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Ionization of Coulomb Systems in $\mathbb{R}^3$ by Time Periodic Forcings of Arbitrary Size

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Abstract: We analyze the long time behavior of solutions of the Schrödinger equation $i\psi_t = (-\Delta - b/r + V(t,x))\psi, x \in \mathbb{R}^3, r = |x|$, describing a Coulomb system subjected to a spatially compactly supported time periodic potential $V(t,x) = V(t + 2\pi/\omega, x)$ with zero time average.

We show that, for any $V(t,x)$ of the form $2\Omega(r)\sin(\omega t - \theta)$, with $\Omega(r)$ nonzero on its support, Floquet bound states do not exist. This implies that the system ionizes, *i.e.* $P(t,K) = \int_K |\psi(t,x)|^2 dx \to 0$ as $t \to \infty$ for any compact set $K \subset \mathbb{R}^3$. Furthermore, if the initial state is compactly supported and has only finitely many spherical harmonic modes, then $P(t,K)$ decays like $t^{-5/3}$ as $t \to \infty$.

To prove these statements, we develop a rigorous WKB theory for infinite systems of ordinary differential equations.

Contents

1. Introduction and Overview of Results . . . 682
   1.1 The Coulomb Hamiltonian . . . 683
   1.2 Setting . . . 683
   1.3 Ionization . . . 684
   1.4 Laplace space formulation . . . 684
   1.5 The homogeneous equation and the PDE-difference equations . . . 685
2. Main Results . . . 686
3. Proofs . . . 686
   3.1 The Hilbert space $\mathcal{H}$ . . . 687
   3.2 Proof of Theorem 2 . . . 687
   3.3 Step 1. Compact operator reformulation . . . 688
   3.4 Restriction to a ball B; Definition of $\mathfrak{C}$ . . . 689

3.5 Step 2. Regularity of $\mathfrak{R}_{\beta,l,m}$ at $p = 0$ and of $\mathfrak{C}_{l,m}$ at $p_1 = 0$ . . . 691
3.6 Compactness . . . 692
3.7 Step 3. The Fredholm alternative . . . 692
3.8 End of proof of Theorem 2 . . . 693
3.9 Proof of Theorem 3 . . . 693
3.10 Proof of ionization for spherically symmetric Ω, Theorem 1 . . . 693
3.11 Asymptotic behavior of v_n in (30) as $n \to -\infty$ . . . 694
4. Proofs of Intermediate Steps . . . 696
4.1 Proof of Proposition 8 . . . 696
4.2 Proof of Proposition 15 . . . 697
4.3 Proof of Proposition 11 . . . 698
4.4 Proof of Proposition 17 . . . 698
4.5 Proof of Proposition 19 . . . 699
4.6 Proof of Proposition 21 and final estimates for Theorem 1 . . . 700
4.7 Proof of Theorem 3 . . . 701
4.8 Connection with the Floquet operator . . . 703
4.9 Differential equation for w . . . 703
4.10 Proof of Proposition 23 . . . 703
4.11 Proof of Theorem 4 . . . 704
4.12 Further results on g_{n_0-k} and h_k . . . 705
4.13 Proof of Lemma 31 . . . 718
4.14 Proof of Lemma 33 . . . 720
4.15 Proof of Lemma 35 . . . 721
5. Appendix . . . 723
5.1 Short proof of the regularity of the unitary propagator . . . 723
5.2 Laplace transform of the Schrödinger equation . . . 724
5.3 Analyticity of $(I - \mathfrak{C}_{l,m})^{-1}$ in X . . . 725
5.4 Coulomb Green's function representation . . . 725
5.5 Dependence of A in Eq. (56) on Z, p . . . 727
5.6 Asymptotics of $w_2(a)$, $w_2'(a)$ for small λ . . . 727
5.7 Stationary phase analysis needed to calculate the ionization rate . . . 732
5.8 Calculation of j_k . . . 734
5.9 Generalizations . . . 735
5.10 Further remarks on the asymptotics . . . 736
Acknowledgements . . . 736
References . . . 737

1. Introduction and Overview of Results

The long time behavior of solutions of the Schrödinger equation of a system with both discrete and continuous spectrum subjected to a time periodic potential is a longstanding problem. Powerful results have been obtained under various assumptions on the potentials, see [5–8,21,32,34,36,37], and references therein. In particular, there are conditional results on the ionization of the Hydrogen atom, subjected to an external time-harmonic dipole field $V(t, x) = E \cdot x \cos \omega t$ if E is sufficiently small, see [43,44].

In addition, Möller and Skibsted proved the equivalence of absence of point spectrum and ionization for a large class of such systems subject to periodic fields [32]. There are also detailed results about the behavior of the wave function for systems subjected to general time periodic potentials, decaying faster than r^{-2}, under the additional assumption of absence of point spectrum of the Floquet operator, see [20].

None of these results however prove or disprove ionization of Coulomb–bound particles subject to time-periodic forcing of fixed amplitude and zero average. In fact, such results have only recently been obtained even for simple model systems, see [11–13,15,16,30] and references cited there. For a periodic dipole field of nonzero average ionization was proved in [33] (we note that the time averaged Hamiltonian has no bound states in this seetting).

What experiments and simplified models show is that the behavior of systems with both discrete and continuous spectrum, subject to time-periodic fields of arbitrary strength, can be very complicated. For amplitudes where perturbation theory is not applicable (such fields are becoming of increasing practical importance in technology), qualitative departures from the behavior at small fields are observed. There are even situations, see *e.g.* [12], where for small enough fields ionization occurs for all initial states while for larger fields there exist localized time–*quasiperiodic* solutions of the Schrödinger equation, *i.e.* Floquet bound states. Though these situations are rather exceptional, constructive methods of analysis are required to determine the outcome in specific settings.

In this paper we prove ionization for Coulomb systems with very special (non-dipole) type of forcings of arbitrary magnitude. This is equivalent to establishing the absence of point and singular continuous spectrum of the corresponding Floquet operators. We also obtain the large time behavior of the wave function. The time decay of the wave function, for compactly supported initial conditions, is of order $t^{-5/6}$. This differs from the $t^{-3/2}$ or, exceptionally, $t^{-1/2}$ power law found for shorter range reference potentials, see [15,20]. The nonperturbative methods include the development of rigorous WKB techniques for infinite systems of ODEs.

1.1. The Coulomb Hamiltonian. In units such that $\hbar^2/2m = 1$, the Coulomb quantum Hamiltonian of a Hydrogen atom (more generally a Rydberg atom) is

$$H_C = -\Delta - \frac{b}{r}, \tag{1}$$

where $b > 0$, $r = |x|$, $x \in \mathbb{R}^3$ and Δ is the Laplacian. It is well known, see e.g. [28], that H_C is self-adjoint on the Sobolev space $H^2(\mathbb{R}^3) = D(-\Delta)$, the domain of $-\Delta$ (cf. also [28], p. 303). The spectrum of H_C consists of isolated eigenvalues $E_n = -b^2/4n^2$, with multiplicity n^2, and an absolutely continuous part, $[0, \infty)$.

1.2. Setting. Our starting point is the time evolution of the wave function $\psi(t, x)$ of the Hydrogen atom described by the Schrödinger equation

$$i\psi_t = H_C\psi + V(t, x)\psi; \quad \psi(0, x) = \psi_0(x) \in H^2(\mathbb{R}^3), \tag{2}$$
$$\text{where } \; V(t, x) = \sum_{j\in\mathbb{Z}} \Omega_j(x)e^{ij\omega t} \text{ is real valued and } \Omega_0 \equiv 0.$$

The operator $H_C + V(t, x)$ satisfies the assumptions of Theorem X.71, p. 290, in [31] v.2.; Theorem X.70, p. 285 also applies in our setting. Thus, for any t, $\psi(t, \cdot) \in H^2(\mathbb{R}^3)$, and the unitary propagator $U(t)$ for (2) is strongly differentiable in t; see § 5.1 for a short proof in our case.

Assumption 1. *The $\Omega_j(x)$, $j \in \mathbb{Z}$ are smooth inside a common compact support, chosen without loss of generality to be the ball $\mathsf{B}_1 \subset \mathbb{R}^3$ of radius 1, and $\sum_{j\in\mathbb{Z}}(1 + |j|)\|\Omega_j\|_{L^\infty(\mathsf{B}_1)} < \infty$.*

1.3. Ionization. We say that the system *ionizes* if the probability to find the particle in any compact set vanishes for large t, *i.e.*, for any $a > 0$ we have

$$P(t, \mathsf{B_a}) = \int_{\mathsf{B_a}} |\psi(t, x)|^2 dx \to 0 \quad as \quad t \to \infty, \tag{3}$$

where $\mathsf{B_a} = \{x : |x| < a\}$. To prove ionization, it clearly suffices to prove (3) for all $a > 1$.

A simple way in which ionization may fail is the existence of a solution of the Schrödinger equation in the form

$$\psi(t, x) = e^{i\phi t} v(t, x) \text{ with } \phi \in \mathbb{R} \text{ and } v \in L^2([0, 2\pi/\omega] \times \mathbb{R}^3) \text{ time-periodic.} \tag{4}$$

Substitution in (2) leads to the equation:

$$Kv = \phi v, \tag{5}$$

where

$$K = i\frac{\partial}{\partial t} - \left(-\Delta - br^{-1} + V(t, x)\right) \tag{6}$$

is the Floquet operator, densely defined on $L^2([0, 2\pi/\omega] \times \mathbb{R}^3)$; $0 \neq v \in L^2$ implies by definition that $\phi \in \sigma_p(K)$, the point spectrum of K.

Somewhat surprisingly, in all studied systems, $\sigma_p(K) \neq \emptyset$ is in fact the *only* possibility for ionization to fail. As we will show this is also true for (2). The proof of ionization also implies that K does not have any singular continuous spectrum. This turns out to be a consequence of the existence of an underlying compact operator formulation, the operator being closely related to K. Generic ionization is then expected since L^2 solutions of the Schrödinger equation of the special form (4) are unlikely. We prove that for $V(t, x) = 2\Omega(r)\sin(\omega t - \theta)$, $\Omega > 0$ on $[0, 1]$ and sufficiently smooth, they do not exist.

1.4. Laplace space formulation. For $\psi \in H^2(\mathbb{R}^3)$, the Laplace transform

$$\hat{\psi}(p, \cdot) := \int_0^\infty \psi(t, \cdot)e^{-pt}dt$$

exists for $p \in \mathbb{H}$, the right half complex plane, and the map $p \to \hat{\psi}(p, \cdot)$ is H^2 valued analytic in Re $p > 0$. The Laplace transform converts the asymptotic problem (3) into an analytical one.

To improve the decay in p of the Laplace transform, it is convenient to write

$$\psi(t,x) = \psi_0(x)e^{-t} + y(t,x). \tag{7}$$

Now, $y(t,x)$ satisfies

$$iy_t - H_C y - V(t,x)y = e^{-t}\left[i\psi_0 + H_C\psi_0 + V(t,x)\psi_0\right] \equiv -y^0(t,x); \quad y(0,x) = 0. \tag{8}$$

Standard arguments (see Appendix 5.2) show that the $t-$Laplace transform of y, $\hat{y}$ is in H^2 and satisfies

$$(H_C - ip)\hat{y}(p,x) = \hat{y}^0(p,x) - \sum_{j\in\mathbb{Z}} \Omega_j(x)\hat{y}(p - ij\omega, x), \tag{9}$$

where

$$\hat{y}^0(p,x) = -\frac{1}{1+p}(i\psi_0 + H_C\psi_0) - \sum_{j\in\mathbb{Z}} \frac{\Omega_j(x)\psi_0(x)}{1+p-ij\omega}. \tag{10}$$

1.5. The homogeneous equation and the PDE-difference equations. The homogeneous system associated to (9) is

$$(-\Delta - b/r - ip)w(p,x) = -\sum_{j\in\mathbb{Z}} \Omega_j(x)w(p - ij\omega, x). \tag{11}$$

Note 2. (i) Clearly, (9) and (11) couple two values of p only if $(p_1 - p_2) \in i\omega\mathbb{Z}$, and are effectively infinite systems of partial differential equations. Setting

$$p = p_1 + in\omega, \text{ with } p_1 \in \mathbb{C} \operatorname{mod}(i\omega), \tag{12}$$

we denote

$$y_n(p_1,x) = \hat{y}(p_1 + in\omega, x),\ y_n^0(p_1,x) = \hat{y}^0(p_1 + in\omega, x),\ w_n(p_1,x) = w(p_1 + in\omega, x).$$

Equations (9) and (11) now become

$$(H_C - ip_1 + n\omega)y_n = y_n^0 - \sum_{j\in\mathbb{Z}} \Omega_j(x)y_{n-j}, \tag{13}$$

$$(H_C - ip_1 + n\omega)w_n = -\sum_{j\in\mathbb{Z}} \Omega_j(x)w_{n-j}. \tag{14}$$

Note 3. Seen as a differential difference equation, the solution $\hat{y}(p,x)$ is then a vector $\{y_n(p_1,x)\}_{n\in\mathbb{Z}}$ and the whole problem depends only parametrically on p_1. We have

$$y_n(p_1 + i\omega, x) = y_{n+1}(p_1,x), \tag{15}$$

and the analysis can be restricted to

$$\mathbb{S}_0 = \{p \in \overline{\mathbb{H}} : \operatorname{Im} p \in [0,\omega)\},$$

where $\overline{\mathbb{H}}$ is the closure of $\mathbb{H}$. There is arbitrariness in the choice of $\mathbb{S}_0$ and, to see analyticity in p_1 on $\partial\mathbb{S}_0$, it is convenient to allow $p_1 \in \overline{\mathbb{H}}$, using (15) to identify different strips of width ω.

2. Main Results

Theorem 1. *Assume $V(t,x) = 2\Omega(r)\sin(\omega t - \theta)$, with $\Omega(r) = 0$ for $r > 1$, $\Omega(r) > 0$ for $r \le 1$ and $\Omega(r) \in C^\infty[0,1]$. Then $\sigma_p(K) = \emptyset$ and ionization always occurs. Furthermore, if $\psi_0(x)$ is compactly supported and has only finitely many spherical harmonics, then $P(t, \mathsf{B}_\mathsf{a}) = O(t^{-5/3})$.*

For the proof, given in § 3.10, § 3.11 and § 4.6, we develop a relatively general rigorous WKB theory for infinite systems of differential equations. This yields the asymptotic behavior of w_n as $n \to -\infty$. The argument relies on Theorems 2 and 3 below.

Remark 4. The condition $\Omega(1^-) \ne 0$ simplifies the arguments but these could accommodate an algebraically vanishing Ω. (We also note that some one-dimensional models with rough Ω such as a δ mass show failure of ionization, see [12,30 and 35].)

We will later derive equivalent systems of integral equations, (22), allowing for a compact operator reformulation of the problem.

Theorem 2. *In the setting § 1.2, assuming spherical symmetry in x of the forcing $V(t,x)$, ionization occurs* iff *for all $p_1 \in \overline{\mathbb{H}}$, (14) has only zero H^2 solutions decaying in $n^{(1)}$. This is true* iff *$\sigma_p(K) = \emptyset$.*

This extends results about absence of singular continuous spectrum of K, [20], to this class of systems, with Coulombic potential and nonanalytic forcing.

The proof is given in § 3.2 and § 3.8.

Properties of Floquet bound states for general compactly supported $V(t,x)$.

Theorem 3. *If there exists an H^2 nonzero solution w of (14) decaying in n,[1] then it has the further property*

$$w_n = \chi_{\mathsf{B}_1} w_n \quad \text{for all } \; n < 0 \tag{16}$$

with χ_A the characteristic function of the set A.

The general idea of the proof is explained in § 3.9 and the details are given in § 4.7.

Note 5. (i) The Sobolev embedding theorem implies that w_n is continuous in x. From (14), w_n is piecewise C^2, implying continuity of ∇w_n up to $\partial \mathsf{B}_1$.

(ii) Equation (16) makes the second order system (14) formally overdetermined since the regularity of w in x imposes both Dirichlet and Neumann conditions on $\partial \mathsf{B}_1$ for $n < 0$. Nontrivial solutions are not, in general, expected to exist.

