{vH} - \lim I_3^v(E, t)$$
$$= \frac{\rho_{-\alpha,-\alpha}(0)}{(2\pi)^2} \int_{-\infty}^\infty d\lambda_2 \int_{-\infty}^\infty d\lambda_1 e^{-i\tau\lambda_1} \frac{2\pi i \nu_0'(E - 2\alpha s)(\lambda_1 + 2\pi i \nu_0'(E - 2\alpha s))}{\lambda_1(\lambda_1 + 2\pi i(\nu_0'(E) + \nu_0'(E - 2\alpha s)))}$$
$$\times \frac{1}{\lambda_2 + f_0(E - 2\alpha s - i0)} \cdot \frac{1}{\lambda_1 + \lambda_2 + f_0(E - 2\alpha s + i0)}.$$

Here integration path in λ_1 encircles zero from above. Computing last integrals by residues we have

$$(5.14) \quad \text{vH} - \lim I_3^v(E, t) = 2\pi\rho_{-\alpha,-\alpha}(0)\left[\frac{\nu_0'(E - 2\alpha s)}{\Gamma_{-\alpha}(E)} - e^{-\tau\Gamma_{-\alpha}(E)}\frac{\nu_0'(E - 2\alpha s)}{\Gamma_{-\alpha}(E)}\right].$$

Treating similarly the term I_2^v in the r.h.s. of (5.10) we obtain

$$(5.15) \quad \text{vH} - \lim I_2^v(E, t)$$
$$= 2\pi\rho_{\alpha,\alpha}(0)\left[\frac{\nu_0'(E)}{\Gamma_\alpha(E)} + e^{-\tau\Gamma_\alpha(E)}\frac{\nu_0'(E + 2\alpha s)}{\Gamma_\alpha(E)}\right] - \rho_{\alpha,\alpha}(0)e^{-2\pi\tau\nu_0'(E + 2\alpha s)}.$$

Now the assertion (5.4) of theorem follows from the (5.10), (5.12), (5.14) and (5.15).

Consider now the off-diagonal entry of (4.1):

$$(5.16) \quad \rho_{\alpha,-\alpha}(E, t) = \frac{\rho_{\alpha,-\alpha}(0)}{(2\pi)^2}\left[\int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} f_\alpha(E, z_1) f_{-\alpha}(E, z_2)\right.$$

$$+ \int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} \frac{v^4 f_{\alpha,-\alpha}(z_1, z_2) f_{-\alpha,\alpha}(z_1, z_2) f_\alpha(E, z_1) f_{-\alpha}(E, z_2)}{1 - v^4 f_{\alpha,-\alpha}(z_1, z_2) f_{-\alpha,\alpha}(z_1, z_2)}\right]$$

$$+ \frac{\overline{\rho_{\alpha,-\alpha}(0)}}{(2\pi)^2} \int_{L_2^v} dz_2 \int_{L_1^v} dz_1 e^{-it\delta z} \frac{v^2 f_{\alpha,-\alpha}(z_1, z_2) f_{-\alpha}(E, z_1) f_\alpha(E, z_2)}{1 - v^4 f_{\alpha,-\alpha}(z_1, z_2) f_{-\alpha,\alpha}(z_1, z_2)}$$

$$= \frac{\rho_{\alpha,-\alpha}(0)}{(2\pi)^2}\left[I_1^v(E, t) + I_2^v(E, t)\right] + \frac{\overline{\rho_{\alpha,-\alpha}(0)}}{(2\pi)^2} I_3^v(E, t).$$

To find the limit of $I_1^v(E, t)$ of (5.16) we change variables to $\zeta_1 = v^{-2}(z_1 - E_\alpha)$, $\zeta_2 = v^{-2}(z_2 - E_{-\alpha})$. This yields

$$(5.17) \qquad\qquad I_1^v(E, t) = \exp(-2\alpha sit) \, J_1^v,$$

where

$$J_1^v = \int_{L_2} d\zeta_2 \frac{e^{i\tau\zeta_2}}{\zeta_2 + f_\alpha(E_{-\alpha} + v^2\zeta_2)} \int_{L_1} d\zeta_1 \frac{e^{-i\tau\zeta_1}}{\zeta_1 + f_{-\alpha}(E_\alpha + v^2\zeta_1)},$$

and by (5.8)

$$(5.18) \quad \text{vH} - \lim J_1^v(E, t) = \int_{L_2} \frac{e^{i\tau\zeta_2} d\zeta_2}{\zeta_2 + f_0(E - 2\alpha s - i0)} \int_{L_1} \frac{e^{-i\tau\zeta_1} d\zeta_1}{\zeta_1 + f_0(E + 2\alpha s + i0)}$$
$$= (2\pi)^2 \exp\{i\tau((f_0(E + 2\alpha s + i0) - f_0(E - 2\alpha s - i0))\}.$$

We have similarly:

$$(5.19) \qquad\qquad\qquad I_2^v = \exp(-2\alpha sit)\, J_2^v,$$

$$J_2^v = \int_{L_1} d\zeta_1 \int_{L_2} d\zeta_2 e^{-i\tau\delta\zeta} \frac{v^4 f_{\alpha,-\alpha} f_{-\alpha,\alpha}}{1 - v^4 f_{\alpha,-\alpha} f_{-\alpha,\alpha}}$$
$$\times \frac{1}{\zeta_1 + f_{-\alpha}(E_\alpha + v^2\zeta_1)} \cdot \frac{1}{\zeta_2 + f_\alpha(E_{-\alpha} + v^2\zeta_2)}.$$