3. Proofs

Outline of the ideas. As in our previous work [10–15], summarized in [18] on simpler systems, we rely on a modified Fredholm theory to prove a dichotomy: there are bound Floquet states, or the system gets ionized. Mathematically the Coulomb potential introduces a number of substantial difficulties compared to the potentials considered before (for references, see e.g. [15]), due to its singular behavior at the origin and, more importantly, its very slow decay at infinity.

[1] For precise conditions, see §3.4 below and the integral form (22).

The slow decay translates into potential-specific corrections at infinity, and standard general methods to show compactness in weighted spaces of the Floquet resolvent, such as those in [20], or our previous ones do not apply. Instead, the asymptotic behavior in the far field of the resolvent has to be calculated in detail. The accumulation of eigenvalues of increasing multiplicity at the top of the discrete spectrum of H_C produces an essential singularity at zero of the Floquet resolvent with a local expansion of the form $\sum_{j,l} p^{l/2} e^{ijA(-ip)^{-1/2}}$ for small p when $\operatorname{Re} p \geq 0$, where $A = \pi b/2$. For sufficiently rapidly decaying potentials the exponentials would be absent. Their presence clearly makes the analysis at $p = 0$ of the Floquet resolvent more delicate and is responsible for the change in the large time asymptotic behavior of the wave function, from $t^{-3/2}$ to $t^{-5/6}$.

We introduce an extended parameter $X = (p^{1/2}, e^{iA(-ip)^{-1/2}})$ and prove analyticity of the solution $\hat{y}$ in X, whose p-counterpart is p small, $\operatorname{Re} p \geq 0$, and similarly in regions near the special points $p \in i\omega\mathbb{Z}$. We reformulate the problem in terms of an integral operator $\mathfrak{C}$, defined in § 3.4, closely related to the Floquet resolvent, shown to be compact in a suitable space and analytic in a variable corresponding to X. Then, by the Fredholm alternative, $(I - \mathfrak{C})^{-1}$ is meromorphic, and in fact analytic in X, since we show absence of eigenvalues of $I - \mathfrak{C}$ for any $p \in \overline{\mathbb{H}}$.

3.1. The Hilbert space $\mathcal{H}$. Let $\mathcal{H}$ be the Hilbert space of sequences $Y = \{y_n\}_{n\in\mathbb{Z}}$, $y_n \in L^2(\mathsf{B}_a)$, with $a > 1$, and with

$$\|Y\|^2 := \|Y\|_a^2 = \sum_{n\in\mathbb{Z}} (1 + |n|)^{4/3} \|y_n\|^2_{L^2(\mathsf{B}_a)} < \infty$$

Note 6. The properties of $(I - \mathfrak{C})^{-1}$ as $\operatorname{Re} p_1 \to 0^+$ ensure that $Y(p_1, \cdot) \in \mathcal{H} \cup H^2$ and is locally integrable in p_1 along $i\mathbb{R}$.

We then extend the stationary phase method to such a setting, cf. § 5.7, to evaluate, asymptotically for large t, the inverse Laplace transform of $\hat{y}$ on $i\mathbb{R}$ and obtain the ionization result and time decay estimates.

To show ionization we then have to rule out the existence of a point spectrum of the Floquet operator, that is the existence of nontrivial solutions of (14). We use the general criterion in Theorem 3 to show that, if there exists a nonzero solution to (14), then a subsequence of $\{w_n\}_{n\in-\mathbb{N}}$ would be singular at $x = 0$, in contradiction with Note 5, (i).

To find the behavior of solutions for large n, we develop a WKB theory for infinite systems of ODEs and find the asymptotic behavior of w_n in n in detail. The formal WKB calculation of the behavior is straightforward algebra, relatively easy even in much more general settings, see § 5.9. Justifying the procedure is however delicate, and a good part of the paper is devoted to that; cf. § 4.11, § 4.12.

The procedure of introducing an enlarged set of parameters with respect to which the solution is regular, when this does not hold in the original parameter, should also be applicable to other problems where complicated singularities arise.

3.2. Proof of Theorem 2. We show that $\hat{y}$ has a limit in L^1_{loc} on $\partial\mathbb{H} = i\mathbb{R}$, where it is smooth except for possible poles and a discrete set of essential (but L^1) singularities. Poles are present iff the integral form (22) of (11) has nontrivial solutions in $\mathcal{H}$. There is sufficient decay in p at infinity, so that, when poles are absent, the Riemann-Lebesgue lemma applies, implying that y decays as $t \to \infty$, proving ionization–since $\psi_0(x)e^{-t}$ obviously goes to zero in this limit. More detailed analysis of the resolvent reveals the nature of the essential singularity at $p = i\omega\mathbb{Z}$. Stationary phase analysis shows a $t^{-5/6}$ decay of the wave function if the initial condition is spatially compactly supported and contains only a finite number of spherical harmonics.

Proposition 7. *Ionization holds for every $\psi_0 \in L^2$ iff it holds for any ψ_0 in a set densely spanning L^2.*

Proof. We make use of the standard triangle inequality argument to estimate $U(t)\psi_0$, where $U(t)$ is the unitary operator associated to the Schrödinger evolution (2). □

We choose ψ_0 in a dense set $C_c^\infty(\mathbb{R}^3)$, the smooth, compactly supported functions in $\mathbb{R}^3$. Define as usual the angular momentum operators

$$-\mathbf{L}^2 = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and

$$L_z = -i\frac{\partial}{\partial\phi}.$$

Let $P_{l,m}$ be the orthogonal projector on $\{\phi : \mathbf{L}^2\phi = l(l+1)\phi,\ L_z\phi = m\phi\}$ for some $m \in \mathbb{Z}$, $|m| \le l \in \mathbb{N} \cup \{0\}$.

Since $\sum_{l,m} P_{l,m} = I$, we can now assume without loss of generality that $\psi_0 \in P_{l,m}\left(C_c^\infty(\mathbb{R}^3)\right)$ if l and m are arbitrary. Likewise, if $P(t, \mathsf{B_a})$ decays like $t^{-5/3}$ when $\psi_0 \in P_{l,m}\left(C_c^\infty(\mathbb{R}^3)\right)$, then the same decay rate clearly holds for any ψ_0 given by a finite linear combination over (l, m) (but not, in general, for any $\psi_0 \in L^2(\mathbb{R}^3)$).

Further notations. As usual we write $\mathbb{D}_\epsilon = \{z : |z| < \epsilon\}$, $\mathbb{D} = \mathbb{D}_1$ and we denote $\mathbb{D}_\epsilon^+ = \mathbb{D}_\epsilon \cap \{z : \arg z \in (-\pi/4, \pi/4)\}$. We also let $\mathcal{I}_\epsilon = i[-\epsilon, 0]$, $\mathbb{H}^{+c} = \{p+c : p \in \mathbb{H}\}$, $\ell_\alpha = \{p : \mathrm{Re}(p) \ge 0, \mathrm{Im}(p) = \alpha\}$ and for a set A, $A_{\backslash \ell_\alpha} = A \backslash \ell_\alpha$. We denote $\mathcal{D} = \mathbb{H} \cup i\mathbb{R}^+$, and $\mathcal{O}(\mathcal{D})$ will denote *some* small open neighborhood of $\mathcal{D}$.

3.3. Step 1. Compact operator reformulation. To investigate the analytic properties of $\hat{\psi}$ it is convenient to introduce a new operator $\mathcal{A}_\beta$ which is a complex perturbation of H_C, having no real eigenvalues. More precisely, define

$$\mathcal{A}_\beta := H_C - i\beta(p)\chi_{\mathsf{B}_a}(r) - ip; \quad \text{(with the understanding that } \mathcal{A}_0 = H_C - ip), \quad (17)$$

where $a > 1$ and

$$\beta = \beta(p) = \begin{cases} c > 0 & \text{if } \mathrm{Im}\, p \in [-\epsilon, p_c] \text{ and } \mathrm{Re}\, p \ge 0 \\ 0 & \text{otherwise} \end{cases}. \quad (18)$$

Here $\epsilon < \omega/2$ is small as required in Proposition 17 below, and we choose p_c so that $p_c/\omega \notin \mathbb{Z}$ and $p_c > -E_0 = b^2/4$, the ground state energy of the unperturbed atom.

Clearly $\mathcal{A}_\beta$ is defined on $D(H_0)$ and $\mathcal{A}_\beta^* = \mathcal{A}_{-\beta} + ip^* + ip$. We rewrite (13) and (14) in the equivalent form

$$\left[H_C - ip_1 + n\omega - i\beta(p)\chi_{\mathsf{B}_a}(r)\right] y_n = y_n^0 - i\beta(p)\chi_{\mathsf{B}_a}(r) y_n - \sum_{j\in\mathbb{Z}} \Omega_j(x) y_{n-j}, \quad (19)$$

$$\left[H_C - ip_1 + n\omega - i\beta(p)\chi_{\mathsf{B}_a}(r)\right] w_n = -i\beta(p)\chi_{\mathsf{B}_a}(r) w_n - \sum_{j\in\mathbb{Z}} \Omega_j(x) w_{n-j}. \quad (20)$$

We show next that $\mathcal{A}_\beta^{-1}$ is analytic in $p \in \mathbb{H}\backslash\{\ell_{p_c} \cup \ell_{-\epsilon}\}$, and sufficiently regular on $i\mathbb{R}$. Since the parameter p_c is artificial, the non-analyticity at $\ell_{p_c} \cup \ell_{-\epsilon}$ of $\mathcal{A}_\beta^{-1}$ is not reflected in the actual solution $\hat{y}$, as discussed in Note 9.

Proposition 8. *There exists an open neighborhood $\mathcal{O}$ of $\mathcal{D}\backslash\left\{\ell_{p_c} \cup \ell_{-\epsilon}\right\}$, not containing the origin 0, such that the operator $\mathfrak{R}_\beta = \mathcal{A}_\beta^{-1}$ exists and is analytic in $p \in \mathcal{O}\backslash(\ell_{p_c} \cup \ell_{-\epsilon})$. Furthermore, for any p for which $\mathfrak{R}_\beta$ exists, we have $\mathfrak{R}_\beta : L^2(\mathbb{R}^3) \to H^2\left(\mathbb{R}^3\right)$.*

The proof is given in § 4.1.

3.4. Restriction to a ball B*; Definition of* $\mathfrak{C}$. To study ionization, we only need to know $y(t,x)$ for x in a fixed (but arbitrary) ball $\mathsf{B}_a \supset \mathsf{B}_1$. Henceforth, to simplify the notation, we write $\mathsf{B}_a = \mathsf{B}$. We shall therefore need to study the properties of $\chi_\mathsf{B}\mathfrak{R}_\beta\chi_\mathsf{B}$. This sandwiched operator (which preserves information about $L^2(\mathbb{R}^3)$ through built-in boundary conditions on $\partial\mathsf{B}$) is the one that we shall most often use below. We recall that $p = p_1 + in\omega$ and $\hat{y}(p_1 + in\omega, x) = y_n(p_1, x)$. Since $\Omega_j(x) = \chi_\mathsf{B}\Omega_j(x)$, (13) implies that for $x \in \mathsf{B}$,

$$y_n(p_1,x) = \chi_\mathsf{B}\mathfrak{R}_\beta y_n^0 + \chi_\mathsf{B}\mathfrak{R}_\beta\chi_\mathsf{B}\left[-i\beta y_n(p_1,x) - \sum_{j\in\mathbb{Z}}\Omega_j(x) y_{n-j}(p_1,x)\right], \quad (21)$$

where we may assume that B contains the support of $\psi_0(x)$, and therefore of y_n^0. Note that $\mathfrak{R}_\beta$ depends on n through $p = in\omega + p_1$. Corresponding to (21), we obtain the homogeneous system:

$$w_n(p_1,x) = \chi_\mathsf{B}\mathfrak{R}_\beta\chi_\mathsf{B}\left[-i\beta w_n(p_1,x) - \sum_{j\in\mathbb{Z}}\Omega_j(x) w_{n-j}(p_1,x)\right]. \quad (22)$$

The elements of $\mathcal{H}$ will be denoted by capital letters, *e.g.* $\{y_n\}_{n\in\mathbb{Z}} =: Y$, $\left\{y_n^0\right\}_{n\in\mathbb{Z}} =: Y_0$. We define the operators $\mathfrak{T}$ on $L^2(\mathsf{B})$ by

$$\{\mathfrak{T}Y_0\}_n = \chi_\mathsf{B}\mathfrak{R}_\beta y_n^0,$$

and $\mathfrak{C}$ on $\mathcal{H}$ by

$$\{\mathfrak{C}Y\}_n = \chi_\mathsf{B}\mathfrak{R}_\beta\chi_\mathsf{B}\left[-i\beta y_n(p_1,x) - \sum_{j\in\mathbb{Z}}\Omega_j(x) y_{n-j}(p_1,x)\right].$$

Then, we rewrite (21) in the form

$$Y = \mathfrak{T}Y_0 + \mathfrak{C}Y. \quad (23)$$

Note 9. We shall see that for any β satisfying (18), Eq. (23) has a unique solution in $\mathbb{H}$ (call it now $Y^{(\beta)}$). Thus, away from the artificial cuts, all these solutions coincide (since the domain of $Y^{(0)}$ corresponding to $c = 0$, contains all of the others). Hence, wherever some $Y^{(\beta)}$ has analytic continuation, so will $Y^{(0)}$.

The homogeneous system corresponding to (23) is given by

$$w = \mathfrak{C}w. \tag{24}$$

Note 10. We have shown (cf. § 1.4 and § 5.2) that $\hat{\psi}$ and the Laplace transform $\hat{y}(p, \cdot) = \mathcal{L}\left(\psi - \psi_0 e^{-t}\right)$ exist for $\mathrm{Re}\, p > 0$. The corresponding $Y = \{y_n\}$, $y_n = \hat{y}(in\omega + p_1, \cdot)$, restricted to B, will therefore satisfy (23) for $\beta = 0$ when $\mathrm{Re}\, p_1 > 0$. It will be shown that (23) has a unique solution $Y \in \mathcal{H}$ for any $\mathrm{Re}\, p_1 \geq 0$, and that Y is analytic in $p_1 \in \mathbb{H}$ and has an L^1_{loc} limit on $i\mathbb{R}$, with sufficient decay in n. The implied decay and regularity properties of $\hat{y}(p, \cdot)$ on $i\mathbb{R}$ show that $\mathcal{L}^{-1}\hat{y} + e^{-t}\psi_0$ (the integration contour taken to be $i\mathbb{R}$) equals ψ for $x \in \mathsf{B}$.

Proposition 11. (Asymptotic behavior of $\chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}}$). *If* $\mathrm{Re}\, p = 0$ *and* $|\mathrm{Im}\, p| \to \infty$ *(see Note 3), then* $\|\chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}}\| = O(|p|^{-1/2})$ (*recall that* $\beta = 0$ *if* $|\mathrm{Im}\, p|$ *is large*). *Moreover, for any* $\epsilon > 0$, $\chi_B\mathfrak{R}_\beta\chi_B$ *is analytic in* p *in an open set containing* $-i(\epsilon, \infty)$.

For $\mathrm{Im}\, p \to +\infty$, the $|p|^{-1/2}$ decay rate follows from the spectral theorem since we are outside the spectrum, while for $\mathrm{Im}\, p \to -\infty$, the rate is obtained using Mourre estimates [27], Theorem 6.1. The rest of the proof involving analyticity is given in § 4.3 and relies on an explicit representation of the resolvent for H_C, see § 5.4. Using spherical symmetry, the explicit Green's function could be avoided, but in view of possible future generalizations to non-spherical $V(t, x)$, we prefer this more delicate approach.

Lemma 12. *We have* $\mathfrak{T}Y_0 \in \mathcal{H}$. *The operators* $\mathfrak{S} := Y \to \{\sum_{j\in\mathbb{Z}} \Omega_j(x) y_{n-j}(p_1, x)\}_{n\in\mathbb{Z}}$ *and* $\mathfrak{C}$ *are bounded in* $\mathcal{H}$.

Proof. We note from Proposition 11 that $\mathfrak{R}_\beta = O\left(|p|^{-1/2}\right)$ for large p, *i.e.* $O\left(|n|^{-1/2}\right)$ for large $|n|$, since $p = in\omega + p_1$. Therefore, from the expression of $\hat{y}_n^0$ in (10),

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} (1 + |n|)^{4/3} \| \{\mathfrak{T}Y_0\}_n \|^2 \\
&\quad \leq \sum_{n\in\mathbb{Z}} (1 + |n|)^{4/3} \left(\|\chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}}\| \left[\frac{\|\psi_0\|_{L^2} + \|\Delta\psi_0\|_{L^2}}{|1 + p_1 + in\omega|} \right] \right. \\
&\qquad \left. + \|\chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}}\| \|\psi_0\|_{L^2} \sum_{j\in\mathbb{Z}} \frac{\|\Omega_j\|_{L^2}}{|1 + p_1 + i(n - j)\omega|} \right)^2 \\
&\quad \leq C \left[1 + \sum_{n\in\mathbb{Z}} (1 + |n|)^{6/7} \left(\sum_{j\in\mathbb{Z}} \frac{\|\Omega_j\|_{L^2}}{(1 + |n - j|)} \right)^2 \right].
\end{aligned}$$

Using (7.12) and (7.13) in [15] with $\gamma = 6/7$, the above is finite since

$$\sum_{j\in\mathbb{Z}} (1 + |j|)^{3/7} \|\Omega_j\|_{L^2} \leq \sum_{j\in\mathbb{Z}} (1 + |j|) \|\Omega_j\|_{L^2} < \infty.$$

The proof that $\mathfrak{S}$ is the same as that of Lemma 27 of [15], with $\gamma = \frac{4}{3}$, replacing absolute values by norms in x. Since $\mathfrak{R}_\beta$ is uniformly bounded (in the operator norm) and acts diagonally in n, $\mathfrak{C}$ is bounded too. □

Lemma 13. *Both $\mathfrak{T}Y_0$ and the operator $\mathfrak{C}$ are analytic in p_1 for*

$$p_1 \in \mathcal{O}\left(\overline{\mathbb{H}}\backslash\left(\{\ell_{p_c} + i\omega\mathbb{Z}\} \cup \{\ell_{-\epsilon} + i\omega\mathbb{Z}\} \cup \{\mathcal{I}_\epsilon + i\omega\mathbb{Z}\}\right)\right).$$

Proof. Propositions 8 and 11 imply that $\mathfrak{R}_\beta$ is analytic in $p \in \mathcal{O}\backslash\{\ell_{p_c} \cup \ell_{-\epsilon}\}$ and in an open set containing $-i(\epsilon, \infty)$. Analyticity of $\mathfrak{T}Y_0$ and $\mathfrak{C}$ follow from their definition (we note $\mathfrak{C}$ is a norm limit of analytic operators: its restrictions to the subspaces with nonzero components for $|n| \le N$ only). □

Remark 14. As shown later, $(I - \mathfrak{C})$ is invertible. Since the solution Y cannot depend on the arbitrary parameters ϵ and p_c (see Note 9), the non-analyticity of $\mathfrak{C}$ and $\mathfrak{T}Y_0$ for

$$p_1 \in \{\ell_{p_c} + i\omega\mathbb{Z}\} \cup \{\ell_{-\epsilon} + i\omega\mathbb{Z}\} \cup \{\mathcal{I}_\epsilon + i\omega\mathbb{Z}\}$$

is not reflected in Y.

Proposition 15. *For* Re $p_1 > 0$ *large enough,* (23) *has a unique solution in $\mathcal{H}$. The inverse Laplace transform in p of $\hat{y}(p, x) =: y_n(p_1, x)$, where $p = in\omega + p_1$, solves the initial value problem* (8) *in* B (*see Note 10*).

The proof is given in § 4.2.

3.5. Step 2. Regularity of $\mathfrak{R}_{\beta,l,m}$ at $p = 0$ and of $\mathfrak{C}_{l,m}$ at $p_1 = 0$. Define $\mathfrak{R}_{\beta,l,m} = P_{l,m}\mathfrak{R}_\beta$ and $\mathfrak{C}_{l,m} = P_{l,m}\mathfrak{C}$.

Note 16. (Compactness versus regularity of $\mathfrak{R}_{\beta,l,m}$). The term $-i\beta\chi_{\mathsf{B}}$ was introduced in § 3.2 to ensure that $\mathfrak{R}_{\beta,l,m}$ is bounded in $\overline{\mathbb{H}}$. Since $-i\beta\chi_{\mathsf{B}}$ is localized in x, the shifts in the poles created by the point spectrum of H_C are smaller as $p \to 0$ (the size of the orbitals of the Hydrogen atom grows when the energy approaches zero.) The resulting integral operators have an essential singularity at $p = 0$. The factor χ_{B} is needed to ensure compactness, simplifying the analysis.

The poles of the resolvent $\mathfrak{R}_{\beta,l,m}$ accumulate at $p = 0$ from $-\mathbb{H}$, along a curve tangent to the positive imaginary p-axis (see Note 58 in §5.6). As a result, while being uniformly bounded, $\mathfrak{R}_{\beta,l,m}$ is not continuous along the imaginary p line at zero but oscillates without limit. Boundedness of $\chi_{\mathsf{B}}\mathfrak{R}_{\beta,l,m}\chi_{\mathsf{B}}$ (which is not difficult to prove) does not ensure boundedness of the solution Y. However, we do have analyticity in an extended, two-dimensional, parameter. Let $\lambda := \sqrt{-ip}$ (with the usual branch of the square root, Im$\lambda < 0$ if $p \in \mathbb{H}$) and let $X := (p^{1/2}, Z)$ with

$$Z = e^{\frac{i\pi b}{2\lambda}}. \tag{25}$$

(The dependence of Z on λ reflects the actual behavior of the solution.) The resolvent is analytic in X and a useful Fredholm alternative can be applied.

For any $a > 1$ we can choose a c in (18) (see § 4.4 below) such that the following statement holds.

Proposition 17 (Analyticity in X). *$\chi_{\mathsf{B}}\mathfrak{R}_{\beta,l,m}\chi_{\mathsf{B}}$ is analytic with respect to X on the compact set $\overline{\mathbb{D}_\epsilon^+}\times\overline{\mathbb{D}}$.*[2]

The proof is given in §4.4.

As a corollary, we have the following regularity property of $\mathfrak{C}_{l,m}$ and $\mathfrak{T}_{l,m}Y_0$. Let

$$\mathbb{S}_{-\epsilon} = \{p : \operatorname{Im} p \in [-\epsilon, \omega-\epsilon), \quad \operatorname{Re} p \geq 0\}. \tag{26}$$

Corollary 18. *For $p_1\in\mathbb{S}_{-\epsilon}$, define $X_1=\left(p_1^{1/2}, Z_1\right)$, where $Z_1=\exp\left[i\pi b/\left(2\sqrt{-ip_1}\right)\right]$. Then, $\mathfrak{T}_{l,m}Y_0$ and $\mathfrak{C}_{l,m}$ are analytic in X_1 on the compact set $\overline{\mathbb{D}_\epsilon^+}\times\overline{\mathbb{D}}$.*

Proof. Note first that Propositions 8 and 11 and the relative arbitrariness in the choice of p_c and ϵ imply that $\chi_B\mathfrak{R}_\beta\chi_B$ is analytic in p in a neighborhood of $p = in\omega$, $n\in\mathbb{Z}\backslash\{0\}$. Since for large $|n|$, $\chi_B\mathfrak{R}_\beta\chi_B = \chi_B\mathfrak{R}_0\chi_B$, its expression as an integral operator involving G in (49) (see Note (28) as well) implies a lower bound of the analyticity radius independent of n. For sufficiently small ϵ for any $n\in\mathbb{Z}$, including $n=0$, then, analyticity of $\chi_B\mathfrak{R}_\beta\chi_B$ in the expanded variable

$$\left(\sqrt{p-in\omega}, \exp\left[\frac{i\pi b}{(2\sqrt{-i(p-in\omega)})}\right]\right)$$

follows in the domain

$$\left\{|p-in\omega|^{1/2}\leq\epsilon;\ \left|\exp\left[\frac{i\pi b}{(2\sqrt{-i(p-in\omega)})}\right]\right|\leq 1\right\}$$

(since Proposition 17 gives analyticity $X\in\overline{\mathbb{D}_\epsilon^+}\times\mathbb{D}$). Analyticity of $\mathfrak{C}_{l,m}$ in $X_1\in\overline{\mathbb{D}_\epsilon^+}\times\overline{\mathbb{D}}$ now follows since $\mathfrak{C}_{l,m}$ is the norm limit of analytic operators (the restrictions of $\mathfrak{C}_{l,m}$ to the subspaces of $\mathcal{H}$ with zero components for $|n|>N$). (See Proposition 11 for the necessary estimates of decay in N.) The analyticity of $\mathfrak{T}_{l,m}Y_0$ follows from its definition. □

3.6. Compactness.

Proposition 19. *$\mathfrak{C}_{l,m}$ is compact in $\mathcal{H}$ (cf. Note 2) for $p_1\in\overline{\mathbb{H}}$.*

The proof is given in §4.5.

3.7. Step 3. The Fredholm alternative. We can now formulate the ionization condition using the Fredholm alternative.

Proposition 20. *If (24) has no nontrivial solution in $\mathcal{H}$ for $p_1\in\mathbb{S}_{-\epsilon}$, then $\left(I-\mathfrak{C}_{l,m}\right)^{-1}$ exists and the system ionizes (cf. (3)).*

---

[2] As usual, by analyticity in a compact set, we mean analyticity in some open set containing the compact set. Analyticity in $\overline{\mathbb{D}_\epsilon^+}\times\overline{\mathbb{D}}$ of course, implies that $\chi_{\mathsf{B}}\mathfrak{R}_{\beta,l,m}\chi_{\mathsf{B}}$ is given by a convergent double series in $p^{1/2}$ and $e^{\frac{i\pi b}{2\lambda}}$.

The first part is simply the Fredholm alternative. Ionization follows from the following proposition. We recall that $\hat{y}(p_1 + in\omega, x) = y_n$ are the components of Y.

Proposition 21. *Assume (24) has no nontrivial solution when $p_1 \in \overline{\mathbb{H}}$. Then, for $\psi_0 \in P_{l,m}\left(C_0^\infty(\mathbb{R}^3)\right)$, the solution $Y \in \mathcal{H}$ to (23) is analytic in $p_1 \in \overline{\mathbb{H}}\backslash\{i\omega\mathbb{Z}\}$ and analytic with respect to X_1 in $\overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$. In particular Y is bounded at $p_1 = 0$. These properties imply sufficient regularity and decay of $\hat{y}(p, x)$ so that the integration contour in $\mathcal{L}^{-1}\hat{y}$ can be taken to be $i\mathbb{R}$. By the Riemann-Lebesgue lemma, $P(t, \mathsf{B}) \to 0$ as $t \to \infty$.*

The proof is given in § 4.6.

3.8. End of proof of Theorem 2. It only remains to make the connection with Floquet theory. This is done in § 4.8.

3.9. Proof of Theorem 3. Equation (14), restricted to B, follows from the homogeneous system $w = \mathfrak{C}w$. Multiplying (14) by $\overline{w_n}$, summing over n, and integrating over $\mathsf{B}_{\tilde{a}}$, where $\tilde{a} \in (1, a]$, we are lead to a nonnegative definite quantity involving $w_n|_{\partial \mathsf{B}_{\tilde{a}}}$ being zero for $n < 0$. Details are given in § 4.7.

3.10. Proof of ionization for spherically symmetric Ω, Theorem 1. We consider the case $V(t, x) = 2\Omega(r)\sin\omega t$, corresponding to $\Omega_1 = -i\Omega$ and $\Omega_{-1} = i\Omega$ (Ω is real valued). The proof in the slightly more general case $2\Omega(r)\sin(\omega t - \theta)$ amounts to replacing t by $t - \theta/\omega$ and $\psi_0(x)$ by $\psi(\theta/\omega, x)$ in our proof. Recall $\mathfrak{C}_{l,m} = P_{l,m}\mathfrak{C}$.[3] We obtain by projection of (23) to $P_{l,m}\left(L^2(\mathbb{R}^3)\right)$,

$$Y = Y_0 + \mathfrak{C}_{lm}Y. \tag{27}$$

The homogeneous equation associated to (27) is

$$w = \mathfrak{C}_{lm}w, \;\; w \in \mathcal{H}. \tag{28}$$

The Fredholm alternative applies and (27) has a unique solution in $\mathcal{H}$ iff (28) implies $w = 0$.

Note 22. By separation of variables in spherical coordinates, we see that $\mathfrak{C}_{l,m}$ can be defined in the same way as $\mathfrak{C}$, replacing $\mathcal{A}_\beta$ in (17) by

$$\mathcal{A}_{\beta,r} = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} - ip_1 + n\omega - i\beta\chi_{\mathsf{B}} \tag{29}$$

and the associated differential-difference systems are obtained by replacing $-\Delta - b/r$ with $-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r}$.

Clearly if there exists a nontrivial solution $w \in \mathcal{H}$ of (28), then, again by elliptic regularity (see Proposition 8), v defined by $w = \mathcal{Y}_{l,m}v(r)$, where $\mathcal{Y}_{l,m}$ are the spherical harmonics, is a nontrivial solution to

$$\mathcal{A}_{\beta,r}v_n = -i\Omega\,(v_{n+1} - v_{n-1}) - i\beta\chi_B v_n \qquad \text{implying } \; \mathcal{A}_{0,r}v_n = -i\Omega\,(v_{n+1} - v_{n-1}) \tag{30}$$

[3] As discussed, it suffices to show ionization on a dense subset of initial conditions.

Proposition 23. *If v satisfies (30), then there exists $n \geq 0$ such that either* (**i**) $v_n(1) \neq 0$*; or* (**ii**) $v_n(1) = 0$*, but* $v_n'(1) \neq 0$*; let n_0 be the smallest such n. By homogeneity, we can assume that* $v_{n_0}(1) = 1$ *in case* (**i**) *and* $v_{n_0}'(1) = -\sqrt{\Omega(1)}$ *in case* (**ii**) (*we use the positivity of* Ω).

The proof is given in § 4.10.

Definition. *We define τ to be* 0 *or* 1 *in case* ***(i)*** *and* 1 *in case* ***(ii)*** *respectively.*

3.11. Asymptotic behavior of v_n in (30) as $n \to -\infty$. In view of Proposition 21 we see that (30) holds the necessary ionization information.

3.11.1. Notation. Let

$$\mathfrak{s}(r) := \int_r^1 \sqrt{\Omega(\rho)}d\rho \quad (r \in (0,1)). \tag{31}$$

By assumption $\Omega > 0$ is smooth and then so is $\mathfrak{s}$. Let

$$n_0 - k,\ \Omega_0 = \Omega(0),\ \Omega_0' = \Omega'(0),\ \mathfrak{s}_0 = \mathfrak{s}(0),\ \alpha = \frac{2\sqrt{\Omega_0}}{\mathfrak{s}_0},\ \zeta = \alpha k r. \tag{32}$$

Denote

$$H_0(\zeta) := \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+1/2}(\zeta); \quad G_0(\zeta) = \sqrt{\frac{\pi}{2}} e^{\zeta} \zeta^{1/2} I_{l+1/2}(\zeta),$$

where $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions of order $l + 1/2$. It follows that for small ζ,

$$H_0(\zeta) \sim 2^{-l} \zeta^{-l} (2l)!/l!.$$

Let $\check{H}(\zeta; k, l)$ be the unique solution of the integral equation

$$\begin{aligned}\check{H}(\zeta; k, l) = G_0(\zeta) &\int_0^{\zeta} e^{-2s} G_0(s) \mathcal{R}(H_0 + k^{-1}\check{H})(s) ds \\ -H_0(\zeta) &\int_{k\alpha}^{\zeta} e^{-2s} H_0(s) \mathcal{R}(H_0 + k^{-1}\check{H})(s) ds\end{aligned} \tag{33}$$

for $\zeta \in [0, k\alpha]$, where the operator $\mathcal{R}$ is defined by

$$(\mathcal{R}f)(\zeta) = 2\left(-\frac{\omega}{2\alpha^2} + \frac{\Omega_0'(1+2\zeta)}{4\alpha\Omega_0} + \frac{\tau}{2}\right) f' - \frac{bf}{\alpha\zeta}.$$

Define

$$H(\zeta) = H(\zeta; k, l) := H_0(\zeta) + k^{-1}\check{H}(\zeta; k, l).$$

It can be checked that H satisfies

$$H'' = 2\left(1 - \frac{\omega}{2k\alpha^2} + \frac{\Omega_0'(1+2\zeta)}{4k\alpha\Omega_0} + \frac{\tau}{2k}\right) H' + \left(\frac{l(l+1)}{\zeta^2} - \frac{b}{\alpha\zeta k}\right) H \tag{34}$$

with the following asymptotic condition[4]

$$H(\zeta) \sim 1 + \frac{l(l+1)}{2\zeta} + \frac{b}{2k\alpha}\log\zeta + O\left(\frac{\log\zeta}{k\zeta}, \frac{1}{\zeta^2}\right) (\zeta, k \to \infty, \zeta \leqslant k\alpha). \tag{35}$$

Remark 24. (i) $H(\zeta; k, l) \sim \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+1/2}(\zeta)(1 + o(1))$ as $k \to \infty$

(ii) From the expression (33) for $\check{H}$ it is seen that as $\zeta \to 0$ we have $\check{H}(\zeta; k, l) \sim \text{const.}\zeta^{-l+1}$ for $l \neq 1$ and $\check{H} \sim$ const. + const $\zeta \log\zeta$ for $l = 1$. For $\tau = 0$ or 1, $\check{H}$ is less singular than H_0 at $\zeta = 0$.