Here we denote $f_{\beta,\gamma} = f_{\beta,\gamma}(E_\alpha + v^2\zeta_1, E_{-\alpha} + v^2\zeta_2)$ (see (4.2)). By using the relations:

$$f_{\alpha,-\alpha} = \frac{\delta f_{\alpha,-\alpha}}{v^2(\delta\zeta + \delta f_{-\alpha,\alpha})}, \quad f_{-\alpha,\alpha} = \frac{\delta f_{-\alpha,\alpha}}{-4\alpha s + v^2(\delta\zeta + \delta f_{\alpha,-\alpha})},$$

where $\delta f_{\beta,\gamma} = f_\beta(E_\alpha + v^2\zeta_1) - f_\gamma(E_{-\alpha} + v^2\zeta_2)$, we obtain

$$(5.20) \quad J_2^v = \int_{L_2} d\zeta_2 \int_{L_1} d\zeta_1 e^{-i\tau\delta\zeta} \frac{v^2 f_{-\alpha,\alpha} \delta f_{\alpha,-\alpha}}{1 - v^4 f_{\alpha,-\alpha} f_{-\alpha,\alpha}} \cdot \frac{1}{\zeta_1 - \zeta_2 + \delta f_{-\alpha,\alpha}}$$
$$\times \frac{1}{\zeta_1 + f_{-\alpha}(E_\alpha + v^2\zeta_1)} \cdot \frac{1}{\zeta_2 + f_\alpha(E_{-\alpha} + v^2\zeta_2)}.$$

Notice that

$$|f_{\alpha,-\alpha}| \leq \frac{2c_0}{v^2(\eta_1 + \eta_2)}, \quad \alpha = \pm.$$

Hence

$$|1 - v^4 f_{\alpha,-\alpha} f_{\alpha,-\alpha}| \geq 1 - \left(\frac{2c_0}{(\eta_1 + \eta_2)}\right)^2 > \frac{1}{2}, \text{ if } \eta_j \geq 2c_0,$$

and integrand of (5.20) is uniformly bounded from above by integrable function $C(|\zeta_1 - \zeta_2||\zeta_1||\zeta_2|)^{-1}$. This allows us to pass to the limit in the integral in (5.20) and obtain that

$$(5.21) \qquad\qquad\qquad \text{vH} - \lim J_2^v(E, t) = 0.$$

Treating similarly the term $I_3^v(E, t)$ of (5.16) we obtain

$$(5.22) \qquad\qquad\qquad \text{vH} - \lim I_3^v(E, t) = 0.$$

Now assertion (5.6) of the theorem following from (5.16)-(5.22). $\square$

According to (5.6) the off-diagonal entry of the reduced density matrix in the van Hove limit does not vanish but just oscillates as const$\cdot e^{-2i\alpha st}$. The exponential that determines these fast oscillations (recall that $t \to \infty$) is the same as in the zero coupling ($\mathcal{S}_2$-isolated) limit of our model (1.5), where the reduced density matrix is

$$\rho_{\alpha\beta}(E_k, t)\,|_{v^2=0} = e^{-its(\alpha-\beta)} \rho_{\alpha\beta}(0),$$

hence is again const·$e^{-2i\alpha st}$ if $\alpha \neq \beta$, $(\alpha = -\beta)$.

In the case where the two-level system models a continuous quantum mechanical degree of freedom associated with a potential with two wells (see e.g. [**8**] for examples and discussion), the above oscillation reflects the phase coherence between the quantum mechanical amplitudes for being in the left and right wells, a pure quantum mechanical effect. In this case our result means that an environment, modeled by a random matrix, does not destroy the quantum mechanical coherence, at least in the weak coupling regime corresponding to the van Hove limit.

However, from the statistical mechanics point of view the absence of decay, moreover, fast oscillations, of the off-diagonal entries of the reduced density matrix seems not too natural. In this connection it worth noting that the fast ("microscopical") oscillating behaviour of $\rho_{\alpha\beta}$, $\alpha \neq \beta$ can be converted into a decaying behaviour by several modification of our initial setting.

One of them is to assume that the spacing $2s$ of our two-level system is random and continuously distributed, although concentrated around a certain $2s_0$. In other words, it is necessary to assume that the two-level system is the subject of a certain (even small) noise.

Another modification is to replace the van Hove limit

$$(5.23) \qquad \lim_{t\to\infty} \rho(E,t)\,|_{v^2=\tau/t}$$

by

$$(5.24) \qquad \lim_{t\to\infty,\triangle t\to\infty} (2\triangle t)^{-1} \int_{t-\triangle t}^{t+\triangle t} \rho(E,t\prime)\,|_{v^2=\tau/t\prime}\,dt\prime.$$

If $\triangle t = t$, we just replace the limit $t \to \infty$ by the Cesaro limit (time average limit), a rather often used procedure in statistical mechanics. However the off-diagonal entry vanishes even for $t \to \infty$, bat $\triangle t/t \to 0$, although with a smaller rate of decay. One can view this as an assumption on a sufficiently large (macroscopic) measurent time: $s^{-1} << \triangle t << t$.

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