Define

$$m_k(r) = \frac{\mathfrak{s}^{2k+\tau}\Omega^{\frac{1}{4}}(1)H(\alpha kr)}{(2k+\tau)!\Omega^{\frac{1}{4}}(r)H(\alpha k)} \exp\left[\frac{\omega}{4}\int_1^r \frac{\mathfrak{s}(s)ds}{\sqrt{\Omega(s)}}\right] := \frac{\mathfrak{s}^{2k+\tau}}{(2k+\tau)!} F_k(r). \tag{36}$$

Note 25. From standard properties of the modified Bessel function $K_{l+1/2}$, it follows that for large enough k, $H(\alpha kr)$ is continuous and nonzero for $r \in (0, 1]$ and that as $r \to 0$, $H(\alpha kr)$ is singular as r^{-l}. Therefore for any k sufficiently large $u_k \equiv r^l m_k$ has a finite limit nonzero limit as $r \to 0^+$.

Definition 26. *With n_0 as in Proposition 23, we define $h_k(r)$ by*

$$w_{n_0-k} = \frac{i^k}{r} m_k(r) h_k(r) \mathcal{Y}_{l,m}. \tag{37}$$

Theorem 4. (Behavior as $k \to +\infty$ (*i.e.* $n \to -\infty$)) *For any sufficiently large k, $u_k := r^l m_k(r)$ is continuous in $r \in (0, 1]$ and $u_k(r) \to const \neq 0$ as $r \to 0^+$. Furthermore, if there is a nontrivial solution to* (30)*, then there exists a subsequence $k_j \to \infty$ such that for any $r \in [0, 1]$,*

$$\lim_{j\to\infty} h_{k,j}(r) = 1. \tag{38}$$

The first part follows simply from Note 25. The rest of the proof is given in § 4.11.

Proposition 27. *There is no nonzero solution of* (28) *in $\mathcal{H}$.*

Indeed, Theorem 4 shows that otherwise $(r^{l+1}v_n)(0) = m_n \neq 0$ for a subsequence of $n < 0$. This implies that the corresponding $w_n(x) \sim m_n r^{-l-1}\mathcal{Y}_{l,m}$ for $r = |x| \to 0$. This singularity is incompatible with $w_n \in H^2$, (see Proposition 8). Thus there is no admissible solution of the homogeneous system and the first part of Theorem 1 follows from Theorem 2 (i). See also the remarks in § 5.10. The result on the decay rate follows from the type of essential singularities of $\hat{y}$ for $p \in i\omega\mathbb{Z}$; see §4.6.

[4] As is common, the notation $O(a, b) \equiv O(|a| + |b|)$; similarly $O(a, b, c) \equiv O(|a| + |b| + |c|)$.

4. Proofs of Intermediate Steps

4.1. Proof of Proposition 8. As mentioned in § 3.3, $\mathcal{A}_\beta$ and $\mathcal{A}_\beta^*$ are adjoints of each-other. They are furthermore densely defined and hence (see, *e.g* [28], Theorems 5.28 and 5.29, p. 168), closed. Once we show that $\mathcal{A}_\beta(p)$ is invertible in $\mathcal{D}$, analyticity of $\mathcal{R}_\beta$ in $\mathcal{O}\backslash(\ell_{p_c} \cup \ell_{-\epsilon})$ follows (the spectrum of the closed operator $H_C - i\beta\chi_{\mathsf{B}}(r)$ is a closed set). Analyticity holds wherever $\mathcal{A}_\beta$ is analytic, [31], Vol. 1, Theorem VIII.2, p. 254).

(1) *Eigenvalues*. We first show below that no $ip \in i\mathcal{D}$ is an eigenvalue of $H_C - i\beta\chi_{\mathsf{B}}(r)$. Assume we had $\mathcal{A}_\beta\psi = 0$. If $\beta = 0$, $\mathcal{A}_0\psi = 0$ implies $ip \in \sigma_p(H_C)$, but, by construction, these values of p correspond to the region where $\beta \neq 0$. So we can assume $\beta > 0$. Then

$$\langle\psi, (-\Delta - ip - br^{-1})\psi\rangle + \langle\psi, -i\beta\chi_{\mathsf{B}}\psi\rangle = 0. \tag{39}$$

Taking the imaginary part of (39) we get

$$\mathrm{Re}\, p\ \langle\psi,\psi\rangle + c\langle\psi, \chi_{\mathsf{B}}\psi\rangle = \mathrm{Re}\, p\ \langle\psi,\psi\rangle + c\langle\chi_{\mathsf{B}}\psi, \chi_{\mathsf{B}}\psi\rangle = 0. \tag{40}$$

If $\mathrm{Re}\, p > 0$ this immediately implies $\psi = 0$. If $\mathrm{Re}\, p = 0$ we get $\chi_{\mathsf{B}}\psi = 0$. But $\chi_{\mathsf{B}}\psi = 0$ implies $0 = \mathcal{A}_\beta\psi = \mathcal{A}_0\psi$. In spherical coordinates the equation $\mathcal{A}_0\psi = 0$ becomes a system of ordinary differential equations

$$\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - br^{-1} - ip\right)\psi_{n,l,m} = 0. \tag{41}$$

Since $\chi_{\mathsf{B}}\psi = 0$, the solution of (41) vanishes identically on $[0, a]$; but then, by standard arguments the solution is identically zero.

If $\mathrm{Im}\, p \notin [-\epsilon, p_c]$, with $p \in \mathcal{D}$, then $\mathcal{A}_\beta = \mathcal{A}_0$, and we are, by construction, outside the spectrum of $\mathcal{A}_0$, and thus $\mathcal{A}_0\psi = 0$ implies $\psi = 0$.

(2) *The range of $\mathcal{A}_\beta$ is dense.* Indeed, the opposite would imply[5] $\mathrm{Ker}(\mathcal{A}_\beta^*) \neq 0$, which leads to the same contradiction as in Step 1 (note that $\mathcal{A}_\beta^*$ is simply $\mathcal{A}_\beta$ with the signs of β and $\mathrm{Re}\, p$ changed at the same time).

(3) *For any $p \in \mathcal{D}$ there is an $\epsilon > 0$ such that $\|\mathcal{A}_\beta\psi\| > \epsilon\|\psi\|$.*

(a) If $\mathrm{Re}\, p > 0$ and $\|\psi\| = 1$, then

$$\begin{aligned}\|\mathcal{A}_\beta\psi\| &\geqslant \left|\langle\mathcal{A}_\beta\psi,\psi\rangle\right| = |\langle\mathcal{A}_0\psi,\psi\rangle - ic\langle\chi_{\mathsf{B}}\psi, \chi_{\mathsf{B}}\psi\rangle| \\ &\geqslant |\mathrm{Re}\, p\langle\psi,\psi\rangle| \geq \mathrm{Re}\, p.\end{aligned} \tag{42}$$

(b) Let now $\mathrm{Re}\, p = 0$, and assume $\mathrm{Im}\, p$ is between two eigenvalues of $-H_C$, the distance to the nearest being $\delta > 0$. To get a contradiction, assume that $\|\psi_j\| = 1$ and $\|\mathcal{A}_\beta\psi_j\| = \epsilon_j \to 0$. Then

$$\begin{aligned}\epsilon_j = \|\mathcal{A}_\beta\psi_j\| &\geqslant \left|\langle\mathcal{A}_\beta\psi_j,\psi_j\rangle\right| = \left|\langle\mathcal{A}_0\psi_j,\psi_j\rangle - ic\langle\chi_{\mathsf{B}}\psi_j, \chi_{\mathsf{B}}\psi_j\rangle\right| \\ &\geqslant c\left|\langle\chi_{\mathsf{B}}\psi_j, \chi_{\mathsf{B}}\psi_j\rangle\right| \to 0,\end{aligned} \tag{43}$$

thus $\chi_{\mathsf{B}}\psi_j \to 0$, and by the definition of $\mathcal{A}_\beta$ and $\mathcal{A}_0$ we get

$$\|\mathcal{A}_0\psi_j\| \to 0, \tag{44}$$

which is impossible, since our assumption and (44) imply noninvertibility of $H_C - ip$ while ip is outside the spectrum of H_C.

5 [28], p. 267.

(c) In the last case, $\operatorname{Re} p = 0$, $\operatorname{Im} p \in \sigma_p(-H_C)$; then if we assume there is a sequence ψ_j, $\|\psi_j\| = 1$ such that $\|\mathcal{A}_\beta \psi_j\| \to 0$ as $j \to \infty$ we get

$$\|\mathcal{A}_\beta \psi_j\| \geqslant |\langle \mathcal{A}_\beta \psi_j, \psi_j\rangle| = \left|\langle \mathcal{A}_0 \psi_j, \psi_j\rangle - ic\langle \chi_{\mathsf{B}}\psi_j, \chi_{\mathsf{B}}\psi_j\rangle\right|$$
$$\geqslant \left|c\langle \chi_{\mathsf{B}}\psi_j, \chi_{\mathsf{B}}\psi_j\rangle\right| \to 0. \tag{45}$$

Since $\|\mathcal{A}_0\psi_j\| \le \|\mathcal{A}_\beta\psi_j\| + c\|\chi_{\mathsf{B}}\psi_j\|$, (45) implies $\|\mathcal{A}_0\psi_j\| \to 0$. On the other hand, with P the orthogonal projection on the finite dimensional eigenspace of H_C corresponding to the eigenvalue ip, we have $\mathcal{A}_0 P = 0 \Rightarrow \mathcal{A}_0 = \mathcal{A}_0(I - P)$ and then since $\mathcal{A}_0\psi_j \to 0$,

$$\|\mathcal{A}_0(I - P)\psi_j\| \to 0. \tag{46}$$

But by definition $\mathcal{A}_0$ is invertible on $(I - P)L^2(\mathbb{R}^3)$ and (46) then implies $\|(I-P)\psi_j\| \to 0$, i.e. $P\psi_j - \psi_j \to 0$. Since $\|\psi_j\| = 1$, $\|P\psi_j\| \to 1$. Then $P\psi_j$ is a bounded sequence in the finite dimensional space $PL^2(\mathbb{R}^3)$, hence we can extract a convergent subsequence, which we may without loss of generality assume to be $P\psi_j$ itself, $P\psi_j \to \psi$, $\|\psi\| = 1$, and also $\psi_j \to P\psi_j \to \psi$, thus $P\psi = \psi$. Therefore, $\mathcal{A}_0\psi = \mathcal{A}_0 P\psi = 0$. Also, since multiplication by $c\chi_{\mathsf{B}}$ is a bounded operator we have $c\chi_{\mathsf{B}}\psi_j \to c\chi_{\mathsf{B}}\psi = 0$, since $c\chi_{\mathsf{B}}\psi_j \to 0$. Therefore, $\|A_\beta\psi\| \le \|\mathcal{A}_0\psi\| + \|c\chi_{\mathsf{B}}\psi\| = 0$ in contradiction to the absence of eigenvalues.

(4) *Definition of the inverse*. This is standard: we let $\psi \in D(\mathcal{A}_\beta)$, $\mathcal{A}_\beta\psi = \phi$ and define $\mathfrak{R}_\beta\phi = \psi$. This is well defined since $\mathcal{A}_\beta\psi_1 = \mathcal{A}_\beta\psi_2$ entails, by Step 1, $\psi_1 = \psi_2$. By Step 2, $\mathfrak{R}_\beta$ is defined on a dense set. By Step 3, for any p there is an $\epsilon > 0$ such that $\|\mathfrak{R}_\beta\| < \epsilon^{-1}$. Thus $\mathfrak{R}_\beta$ extends by density to $L^2(\mathbb{R}^3)$ and by construction $\mathcal{A}_\beta\mathfrak{R}_\beta\phi = \phi$ whenever $\mathfrak{R}_\beta\phi \in D(\mathcal{A}_\beta)$. Conversely, if $\phi \in D(\mathcal{A}_\beta)$, and $\mathcal{A}_\beta\phi = u$ then $\mathfrak{R}_\beta u = \phi$ entailing $\mathfrak{R}_\beta\mathcal{A}_\beta\phi = \phi$ on the dense set $\mathrm{D}(\mathcal{A}_\beta)$.

For the regularity of $\mathfrak{R}_\beta$ in x, we first note that if we define $\mathcal{Q} = (I - \Delta)^{-1}$, we have the following identity:

$$\mathfrak{R}_\beta = \mathcal{Q}\left[1 - \left(\frac{b}{r} + i\beta\chi_B + ip + 1\right)\mathcal{Q}\right]^{-1}. \tag{47}$$

It is clear that if $\phi \in L^2(\mathbb{R}^3)$, $\mathcal{Q}\phi \in H^2(\mathbb{R}^3)$ and so $(b/r - i\beta\chi_B + ip + 1)\,\mathcal{Q}\phi \in L^2$. Therefore, from (47), $\mathfrak{R}_\beta : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$.

4.2. Proof of Proposition 15. The shift operator S, defined by $(SY)_j = y_{j+1}$, is quite straightforwardly shown to be bounded in $\mathcal{H}$: the proof of Lemma 27 in [15] goes through without changes. By the second resolvent identity we have

$$\mathfrak{R}_\beta = (1 - i\beta\mathfrak{R}_0\chi_{\mathsf{B}})^{-1}\,\mathfrak{R}_0.$$

Since $-\Delta - br^{-1}$ is self-adjoint, we have by the spectral theorem, for some $C > 0$ independent of p,

$$\|(-\Delta - br^{-1} - ip)^{-1}\|_{L^2(\mathbb{R}^3)} \leqslant C(\operatorname{Re} p)^{-1}, \tag{48}$$

and thus $\|\mathfrak{R}_\beta\|_{L^2(\mathsf{B})} \leqslant C_1(1 + |\operatorname{Re} p|)^{-1}$. Since $\mathfrak{R}_\beta$ is diagonal (in n) and $\mathfrak{S}$ is bounded (cf. Lemma 12), we have $\|\mathfrak{C}\|_{\mathcal{H}} \leqslant C_2(1 + |\operatorname{Re} p_1|)^{-1}$. Thus $\|\mathfrak{C}\|_{\mathcal{H}}$ is small for large $\operatorname{Re} p_1$, and therefore $(I - \mathfrak{C})Y = Y^{(0)}$ has a unique solution $Y \in \mathcal{H}$ and the proof follows.

4.3. Proof of Proposition 11.

Proof. The estimate $\|\chi_{\mathsf{B}}\mathfrak{R}_0\chi_{\mathsf{B}}\| = O(p^{-1/2})$ is shown right after the statement of Proposition 11.

We now consider the analyticity of $\mathfrak{R}_\beta$ in an open set on the imaginary p axis for $\mathrm{Im}\, p < -\epsilon$. There, $\beta = 0$ and $\chi_B\mathfrak{R}_\beta\chi_B = \chi_B\mathfrak{R}_0\chi_B$ is manifestly analytic from its representation as an integral operator, whose kernel G is given below (see [26] and Appendix §5.4 for details).

With $k = \sqrt{ip}$ (using the principal branch of the square root), and $\nu = b/(2k)$,

$$G(x, x'; k) = \frac{ik(\eta-\xi)I(-ik\xi)J(-ik\eta) - k^2\xi\eta[I(-ik\xi)\dot{J}(-ik\eta) - J(-ik\eta)\dot{I}(-ik\xi)]}{\Gamma(1-i\nu)\Gamma(1+i\nu)} \times \frac{e^{\frac{ik}{2}(\xi+\eta)}}{4\pi|x-x'|}, \tag{49}$$

where

$$\xi = |x| + |x'| + |x-x'|, \eta = |x| + |x'| - |x-x'|, \tag{50}$$

$$I(z_1) = \int_0^{i\infty} e^{-z_1 t}t^{-i\nu}(1+t)^{i\nu}dt, \quad \dot{I}(z_1) = -\int_0^{i\infty} e^{-z_1 t}t^{1-i\nu}(1+t)^{i\nu}dt \tag{51}$$

$$J(z_2) = \int_0^1 e^{z_2 t}t^{-i\nu}(1-t)^{i\nu}dt; \; \dot{J}(z_2) = \int_0^1 e^{z_2 t}t^{1-i\nu}(1-t)^{i\nu}dt.$$

□

Further properties of function G are discussed in §5.4.

Note 28. Note that (49) still holds for $p \in i\mathbb{R}^+$, with $k = \sqrt{ip}$, with the choice $\arg k = \pi/2$ for $p \in i\mathbb{R}^+$, and with the upper limits $i\infty$ in (51) replaced by $+\infty$.

4.4. Proof of Proposition 17. The function $f = \mathfrak{R}_{\beta,l,m}\chi_{\mathsf{B}}g$ is the solution of the equation

$$\begin{aligned} \left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} + \lambda^2 - ic\right) f &= \quad g; \;\; r \leqslant a, \\ \left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} + \lambda^2\right) f &= \quad 0; \;\; r > a, \end{aligned} \tag{52}$$

such that f decays at infinity, is regular at the origin and C^1 at $r = a$. We note $\lambda = \sqrt{-ip}$ is in the closure of the fourth quadrant for $\mathrm{Re}\, p \geqslant 0$. We let $\alpha = \sqrt{\lambda^2 - ic}, \kappa_1 = b/(2\alpha)$, $\kappa = b/(2\lambda)$, $\mu = 2l + 1$ and define (in terms of the Whittaker functions $\mathfrak{M}$ and $\mathfrak{W}$ [6])

$$\begin{aligned} m_1(s) := s^{-1}\mathfrak{M}_{\kappa_1,\mu/2}(2\alpha s); \quad w_1(s) := s^{-1}\mathfrak{W}_{\kappa_1,\mu/2}(2\alpha s); \\ w_2(s) := s^{-1}\mathfrak{W}_{\kappa,\mu/2}(2\lambda s). \end{aligned} \tag{53}$$

[6] See [9], pp. 60, Eq. (1) and pp. 63, Eq. (5).

For $r > a$ we have $f = \mathrm{B}w_2(r)$ since $r^{-1}\mathfrak{M}_{\kappa,\mu/2}(2\lambda r)$ grows with r as $r \to \infty$. For $r \leqslant a$ we must have

$$f = Am_1 + f_0, \tag{54}$$

where, using standard results about the Wronskian of $\mathfrak{M}$ and $\mathfrak{W}$, see [9], pp. 25 and [1], pp 505, 508, we have

$$\frac{2\alpha\Gamma(1+\mu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\kappa_1\right)} f_0 = w_1(r)\int_0^r \chi_{[0,a]}(s)s^2 m_1(s)g(s)ds$$
$$+m_1(r)\int_r^a s^2 w_1(s)\chi_{[0,a]}(s)g(s)ds. \tag{55}$$

The integral representations of the functions $\mathfrak{M}$ and $\mathfrak{W}$ entail immediately that the functions f_0, f and $\mathfrak{M}_{\kappa_1,\mu/2}(2\alpha r)$ depend analytically on λ for small λ. Continuity of f and f' at $a \geqslant 1$ imply that A defined in (54), is given by

$$A = \frac{f_0(a)w_2'(a) - f_0'(a)w_2(a)}{m_1'(a)w_2(a) - m_1(a)w_2'(a)}. \tag{56}$$

In § 5.5 it is shown that that A is analytic in $(\lambda, \exp[i\pi b/(2\lambda)])$ in a domain corresponding to λ small in the closure of the fourth quadrant, if a and c are chosen large enough. It follows that resolvent $\mathfrak{R}_{\beta,m,n}$ is analytic in X for $X = \left(\sqrt{p}, \exp\left[i\pi b/\left(2\sqrt{-ip}\right)\right]\right) \in \overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$ for small ϵ.

4.5. Proof of Proposition 19. By adding and subtracting 1 from $\mathcal{A}_\beta$ and using the second resolvent formula, whenever everything is well defined, we have

$$\chi_{\mathsf{B}}\mathcal{A}_\beta^{-1}\chi_{\mathsf{B}} =: \chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}} = \chi_{\mathsf{B}}(-\Delta+1)^{-1}\chi_{\mathsf{B}}$$
$$-\chi_{\mathsf{B}}\mathfrak{R}_\beta(-br^{-1} - i\beta\chi_{\mathsf{B}} - 1 - ip)(-\Delta+1)^{-1}\chi_{\mathsf{B}}. \tag{57}$$

The Green's function for $-\Delta + 1$ is

$$G(x,y) = \frac{1}{4\pi|x-y|}e^{-|x-y|}. \tag{58}$$

Now if $\|\phi_j\|_{L^2(\mathsf{B})} \leqslant 1$ then the functions $f_j = (-\Delta+1)^{-1}\chi_{\mathsf{B}}\phi_j$ are seen by straightforward calculation to be equicontinuous on the one point compactification of $\mathbb{R}^3$. A subsequence, without loss of generality assumed to be the f_j's themselves, converges in $L^2(\mathbb{R}^3)$ as well (to a function with exponential decay, since there is a $\delta_1 > 0$ small enough and independent of j so that $e^{\delta_1|x|}(-\Delta+1)^{-1}\chi_{\mathsf{B}}\phi_j)$ is also equicontinuous on the compactification of $\mathbb{R}^3$). In particular, $\chi_{\mathsf{B}}(-\Delta+1)^{-1}\chi_{\mathsf{B}}$ is compact.

Now f_j converge in the sup norm with weight $e^{\delta_1|x|}$, and thus $(-br^{-1} - i\beta\chi_{\mathsf{B}} - 1 - ip)f_j$ converge in $L^2(\mathbb{R}^3)$. Since $\mathfrak{R}_\beta$ is bounded, compactness of $\chi_{\mathsf{B}}\mathfrak{R}_\beta\chi_{\mathsf{B}}$ follows.

By Proposition 11, and the previous argument, $\mathfrak{C}$ is a norm limit of compact operators (the truncations of $\mathfrak{C}$ to the subspaces of $\mathcal{H}$ with vanishing components for $|n| > N$). Therefore, $\mathfrak{C}_{l,m} = P_{l,m}\mathfrak{C}$ is also compact.

4.6. Proof of Proposition 21 and final estimates for Theorem 1. If (24) has no nontrivial solution for any $p_1 \in \overline{\mathbb{H}}$, then compactness of $\mathfrak{C}_{l,m}$ implies that $\left(I - \mathfrak{C}_{l,m}\right)^{-1}$ exists. Lemma 13, Corollary 18 and Proposition 11 give the analytic and continuity properties of $\mathfrak{C}_{l,m}$ and $\mathfrak{T}_{l,m} Y_0$. Analyticity of $\left(I - \mathfrak{C}_{l,m}\right)^{-1}$ in X_1 for $X_1 \in \overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$, follows in a standard way from analyticity of $\mathfrak{C}_{l,m}$ and the second resolvent formula,

$$A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1} \tag{59}$$

(see § 5.3). The same resolvent identity can be applied to show analyticity of $(I - \mathfrak{C}_{l,m})^{-1}$ with respect to p_1 in a neighborhood of

$$\overline{\mathbb{H}} \backslash \left\{(\ell_{p_c} + i\omega\mathbb{Z}) \cup (\ell_{-\epsilon} + i\omega\mathbb{Z}) \cup (\mathcal{I}_\epsilon + i\omega\mathbb{Z})\right\}.$$

Hence, the solution $Y = \left(I - \mathfrak{C}_{l,m}\right)^{-1} \mathfrak{T}_{l,m} Y_0$ is analytic for $p_1 \in \overline{\mathbb{H}} \backslash \{i\omega\mathbb{Z}\}$, since ϵ, β and p_c are artificially introduced parameters the value of which cannot affect Y, since Y_0 is independent of these choices (see Remark 14.)

The function $\hat{y}(p; x) = y_n(p_1, x)$, with $p = in\omega + p_1$, is analytic in p for $p \in i\mathbb{R} \backslash i\omega\mathbb{Z}$ and by analyticity of Y in X_1, boundedness at $p = i\omega\mathbb{Z}$ follows. In particular, as $p \to in\omega$ from the right half-plane, $\hat{y}(p, x)$ is analytic in the extended variable

$$\left((p - in\omega)^{1/2}, \exp\left[\frac{i\pi b}{2\sqrt{-i(p - in\omega)}}\right]\right). \tag{60}$$

The regularity properties of Y in p and the decay properties in $|n|$ of its components y_n for large $|n|$, simply stemming from $Y \in \mathcal{H}$, imply that $y(t, x)$ can be expressed as an inverse Laplace transform of $\hat{y}(p, x)$ on $i\mathbb{R}$. We now show that

$$P(t, \mathsf{B}) = \|\psi_0(x)e^{-t} + y(x, t)\|^2_{L^2(\mathsf{B})} \le 2e^{-2t}\|\psi_0\|^2_{L^2(\mathsf{B})} \\ +2\|y(x, t)\|^2_{L^2(\mathsf{B})} \to 0 \quad \text{as } t \to \infty.$$

We note that

$$\int_{\mathsf{B}} dx |y(t, x)|^2 = \int_{\mathsf{B}} dx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it(s-s')} \hat{y}(is, x)\overline{\hat{y}(is', x)} ds ds' \\ = \int_{-\infty}^{\infty} e^{i\tilde{s}t} \left\{\int_{-\infty}^{\infty} \left[\int_{\mathsf{B}} \hat{y}(i\tilde{s} + is', x)\overline{\hat{y}(is', x)} dx\right] ds'\right\} d\tilde{s}. \tag{61}$$

So, in order to show ionization, it suffices from the Riemann-Lebesgue Lemma to show that

$$\int_{-\infty}^{\infty} \left[\int_{\mathsf{B}} \overline{\hat{y}(is', x)}\hat{y}(is' + i\tilde{s}, x) dx\right] ds'$$

is in $L^1(d\tilde{s})$. This follows from Cauchy-Schwarz, since

$$\int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} \left[\int_{\mathsf{B}} |\hat{y}(is', x)||\hat{y}(is' + i\tilde{s}, x)| dx\right] ds'\right\} d\tilde{s} \le \left(\int_{\mathbb{R}} \|\hat{y}(is', \cdot)\|_{L^2(\mathsf{B})} ds'\right)^2. \tag{62}$$

However,

$$\begin{aligned}\int_{\mathbb{R}} \|\hat{y}(is', \cdot)\|_{L^2(\mathsf{B})} ds' &= \int_0^{\omega} \left[\sum_{n\in\mathbb{Z}} \|y_n(iq, \cdot)\|_{L^2(\mathsf{B})} \right] dq \\ &\leq C \int_0^{\omega} \sum_{n\in\mathbb{Z}} (1+|n|)^{4/3} \|y_n(iq, \cdot)\|^2_{L^2(\mathsf{B})} dq \\ &\leq C \int_0^{\omega} \|Y(iq, \cdot)\|^2_{\mathcal{H}} dq < \infty, \end{aligned} \tag{63}$$

since Y is bounded in $p_1 = iq$ for $q \in [0, \omega]$.

Since $\hat{y}(p_1 + in\omega, x), n \in \mathbb{Z}$, is analytic in the variable (60), standard stationary phase analysis (see Appendix §5.7) shows that $y(t, x) = O(t^{-5/6})$, and hence $P(t, \mathsf{B}) = O(t^{-5/3})$ as $t \to \infty$.

4.7. Proof of Theorem 3. Since (14) (restricted to B) follows from the homogeneous system $w = \mathfrak{C}w$ (see also Proposition 8 for the necessary regularity), we look for a nontrivial solution of (14) in $\mathcal{H}$. We multiply (14) by $\overline{w}_n$, integrate over the ball $\mathsf{B}_{\tilde{a}}$ (of radius $\tilde{a} \in (1, a]$), sum over n (this is legitimate since $w \in \mathcal{H}$) and take the imaginary part of the resulting expression. Noting that

$$\begin{aligned}\overline{\sum_{j,n\in\mathbb{Z}} \Omega_j(x) w_{n-j} \overline{w_n}} &= \sum_{j,n\in\mathbb{Z}} \Omega_{-j} \overline{w}_{n-j} w_n = \sum_{j,n\in\mathbb{Z}} \Omega_j \overline{w}_{n+j} w_n \\ &= \sum_{j,m\in\mathbb{Z}} \Omega_j(x) \overline{w_m} w_{m-j} \end{aligned} \tag{64}$$

so the sum (64) is real, we get from (14),

$$\begin{aligned}0 &= \mathrm{Im}\left(+ip_1 \sum_{n\in\mathbb{Z}} \int_{\mathsf{B}_{\tilde{a}}} |w_n(x)|^2 dx + \int_{\mathsf{B}_{\tilde{a}}} \sum_{n\in\mathbb{Z}} dx \overline{w}_n \Delta w_n \right) \\ &= +\mathrm{Re}\, p_1 \sum_{n\in\mathbb{Z}} \int_{\mathsf{B}_{\tilde{a}}} |w_n(x)|^2 dx + \frac{1}{2i} \int_{\partial\mathsf{B}_{\tilde{a}}} \left(\sum_{n\in\mathbb{Z}} \overline{w}_n \nabla w_n - w_n \nabla \overline{w}_n \right) \cdot \mathbf{n}\, dS. \end{aligned} \tag{65}$$

It is convenient to decompose w_n using spherical harmonics. We write

$$w_n = \sum_{l\geqslant 0, |m|\leqslant l} R_{n,l,m}(r) \mathcal{Y}_l^m(\theta, \phi). \tag{66}$$

The last integral in (65), including the prefactor, then equals

$$\begin{aligned}&-\frac{i}{2} a^2 \sum_{n\in\mathbb{Z}} \sum_{m,l} \left[\overline{R}_{n,m,l} R'_{n,m,l} - \overline{R'}_{n,m,l} R_{n,m,l} \right] \\ &= -\frac{i}{2} a^2 \sum_{n\in\mathbb{Z}} \sum_{m,l} \mathcal{W}[\overline{R}_{n,m,l}, R_{n,m,l}], \end{aligned} \tag{67}$$

where $\mathcal{W}[f, g]$ is the Wronskian of f and g. On the other hand, we have outside of $\mathsf{B}_{\tilde{a}}$,

$$\Delta w_n + br^{-1}w_n + (ip_1 - n\omega)w_n = 0, \tag{68}$$

and then by (66), the $R_{n,l,m}$ satisfy for $r > a$ the equation

$$R'' + \frac{2}{r}R' + br^{-1}R - \frac{l(l+1)}{r^2}R = (-ip_1 + n\omega)R, \tag{69}$$

where we have suppressed the subscripts. Let $g_{n,l,m} = rR_{n,l,m}$. Then for the $g_{n,l,m}$ we get

$$g'' - \left[\frac{l(l+1)}{r^2} - ip_1 + n\omega - br^{-1}\right]g = 0. \tag{70}$$

Thus

$$\overline{R}R' = \frac{\overline{g}g'}{r^2} - \frac{|g|^2}{r^3} \tag{71}$$

and

$$r^2\mathcal{W}[\overline{R}, R] = \mathcal{W}[\overline{g}, g] =: \mathcal{W}_n. \tag{72}$$

Multiplying (70) by $\overline{g}$, and the conjugate of (70) by g and subtracting, we get for $r > a$,

$$\mathcal{W}_n' = -i(p_1 + \overline{p_1})|g|^2 = -2i|g|^2\text{Re}\, p_1. \tag{73}$$

Remark 29. Direct estimates using the Green's function representation (49) imply that

$$w_n(x) = \frac{e^{-\kappa_n r}}{r^{1+\frac{b}{2\kappa_n}}}\left(c_n(\theta, \phi) + O(r^{-1})\right) \quad \text{as } r \to \infty \tag{74}$$

with $c_n(\theta, \phi)$ independent of r and with

$$\kappa_n = \sqrt{-ip_1 + n\omega} \text{ (when Re } p_1 > 0, \kappa_n \text{ is in the fourth quadrant when } n < 0). \tag{75}$$

(i) We first take Re $p_1 > 0$, to illustrate the argument. Using (74) we get

$$g \sim Ce^{-\kappa_n r}r^{-\frac{b}{2\kappa_n}}(1 + o(1)) \quad \text{as } \; r \to \infty. \tag{76}$$

There is a one-parameter family of solutions of (70) satisfying (76) and the asymptotic expansion can be differentiated [42]. We assume, to get a contradiction, that there exist $n < 0$ for which $g = g_n \neq 0$. For these n we have, using (76), differentiability of this asymptotic expansion and the definition of κ_n that

$$\frac{1}{2i}\lim_{r\to\infty} |g_n|^{-2}\mathcal{W}_n = -\text{Im}\,\kappa_n \; > 0. \tag{77}$$

It follows from (73) and (77) that $\mathcal{W}_n/(2i)$ is strictly positive for all $r > a$ (by monotonicity and positivity at infinity) and all n for which $g_n \neq 0$. This implies that the last term in (65) is a sum of nonnegative terms which shows that (65) cannot be satisfied nontrivially.

(ii) Re $p_1 = 0$. For $n < 0$, we use Remark 29 (and differentiability of the asymptotic expansion as in Case (i)) to calculate $\mathcal{W}_n$ in the limit $r \to \infty$: $\mathcal{W}_n = 2i|c_n|^2|\kappa_n|(1+o(1))$.

Since for Re $p_1 = 0$, $\mathcal{W}_n$ is constant, cf. (73), it follows that $\mathcal{W}_n = 2i|c_n|^2|\kappa_n||g_n|^2$ exactly. Thus, (65) cannot be non-trivially satisfied, implying that

$$w_n(x) = 0 \quad \text{for all} \;\; n < 0 \quad \text{and} \quad |x| = r = \tilde{a} \in (1, a]. \tag{78}$$

For $\tilde{a} > r > 1$ (where $V(t, x) = 0$) we have $\mathfrak{O}w_n = 0$, where $\mathfrak{O}$ is the elliptic operator $-\Delta - b/r - ip_1 + n\omega$. The proof that $w_n(x) = 0$ for $r > 1$ then follows immediately from (78), by standard unique continuation results [17,23] (in fact, $\mathfrak{O}$ is analytic hypo-elliptic). See also Note 5.

4.8. Connection with the Floquet operator. It is easy to check that the discrete time-Fourier transform of the eigenvalue equation for the Floquet operator, Eq. (5), $Kv = \phi v$, with $p_1 = i\phi$, coincides with (14), the differential version of the homogeneous equation associated to (23). Now, (78) shows that a solution of (14) is an eigenvector of K.

In the opposite direction the existence of a Floquet eigenfunction entails failure of ionization since it implies the existence of a solution of (2) for which the absolute value is time-periodic.

4.9. Differential equation for w. We seek to show that the only solution to the homogeneous system

$$w = \mathfrak{C}_{l,m} w \tag{79}$$

in the space $\mathcal{H}$ is $w = 0$. Since w is piecewise C^2 (see Note 5), (79) implies that the components of $w = \left\{\mathcal{Y}_{l,m} r^{-1} g_n(r)\right\}_{n\in\mathbb{Z}}$ satisfy the differential-difference system (see Note 5):

$$\frac{d^2}{dr^2} g_n - \left(-br^{-1} + n\omega - ip_1 + \frac{l(l+1)}{r^2}\right) g_n = i\Omega\,(g_{n+1} - g_{n-1}). \tag{80}$$

First, we notice that for $n < 0$, Theorem 2 implies that $g_n(r) = 0$ for $r \geqslant 1$. Thus $g_n(1) = 0$, $g_n'(1) = 0$ for all $n < 0$.

4.10. Proof of Proposition 23. The gist of the proof is that contractive mapping arguments show that if the statement was false then the solution would vanish.

Lemma 30. *If $Y \neq 0$, then there exists some $n_0 \geqslant 0$ so that either $g_{n_0}(1) \neq 0$ or $g'_{n_0}(1) \neq 0$.* (*As before, in the sequel, we shall define n_0 to be the smallest such integer*).

Proof. To get a contradiction, assume the statement is false. Since the functions w_n are in the domain of Δ (see Note 5), then, in particular, for any n, g_n is continuous in r. Thus, the set $Z_n := \{r : g_n(r) = 0\}$ is closed and so is the (possibly empty) left connected component of 1 in Z_n, call it K_n. Let

$$K = \bigcap_{n\in\mathbb{Z}} K_n.$$

Assume to get a contradiction that K is nonempty: let then $K = [a, 1]$. If $a = 0$, then $Y \equiv 0$ since $g_n(1) = 0$, $g_n'(1) = 0$ imply $g_n(r) = 0$ for $r > 1$. Then $Y \neq 0$ implies

$a > 0$. We first take $0 < a < 1$. We write the differential equation for $g_n(r)$ in integral form and use the conditions $g_n(a) = 0 = g_n'(a)$, since g_n vanishes on $[a, 1]$:

$$e^{\sqrt{n\omega}r} g_n(r) = \int_r^a \left(\frac{[1 - e^{-2\sqrt{n\omega}(s-r)}]}{2\sqrt{n\omega}} \right) e^{\sqrt{n\omega}s} \times \left\{ \left[\frac{l(l+1)}{s^2} - ip_1 + \tilde{V}(s) \right] g_n(s) - i\Omega(s) \left(g_{n-1}(s) - g_{n+1}(s) \right) \right\} ds. \tag{81}$$

Consider the Banach space of sequences

$$\{g_n(r)\}_{n=-\infty}^{\infty}$$

in the norm

$$\sup_{n\in\mathbb{Z}, r\in[a-\epsilon,a]} \left| e^{\sqrt{n\omega}r} g_n(r) \right|.$$

It is easy to see that the rhs of (81) is a contractive mapping if ϵ is small enough and then $g_n(r) = 0$ for $r \in [a - \epsilon, a]$ contradicting the definition of a. The same is true if $a = 1$, since $g_n(1) = 0$ and $g_n'(1) = 0$ would imply, with the same proof as before, that $g_n = 0$ for $r \in [1 - \epsilon, 1]$, for some $\epsilon > 0$, contradicting the definition of a. □

4.11. Proof of Theorem 4. For a heuristic discussion see § 5.9. The proof is by rigorous WKB. The fact that there are two competing potentially large variables, k and $1/r$ makes it necessary to rigorously match two regimes. First, note that (37) implies

$$g_{n_0-k}(r) = i^k m_k(r) h_k(r). \tag{82}$$

We need a few more preliminary results.

Lemma 31. *For any $\epsilon_1 > 0$, there exists $C_3 > 0$ independent of k and ϵ_1 so that for $k \geqslant k_0 = C_3\epsilon_1^{-1}$, and for $r \in [\epsilon_1, 1]$,*

$$\sup_{\epsilon_1 \le r \le 1} |h_k'| \leqslant C_4 k_0 \left(\frac{k_0}{k} \right)^{1/2}, \tag{83}$$

where C_4 is independent of ϵ_1 and k.

The proof of Lemma 31 is given in §4.13.

Definition 32. *For fixed ϵ, we define $L_\epsilon = \alpha C_3 (2C_4C_3/\epsilon)^2$, with C_3 and C_4 defined in Lemma 31, and $\zeta = \alpha kr$, where α is given in (32). We will take ϵ small enough so that $L_\epsilon \geq C_3\alpha$.*

Finally, in what follows, c_* is a positive "generic" constant, the value of which is immaterial.

Lemma 33. *For $\epsilon > 0$ small enough and $k\alpha r = \zeta \in [L_\epsilon, k\alpha]$, we have*

$$|h_k(r) - 1| \leqslant \epsilon. \tag{84}$$

The proof of Lemma 33 is given in § 4.14.

Definition 34. *Let* $\tilde{h}_k(\zeta) = h_k(\zeta/(\alpha k))$.

Lemma 35. *For any small* $\epsilon > 0$*, there exists a subsequence* $S = \{\tilde{h}_{k_j}\}_{j\in\mathbb{N}}$ *that converges to a continuous function* $\tilde{h}$ *for* $\zeta \in [0, L_\epsilon]$*. For the limiting function* $\tilde{h}(\zeta)$*, we have* $|\tilde{h}(\zeta) - 1| \le 4\epsilon$ *for* $\zeta \in [0, L_\epsilon]$.

The proof of this proposition is given in § 4.15.

Proposition 36. *For any* $r \in [0, 1]$*,* $\lim_{j\to\infty} h_{k,j}(r) = 1$.

Proof. From Lemma 35 and Lemma 33 it follows that for any $r \in [0, 1]$ and any $\epsilon > 0$ we have $\lim_{j\to\infty} |h_{k_j}(r) - 1| \le 4\epsilon$. □

The proof of Theorem 4 now follows from the definition of h_k in (36), Remark 24, Note 5 and Proposition 36.

4.12. Further results on g_{n_0-k} and h_k.

Lemma 37. *For any* $j, k \in \mathbb{N} \cup \{0\}$ *we have, at* $r = 1$*, i.e. at* $\mathfrak{s} = 0$*,*

$$\frac{\partial^{j+\tau} g_{n_0-k}}{\partial \mathfrak{s}^{j+\tau}}|_{\mathfrak{s}=0} = \delta_{j,2k}\, i^k \quad \text{for } 0 \leqslant j \leqslant 2k$$

Proof. In case **(i)** (corresponding to $\tau = 0$), note that (80) may be rewritten, cf. (31), as

$$(g_{n_0-k})_{\mathfrak{s}\mathfrak{s}} - \frac{\Omega'}{2\Omega^{3/2}}(g_{n_0-k})_{\mathfrak{s}} + \frac{Q_k}{\Omega} g_{n_0-k} = i\left(g_{n_0-k+1} - g_{n_0-k-1}\right), \tag{85}$$

where

$$Q_k = \frac{b}{r} + (k - n_0)\omega + i p_1 - \frac{l(l+1)}{r^2} \tag{86}$$

Since $g_{n_0-k}(1) = 0 = g'_{n_0-k}(1)$ for all $k \geqslant 1$, while $g_{n_0}(1) = 1$, the statement follows from (85) for any $0 \leqslant j \leqslant 2$, if $2k \geqslant j$. Assuming the statement holds for some $j \geqslant 2$ for $2k \geqslant j$, we prove it for $(j + 1)$ for $2k \geqslant (j + 1)$.

Taking $(j - 1)$ derivatives in $\mathfrak{s}$ of (85) at $\mathfrak{s} = 0$, we obtain

$$\frac{\partial^{j+1} g_{n_0-k}}{\partial \mathfrak{s}^{j+1}} = i\frac{\partial^{j-1}}{\partial \mathfrak{s}^{j-1}} g_{n_0-(k-1)} - i\frac{\partial^{j-1}}{\partial \mathfrak{s}^{j-1}} g_{n_0-(k+1)} + L$$

where L is a linear combination of derivatives of g_{n_0-k} up to order j, which are all zero since $2k \geqslant (j + 1) > j$. The first two terms on the rhs give a contribution of $i i^k \delta_{(j-1),2(k-1)} + 0$ since $2k \geqslant (j+1)$ implies $2(k-1) \geqslant (j-1)$ and $2(k+1) > (j-1)$ completing the inductive step.

In case **(ii)** (corresponding to $\tau = 1$): since $g_{n_0}(1) = 0$ and $g_{n_0-k}(1) = 0 = g'_{n_0-k}(1)$ for all $k \geqslant 1$, it follows from (85) that $g''_{n_0-k} = 0$ for all $k \geqslant 1$ implying the conclusion for $j = 0$ and $j = 1$. By taking an additional derivative of (85) with respect to $\mathfrak{s}$ and evaluating at $\mathfrak{s} = 0$, we obtain

$$\frac{\partial^3 g_{n_0-k}}{\partial \mathfrak{s}^3} = i\delta_{2,2k}\frac{\partial}{\partial \mathfrak{s}} g_{n_0}|_{\mathfrak{s}=0} = i\delta_{2,2k}\frac{g'_{n_0}(1)}{-\sqrt{\Omega(1)}} = i\delta_{2,2k}$$

so the statement holds for $j = 2$ and any k with $2k \geqslant j$. The rest of the proof is very similar to that for $\tau = 0$. □

Let $\psi_{1,k}$, $\psi_{2,k}$ be two independent solutions of

$$\mathcal{L}_k \psi = 0 \ \ ; \text{and } W_k = \psi_{1,k}(r)\psi_{2,k}'(r) - \psi_{2,k}(r)\psi_{1,k}'(r) \tag{87}$$

where

$$\mathcal{L}_k \psi = \psi'' + Q_k \psi \tag{88}$$

From the form of the equation we see that W_k is independent of r.

Lemma 38. *For $n = n_0 - k$, $k \geqslant 1$, the system (80) is equivalent to*

$$g_{n_0-k}(r) = i \int_r^1 \Omega(s) \left(g_{n_0-k+1}(s) - g_{n_0-k-1}(s) \right) G_k(r,s) ds \quad k \geqslant 1 \tag{89}$$

where

$$G_k(r,s) = W_k^{-1} [\psi_{1,k}(r)\psi_{2,k}(s) - \psi_{2,k}(r)\psi_{1,k}(s)] \tag{90}$$

Proof. The proof simply follows from variation of parameters, the two boundary conditions at $r = 1$ and $g_{n_0-k}(1) = g_{n_0-k}'(1) = 0$. □

Definition 39. *Define*

$$j_k = \frac{\mathfrak{s}}{m_k} \left[\mathcal{L}_k m_k - \Omega m_{k-1} \right] \tag{91}$$

Lemma 40. *For $k \geq 1$, there exist constants C_1, C_2 and c_*, independent of k so that for any $r \in (0, 1]$ we have $|j_k| \leq c_*$. For $r \geq \frac{1}{k}$, we have $|j_k'(r)| \leq C_1 / \left(k r^2 \right) + C_2$*

Proof. In the Appendix, (253), we obtain an explicit expression for j_k. Routine asymptotics for large k in different regimes of $r \in (0, 1]$, discussed in the Appendix §5.8, show that $k^2 j_k^{(2)} + k j_k^{(1)} = O(1)$ in all cases and hence $j_k = O(1)$. In fact, as $r \to 0$ and $k \to \infty$ with $\zeta = k\alpha r = O(1)$ fixed, we have $j_k \to g(\zeta)$, where $g(\zeta)$ is bounded. Also taking the r- derivative of j_k for $r = O(1)$ not small, we get $j_k'(r) = O(1)$. When $r \ll 1$, the asymptotics in the regime $\frac{1}{k} \ll r \ll 1$ gives $j_k = O\left(\zeta^{-1}\right) = O\left(1/(kr)\right)$. Since the asymptotics is differentiable, we have $j_k'(r) = O\left(1/(kr^2)\right)$. Finally, we look at $\zeta = O(1)$, $\zeta \geq 1$. Since $\frac{d}{dr} j_k = k \frac{d}{d\zeta} j_k \sim k g'(\zeta)$, where $\zeta^2 g'(\zeta)$ is bounded for all ζ, it follows that $|j_k'(r)| \leq C_1 / \left(kr^2\right) + C_2$ for $r \geq 1/k$. □

Lemma 41. *For $k \geqslant 1$, $h_k(r)$ defined in (82) satisfies the system of differential equations:*

$$h_k'' + 2h_k' \frac{m_k'}{m_k} + \left(\frac{\Omega m_{k-1}}{m_k} + \frac{j_k}{\mathfrak{s}} \right) h_k = \Omega \left(\frac{m_{k-1}}{m_k} h_{k-1}(r) + \frac{m_{k+1}}{m_k} h_{k+1}(r) \right), \tag{92}$$

and the system of integral equations (89) is equivalent, for $k \geqslant 1$, to

$$\begin{aligned} h_k(r) = & \int_r^1 \frac{\Omega(s) m_{k-1}(s)}{m_k(r)} G_k(r,s) h_{k-1}(s) ds \\ & + \int_r^1 \frac{\Omega(s) m_{k+1}(s)}{m_k(r)} G_k(r,s) h_{k+1}(s) ds := \mathcal{A}_k h_{k-1} + \mathcal{H}_k h_{k+1}. \end{aligned} \tag{93}$$

Proof. This simply follows by substituting $g_{n_0-k}(r) = i^k m_k(r) h_k(r)$ into (80) and (89), and using

$$\frac{m_k''}{m_k} + Q_k = \frac{\mathcal{L}_k m_k}{m_k} = \frac{\Omega m_{k-1}}{m_k} + \frac{j_k}{\mathfrak{s}},$$

in turn a consequence of Lemma 40. □

Remark 42. Let now $r \in [\hat{\epsilon}, 1]$, where $\hat{\epsilon} \geqslant C_2 k^{-1}$ for sufficiently large C_2 independent of k. It is convenient to rewrite $\mathcal{A}_k$ and $\mathcal{H}_k$ in (93) in terms of $\mathfrak{s}$ (see (3.11.1)). Furthermore, changing the variable of integration from s to $t = \mathfrak{s}(s)/\mathfrak{s}(r)$, we obtain

$$[\mathcal{A}_k h_{k-1}](\mathfrak{s}) = (2k+\tau)(2k+\tau-1)\int_0^1 t^{2k-2+\tau} T_k(\mathfrak{s}, t) h_{k-1}(\mathfrak{s}t)dt, \tag{94}$$

where, using (36), we get

$$T_k(\mathfrak{s}, t) = \frac{\sqrt{\Omega(r(\mathfrak{s}t))} F_{k-1}(r(\mathfrak{s}t)}{\mathfrak{s} F_k(r(\mathfrak{s}))} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) \tag{95}$$

and

$$[\mathcal{H}_k h_{k+1}](\mathfrak{s}) = \frac{\mathfrak{s}^3}{(2k+2+\tau)(2k+1+\tau)} \int_0^1 \sqrt{\Omega(r(\mathfrak{s}t)} t^{2k+2+\tau} \times \frac{F_{k+1}(r(\mathfrak{s}t)}{F_k(r(\mathfrak{s}))} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) h_{k+1}(\mathfrak{s}t)dt. \tag{96}$$

In evaluating $\mathcal{A}_k$ for large k, it is useful to calculate the Taylor expansion of $T_k(\mathfrak{s}, t)$ and its $\mathfrak{s}$ derivative at $t = 1$. To do so, we first note that

$$\frac{\partial T_k}{\partial t} = \left(-\frac{\Omega'(r')F_{k-1}(r')}{2\Omega(r')F_k(r)} - \frac{F_{k-1}'(r')}{F_k(r)}\right) G_k(r, r') - \frac{F_{k-1}(r')}{F_k(r)} \frac{\partial G_k}{\partial r'}(r, r'), \tag{97}$$

where, to simplify notation, we wrote $r(\mathfrak{s}) = r$ and $r(t\mathfrak{s}) = r'$ and used $\partial_{r'}\mathfrak{s}(r') = -\sqrt{\Omega(r')}$. From (87) and (90) we get $G_k(r, r) = 0$ and $\partial_{r'} G_k(r, r') = 1$ at $r' = r$; (97) implies

$$\left.\frac{\partial T_k}{\partial t}\right|_{t=1} = -\frac{F_{k-1}(r)}{F_k(r)}. \tag{98}$$

Using (97), taking an additional derivative with respect to t, using also (86) and (88) to see that $\partial_{r'r'} G_k = -Q_k G_k$, we obtain

$$\left.\frac{\partial^2 T_k}{\partial t^2}\right|_{t=1} = \frac{F_{k-1}(r)}{F_k(r)} \left(\frac{\xi \Omega'(r)}{2\Omega^{3/2}(r)} + \frac{2\xi F_{k-1}'(r)}{\sqrt{\Omega(r)} F_{k-1}(r)}\right). \tag{99}$$

A similar calculation can be carried out for the third derivative. We only write down the potentially largest term in the regime $kr \geq C_2$ (for large k and small r)

$$\left.\frac{\partial^3 T_k}{\partial t^3}\right|_{t=1} = \frac{\xi^2 F_{k-1}(r) Q_k(r)}{\Omega(r) F_k(r)} + O\left(1, \frac{1}{kr^3}\right) = \frac{\xi^2 F_{k-1}(r)}{\Omega(r) F_k(r)} \left(k\omega - \frac{l(l+1}{r^2}\right) + O\left(1, \frac{1}{kr^3}\right). \tag{100}$$

Note that if kr is sufficiently large, (35) gives

$$\frac{F_{k-1}(r)}{F_k(r)} = \frac{H(\alpha(k-1)r)H(\alpha k)}{H(\alpha kr)H(\alpha(k-1)} = 1 + O(k^{-2}r^{-1}) \tag{101}$$

and

$$\frac{F'_{k-1}(r)}{F_{k-1}(r)} \sim -\frac{l(l+1)}{2\alpha kr^2} + O\left(\frac{1}{k^2r^3}\right). \tag{102}$$

Note also that (32) implies $\alpha - 2\sqrt{\Omega(r)}/(\mathfrak{s}(r)) = O(r)$ for small r. Including all terms that become important when r is small, we note that in the regime when kr is sufficiently large, we have

$$T_k = (1-t) + \left(-\frac{k}{4}f_1 + \frac{f_2}{r^2}\right)\left(\frac{2}{3}(1-t)^3 - \frac{(1-t)^2}{k}\right)$$
$$+O\left(\frac{(1-t)^4}{r^3}, \frac{(1-t)^3}{kr^3}, \frac{(1-t)^3}{r}, \frac{(1-t)^2}{kr}, \frac{(1-t)^2}{k^2r^3}\frac{(1-t)}{k^2r}\right), \tag{103}$$

$$\frac{\partial T_k}{\partial \mathfrak{s}} = \left(-\frac{k}{4}f'_1 + \frac{f_3}{r^3}\right)\left(\frac{2}{3}(1-t)^3 - \frac{(1-t)^2}{k}\right)$$
$$+O\left(\frac{(1-t)^4}{r^4}, \frac{(1-t)^3}{kr^4}, \frac{(1-t)^3}{r^2}, \frac{(1-t)^2}{kr^2}, \frac{(1-t)^2}{k^2r^4}, \frac{(1-t)}{k^2r^2}\right), \tag{104}$$

where

$$f_1(\mathfrak{s}) = \frac{\omega\mathfrak{s}^2}{\Omega(r(\mathfrak{s}))}, \tag{105}$$

$$f_2(\mathfrak{s}) = \frac{l(l+1)\mathfrak{s}^2}{4\Omega}, \tag{106}$$

$$f_3(\mathfrak{s}) = \frac{l(l+1)\mathfrak{s}^2}{2\Omega^{3/2}}. \tag{107}$$

When $r \in [0, \hat{\epsilon}]$, for $\hat{\epsilon} = C_2/k$, it is sometimes more convenient to express $\mathcal{A}_k$ in terms of $\zeta = k\alpha r$. For that purpose, we define

$$Q(\zeta) = -2k\log\left[1 - \frac{\mathfrak{s}(0) - \mathfrak{s}}{\mathfrak{s}(0)}\right] - \log\left[\left(\frac{\Omega(0)}{\Omega(r)}\right)^{\frac{1}{4}}\exp\left(\frac{1}{4}\int_0^r dr' \frac{\omega\mathfrak{s}(r')}{\sqrt{\Omega(r')}}\right)\right], \tag{108}$$

where we recall the relation (31) between $\mathfrak{s}$ and $r = \zeta/(k\alpha)$, $\zeta \in [0, k\alpha\epsilon]$. A series expansion in k^{-1} leads to

$$Q(\zeta) = \zeta - \frac{\zeta}{k}\left(\frac{\omega}{2\alpha^2} - \frac{\Omega'(0)}{4\Omega(0)\alpha}\right) + \frac{\zeta^2}{4k}\left(1 + \frac{\Omega'(0)}{\alpha\Omega(0)}\right) + O\left(\frac{\zeta^3}{k^2}, \frac{\zeta^2}{k^2}\right). \tag{109}$$

We choose $\hat{\epsilon}_1 = \tilde{C}_2 k^{-1}\log k$, for some k-independent $\tilde{C}_2$ (chosen more precisely later). We define $\hat{\delta}_1$, dependent of r, so that

$$(1 - \hat{\delta}_1)\mathfrak{s}(r) = \mathfrak{s}(\epsilon_1). \tag{110}$$

From (31), it follows that for sufficiently large $\tilde{C}_2$ we have

$$\hat{\delta}_1 \geqslant 1 - \frac{\mathfrak{s}(\epsilon_1)}{\mathfrak{s}(\epsilon)} \geqslant \frac{(5+l)\log k}{(4k+2\tau)k}. \tag{111}$$

It follows from the definition of $\mathcal{A}_k$ in (93) that for $r \in [0, \hat{\epsilon}]$, *i.e.* $\zeta \in [0, k\alpha\hat{\epsilon}]$,

$$\begin{aligned}\left[\mathcal{A}_k h_{k-1}\right](\zeta) = &\int_\zeta^{k\alpha\hat{\epsilon}_1} e^{-Q(\eta)+Q(\zeta)}\left(1+\frac{a_1}{k}\right)\frac{H(\eta(1-k^{-1}))}{H(\zeta)}\mathcal{G}(\zeta,\eta)h_{k-1}(\eta(1-k^{-1}))d\eta\\ &+(2k+\tau)(2k+\tau-1)\\ &\times\int_{\hat{\epsilon}_1}^1\left[\frac{\mathfrak{s}(r')}{\mathfrak{s}(r)}\right]^{2k-2+\tau}\Omega(r')G_k\left(r,r'\right)\frac{F_{k-1}(r')}{\mathfrak{s}^2F_k(r)}h_{k-1}(r')dr'\\ =:&\ [\mathcal{A}_k^0 h_{k-1}](\zeta) + [\mathcal{A}_k^1 h_{k-1}](r),\end{aligned} \tag{112}$$

where $\mathcal{G}(\zeta,\eta)$ is defined by

$$\mathcal{G}(\zeta,\eta) = k\alpha G_k(r(\zeta), r(\eta)), \tag{113}$$

while

$$\begin{aligned}\frac{a_1(\eta,\zeta)}{k} = &\frac{H(\alpha k)}{H(\alpha(k-1))}\left(1+\frac{\tau}{2k}\right)\left(1+\frac{\tau-1}{2k}\right)\frac{\mathfrak{s}^2(0)\Omega(\eta/(k\alpha))}{\mathfrak{s}^2(\eta/(k\alpha))\Omega(0)}\\ &\times\left(\frac{\mathfrak{s}(\eta/(k\alpha))}{\mathfrak{s}(\zeta/(k\alpha))}\right)^\tau - 1,\end{aligned} \tag{114}$$

while for large k and $0 < \zeta \le \eta \le \hat{\epsilon}_1\alpha$ we have

$$a_1(\eta,\zeta) = \tau - \frac{1}{2} + \left(1+\frac{\Omega'(0)}{\alpha\Omega(0)}\right)\eta + \frac{\tau}{2}(\zeta-\eta) + O\left(\frac{\eta^2}{k},\frac{\eta}{k}\right). \tag{115}$$

Similarly, for $k\alpha r = \zeta \in (0, k\hat{\epsilon}_1\alpha)$, we define

$$\frac{b_1(\eta,\zeta)}{k} = \frac{H(\alpha k)}{H(\alpha(k+1))}\frac{\mathfrak{s}^2(\eta/(k\alpha))\Omega(\eta/(k\alpha))}{\mathfrak{s}^2(0)\Omega(0)}\left(\frac{\mathfrak{s}(\eta/(k\alpha))}{\mathfrak{s}(\zeta/(k\alpha))}\right)^\tau - 1. \tag{116}$$

We then have

$$\begin{aligned}[\mathcal{H}_k h_{k+1}](\zeta) = &\frac{\Omega(0)\mathfrak{s}^2(0)}{\alpha^2k^2(2k+2+\tau)(2k+1+\tau)}\\ &\times\int_\zeta^{k\alpha\hat{\epsilon}_1} e^{-Q(\eta)+Q(\zeta)}\left(1+\frac{b_1}{k}\right)\frac{H(\eta(1+k^{-1}))}{H(\zeta)}\mathcal{G}(\zeta,\eta)h_{k+1}(\eta(1+k^{-1}))d\eta\\ &+\frac{\mathfrak{s}^2}{(2k+2)(2k+1+2\tau)}\\ &\times\int_0^{1-\hat{\delta}_1}\sqrt{\Omega(r(\mathfrak{s}t))}G_k\left(r(\mathfrak{s}),r(\mathfrak{s}t)\right)t^{2k+2+\tau}\frac{F_{k+1}(r(\mathfrak{s}t))}{F_k(r(\mathfrak{s}))}h_{k+1}(\mathfrak{s}t)dt\\ =:&\ [\mathcal{H}_k^0 h_{k+1}] + [\mathcal{H}_k^1 h_{k+1}].\end{aligned} \tag{117}$$

Lemma 43. *For $k \geqslant 2$ and $k_1 \in \{k-1, k, k+1\}$ we have*

(1) *If $r \in (0,1)$ and $s \in (r, r+\delta)$, where $\delta \leq \min\left\{C_2 k^{-1}\log k, 1-r\right\}$, then*

$$\left|G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right| \leqslant \frac{c_*}{k^{1/2}}, \quad \left|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right)\right| < c_* k^{1/2}.$$

(2) *If $r \in (0,1)$, $\delta \leq C_2 k^{-1}\log k$ with $r+\delta < 1$, then for $s \in (r+\delta, 1)$,*

$$\left|G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right| < c_* k^{l/2-1/2}, \left|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right)\right| < c_* k^{l/2+1/2}.$$

Proof. It suffices to find bounds for $G_k(r,s)H(\alpha k_1 s)/H(\alpha k r)$ since the other functions involved are regular everywhere for $r, s \in [0,1]$, see (36). We first consider $k \to +\infty$.

It is easily verified that $\mathcal{G}(\zeta,\eta)$, defined in (113), is the Green's function (see (86), (88)) for

$$\mathcal{L} := \Psi \mapsto \Psi'' - \frac{l(l+1)}{\zeta^2}\Psi + \frac{\Psi}{k}\left[\frac{\omega}{\alpha^2} + \frac{b}{\alpha\zeta}\right] + \frac{\Psi}{k^2\alpha^2}\left[ip_1 - n_0\omega\right] \tag{118}$$

and is given by

$$\mathcal{G}(\zeta,\eta) := k\alpha G_k(r(\zeta), r(\eta)) = \frac{\Psi_1(\zeta)\Psi_2(\eta) - \Psi_2(\zeta)\Psi_1(\eta)}{W}, \tag{119}$$

where Ψ_1, Ψ_2 are two independent solutions of $\mathcal{L}\Psi = 0$ and $W = \Psi_1(\zeta)\Psi_2'(\zeta) - \Psi_2(\zeta)\Psi_1'(\zeta)$ is their constant Wronskian.

Standard asymptotic results show there exist two independent solutions Ψ_1, Ψ_2 such that for large k, we have uniformly in $z \in [0, \sqrt{\omega k}]$,

$$\Psi_1 \sim -\frac{2^l l!}{(2l)!}\sqrt{\frac{\pi z}{2}}Y_{l+1/2}(z)\ ; \text{ where } z = \sqrt{\frac{\omega}{\alpha^2 k}}\zeta = \sqrt{\omega k}r, \tag{120}$$

$$\Psi_2 \sim \frac{2^{-l-1}(2l+2)!}{(l+1)!}\sqrt{\frac{\pi z}{2}}J_{l+1/2}(z). \tag{121}$$

The Wronskian W is asymptotic, for large k, to $(2l+1)\sqrt{\omega}/\sqrt{\alpha^2 k}$. The expressions (120) and (121) may also be used to determine the asymptotics of Ψ_1' and Ψ_2'. Using (119), (120), (121) and (36) and the bounds on W, with $l_1 = l + \frac{1}{2}$ it follows that

$$\left|\frac{F_{k_1}(s)}{F_k(r)}G_k(r,s)\right| \leqslant \frac{c_*|zz'|^{1/2}}{k^{1/2}}\left|\frac{H\left(\alpha\sqrt{\frac{k_1}{\omega}}z'\right)}{H\left(\alpha\sqrt{\frac{k}{\omega}}z\right)}\left[Y_{l_1}(z)J_{l_1}(z') - J_{l_1}(z)Y_{l_1}(z')\right]\right|, \tag{122}$$

where $z' = \eta\sqrt{\omega}/\sqrt{\alpha^2 k} = \sqrt{\omega k}s$. A similar bound holds for

$$\left|\frac{\partial}{\partial r}\left\{\frac{F_{k_1}(s)}{F_k(r)}G_k(r,s)\right\}\right|.$$

We now prove part (1). We break this case up into two subcases: (a) $r \in [k^{-2/3}, 1]$ and (b) $r \in [0, k^{-2/3}]$. In case (a), we note that $s \in [r, r+\delta]$ implies s/r and therefore

$1 \leq z'/z = O(1)$. The function H in (122) is close to 1 because its argument is large. Furthermore, note that $\sqrt{z}Y_{l+1/2}(z)$ and $\sqrt{z}J_{l+1/2}(z)$ are bounded for large z, while they are asymptotic to constant multiples of z^{-l} and z^{l+1} for small z. Using (122), part 1 of the lemma follows by inspection in case (a). For case (b), (122) further simplifies since z,z' are small and

$$\frac{H(k_1\eta/k)}{H(\zeta)}G_k(r(\zeta), r(\eta)) = \frac{H(k_1\eta/k)}{k\alpha H(\zeta)}\mathcal{G}(\zeta, \eta)$$
$$\sim \frac{H(k_1\eta/k)\left(\eta^{l+1}\zeta^{-l} - \zeta^{l+1}\eta^{-l}\right)}{k\alpha H(\zeta)(2l+1)}. \tag{123}$$

When $\zeta \in [\log k, \alpha k^{1/3}]$ and $\eta \in [\zeta, \zeta + \alpha k\delta]$, we have $1 \leq [\eta/\zeta]^l \leq c_*$ and therefore

$$\left|\frac{H(k_1\eta/k)}{H(\zeta)}G_k(r(\zeta), r(\eta))\right| = \left|\frac{H(k_1\eta/k)}{k\alpha H(\zeta)}\mathcal{G}(\zeta, \eta)\right| \leqslant \frac{c_*}{k^{1/2}}.$$

The same inequality holds if $\zeta \in [0, \log k]$, since $\eta \in [\zeta, (C_2+1)\log k]$ since in this regime $\zeta^{-l}/H(\zeta)$ is bounded and the logarithmic growth in k of terms involving η can be bounded by, say, $k^{1/2}$, while for small η, $\eta^l H(k_1\eta/k)$ is bounded. The bounds on derivatives follow in a similar manner using $\frac{d}{dr} = k\alpha\frac{d}{d\zeta}$.

Part 2 (which is only relevant for $r + \delta \leqslant 1$) follows similarly on careful inspection of (122), from the asymptotic behavior in different regimes of z and z'. □

Lemma 44. *Let $r \in (0, \hat{\epsilon}]$, with $\hat{\epsilon}_1 = \frac{C_2}{k}\log k$. We choose C_2 large enough so that $\frac{\mathfrak{s}(\epsilon_1)}{\mathfrak{s}(r)} = (1-\delta_1) \leq \frac{(5+l)\log k}{4k+2\tau}$. Then $|[\mathcal{A}_k^1 f](r)| \leqslant c_* k^{l/2-1/2}(1-\delta_1)^{2k-2+\tau}\|f\|_\infty \leqslant c_* k^{-3}\|f\|_\infty$ and $|\frac{d}{dr}[\mathcal{A}_k^1 f](r)| \leqslant c_* k^{l/2+1/2}(1-\delta_1)^{2k-2+\tau}\|f\|_\infty \leqslant c_* k^{-2}\|f\|_\infty$.*

Proof. Consider $\mathcal{A}_k^1$ given by (112). We note that $\mathfrak{s}^{-2}\Omega(s)$ and its $r-$derivative are bounded, while $G_k(s,r)F_k(s)/F_k(r)$ and its r-derivative are bounded by $c_* k^{l/2-1/2}$ and $c_* k^{l/2+1/2}$ respectively for any τ (cf. Lemma 43). Further $|\mathfrak{s}(s)/\mathfrak{s}(r)| \leqslant (1-\delta_1)$ and from (111), we have

$$(1-\delta_1)^{2k-2+\tau} \leqslant \frac{c_*}{k^{l/2+5/2}},$$

and the lemma follows. □

Remark 45. Since for $r \in (0, \hat{\epsilon}]$, the bound in Lemma 44 on $\mathcal{A}_k^1$ is $O(k^{-2})$, we will see later that $\mathcal{A}_k$ is dominated by $\mathcal{A}_k^0$ (defined in (112)) as $k \to \infty$.

Lemma 46. *Define $\mathcal{G}_0(\zeta, \eta) = \lim_{k\to\infty}\mathcal{G}(\zeta, \eta)$ and $H_0(\zeta) = \lim_{k\to\infty} H(\zeta)$, where $\zeta, \eta \ll k^{1/2}$ as $k \to \infty$. Then,*

$$\int_\zeta^\infty e^{-\eta+\zeta}\mathcal{G}_0(\zeta, \eta)\frac{H_0(\eta)}{H_0(\zeta)}d\eta = 1, \tag{124}$$

$$\int_\zeta^\infty e^{-\eta+\zeta}\mathcal{G}_{0\zeta}(\zeta, \eta)\frac{H_0(\eta)}{H_0(\zeta)}d\eta = -1 + \frac{H_0'(\zeta)}{H_0(\zeta)}. \tag{125}$$

Proof. Using (120) and (121) and the behavior of Bessel functions for small argument, [1], it follows that for $\zeta, \eta \ll k^{1/2}$ we have

$$\mathcal{G}_0(\zeta,\eta) = \lim_{k\to\infty} \mathcal{G}(\zeta,\eta) = \frac{\eta^{l+1}\zeta^{-l} - \zeta^{l+1}\eta^{-l}}{2l+1} \tag{126}$$

and $H_0(\zeta) = \lim_{k\to\infty} H(\zeta) = \sqrt{\frac{2}{\pi}}\zeta^{1/2}e^{\zeta}K_{l+1/2}(\zeta)$. Now, using the modified Bessel function equation, it is easily verified that $f(\zeta) = e^{-\zeta}H_0(\zeta)$ satisfies

$$f'' - \frac{l(l+1)}{\zeta^2} f = f$$

with $f(\zeta) \sim e^{-\zeta}$ as $\zeta \to \infty$. Using variation of parameters to invert the left hand side of the above equation, and using the boundary conditions at ∞ we obtain

$$f(\zeta) = \int_\zeta^\infty \mathcal{G}_0(\zeta,\eta) f(\eta) d\eta.$$

Dividing through by $f(\zeta)$, the first identity in the lemma follows. By differentiating the first identity with respect to ζ, and using the first identity in the resulting expression, we obtain the second identity. □

Lemma 47. *For any $r \in (0,1)$,*

$$\left|\mathcal{A}_k[1](r) - 1\right| = \left|\int_r^1 \Omega(s)\frac{m_{k-1}(s)}{m_k(r)} G_k(r,s)ds - 1\right| \leqslant \frac{c_*}{k^2}. \tag{127}$$

For $\frac{1}{k} \le r \le \frac{1}{2}$ we get

$$\left|\frac{d}{dr}\mathcal{A}_k[1](r)\right| = \left|\frac{d}{dr}\int_r^1 \Omega(s)\frac{m_{k-1}(s)}{m_k(r)} G_k(r,s)ds\right| \leqslant \frac{c_*}{k^2} + \frac{c_*}{k^3 r^2}, \tag{128}$$

while for any $r \in [0, \frac{1}{2}]$,

$$\int_r^1 \Omega(s)\left|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{m_{k-1}(s)}{m_k(r)}\right)\right| ds \leqslant c_* k. \tag{129}$$

Proof. Recalling the definition (93), it follows from (39) and Lemma 40 that

$$\mathcal{L}_k m_k - \Omega m_{k-1} = \frac{j_k(r)}{\mathfrak{s}} m_k, \tag{130}$$

where $j_k(r) = O(1)$ as $k \to +\infty$ for any $r \in [0,1]$. We can check from (36) that $m_k(1) = 0$, $m_k'(1) = 0$ for $k \geqslant 1$. From (130), inversion of $\mathcal{L}_k$ yields

$$m_k(r) = \int_r^1 G_k(r,s)\left\{\Omega(s)m_{k-1}(s) + \frac{j_k(s)}{\mathfrak{s}(s)} m_k(s)\right\} ds. \tag{131}$$

Therefore,

$$\int_r^1 G_k(r,s)\frac{\Omega(s)m_{k-1}(s)}{m_k(r)} ds = 1 - \int_r^1 G_k(r,s)\frac{j_k(s)m_k(s)}{\mathfrak{s}(s)m_k(r)} ds. \tag{132}$$

First, we choose $\hat{\delta}_1$ so that $1-\hat{\delta}_1 = (5+l)\log k/(4k+2\tau)$. We then define $\hat{\delta}$ so that $(1-\hat{\delta}_1)\mathfrak{s}(r) = \mathfrak{s}(r+\hat{\delta})$. It is clear that for large k we have $\hat{\delta} \sim (5/2+l/2)\,\mathfrak{s}(r)\log k/\left((2k+\tau)\sqrt{\Omega(r)}\right)$. Lemma 43, and the fact that $k^{l/2-1/2}(1-\hat{\delta}_1)^{2k+1+\tau}/(2k+1+\tau) \leqslant \frac{1}{k^3}$ give

$$\left|\int_{r+\hat{\delta}}^1 G_k(r,s)\frac{j_k(s)m_k(s)}{\mathfrak{s}(s)m_k(r)}ds\right| \leqslant \left|\int_0^{1-\hat{\delta}_1} t^{2k+\tau} \times \frac{F_k(r(\mathfrak{s}t)}{\sqrt{\Omega(r(\mathfrak{s}t))}F_k(r(\mathfrak{s})}G_k(r(\mathfrak{s}),r(\mathfrak{s}t))j_k(r(\mathfrak{s}t))dt\right| \leqslant \frac{c_*}{k^3}\|j_k\|_\infty \leqslant \frac{c_*}{k^3}. \tag{133}$$

Now, consider the contribution from $\int_r^{r+\hat{\delta}}$. There are again two cases: (i) $1 \geqslant r \geqslant k^{-2/3}$ and (ii) $0 < r \leqslant k^{-2/3}$.

In the first case, Taylor expanding $G_k(r,s)$ near $s=r$ we get $G_k = (s-r)+O((s-r)^3 Q_k) = \sqrt{\Omega(r)}\mathfrak{s}(1-t) + O(k^{4/3}(1-t)^3, (1-t)^2)$. Hence,

$$\left|\int_r^{r+\hat{\delta}} G_k(r,s)\frac{j_k(s)m_k(s)}{\mathfrak{s}(s)m_k(r)}ds\right| \leqslant c_*\|j_k\|_\infty \int_{1-\hat{\delta}_1}^1 t^{2k+\tau-1}(1-t)dt \leqslant \frac{c_*}{k^2}. \tag{134}$$

For the case (ii), we rewrite the integral in terms of $\zeta = k\alpha r$, to obtain

$$\begin{aligned}\left|\int_r^{r+\hat{\delta}} G_k(r,s)\frac{j_k(s)m_k(s)}{\mathfrak{s}(s)m_k(r)}ds\right| &\leqslant \frac{c_*}{k^2}\|j_k\|_\infty \int_\zeta^{\zeta+k\alpha\hat{\delta}} e^{-Q(\eta)+Q(\zeta)}\mathcal{G}(\zeta,\eta)\frac{H(\eta)}{H(\zeta)}d\eta \\ &\leqslant \frac{c_*}{k^2}\int_\zeta^{\zeta+k\alpha\hat{\delta}} d\eta e^{-\eta+\zeta}\mathcal{G}_0(\zeta,\eta)\frac{H_0(\eta)}{H_0(\zeta)} \\ &\leqslant \frac{c_*}{k^2}\int_\zeta^{\infty} d\eta e^{-\eta+\zeta}\mathcal{G}_0(\zeta,\eta)\frac{H_0(\eta)}{H_0(\zeta)} \leqslant \frac{c_*}{k^2}\end{aligned} \tag{135}$$

by Lemma 46. Using (132) and (135), the first part follows.

To prove (128), we note that if C_3 is large and $rk > C_3$, Taylor expansion gives

$$\begin{aligned}U_1(\mathfrak{s},t) &:= \frac{F_k(r(\mathfrak{s}t)}{\sqrt{\Omega(r(\mathfrak{s}t))}F_k(r(\mathfrak{s})}G_k(r(\mathfrak{s}),r(\mathfrak{s}t)) \\ &= f_4(\mathfrak{s})(1-t) + O\left((1-t)^2, k(1-t)^3, \frac{(1-t)^3}{r^2}, \frac{(1-t)^2}{kr^2}\right)\end{aligned} \tag{136}$$

for $f_4 = -\mathfrak{s}/\Omega(r(\mathfrak{s}))$, while

$$\frac{\partial}{\partial\mathfrak{s}}U_1(\mathfrak{s},t) = f_4'(\mathfrak{s})(1-t) + O\left((1-t)^2, k(1-t)^3, \frac{(1-t)^3}{r^3}, \frac{(1-t)^2}{kr^3}\right).$$

From (132) we note that

$$\frac{d}{dr}\mathcal{A}_k[1](r) = -\sqrt{\Omega(r(\mathfrak{s}))}\left\{\int_{1-\hat{\delta}_1}^{1} t^{2k+\tau+1}\frac{j_k'(r(\mathfrak{s}t))}{\sqrt{\Omega(r(\mathfrak{s}t))}}U_1(\mathfrak{s},t)dt - \int_{1-\hat{\delta}_1}^{1} t^{2k+\tau} j_k(r(\mathfrak{s}t))U_{1,\mathfrak{s}}(\mathfrak{s},t)dt\right\} - \frac{d}{dr}\int_{r+\hat{\delta}}^{1}\left(\frac{\xi(s)}{\xi(r)}\right)^{2k+\tau}\frac{j_k(s)F_k(s)}{\xi(s)F_k(r)}G_k(r,s)ds. \quad (137)$$

We note further that

$$-\frac{d}{dr}\int_{r+\hat{\delta}}^{1}\left(\frac{\xi(s)}{\xi(r)}\right)^{2k+\tau}\frac{j_k(s)F_k(s)}{\xi(s)F_k(r)}G_k(r,s)ds = -\int_{r+\hat{\delta}}^{1}\left(\frac{\xi(s)}{\xi(r)}\right)^{2k+\tau}\frac{j_k(s)}{\xi(s)}\frac{\partial}{\partial r}\left[\frac{F_k(s)}{F_k(r)}G_k(r,s)\right]ds + (2k+\tau)\frac{\xi'(r)}{\xi(r)}\int_{r+\hat{\delta}}^{1}\left(\frac{\xi(s)}{\xi(r)}\right)^{2k+\tau}\frac{j_k(s)F_k(s)}{\xi(s)F_k(r)}G_k(r,s)ds + \left(\frac{\xi(r+\hat{\delta})}{\xi(r)}\right)^{2k+\tau}\frac{j_k(r+\hat{\delta})F_k(r+\hat{\delta})}{\xi(r+\hat{\delta})F_k(r)}G_k(r,r+\hat{\delta})\left(1+\hat{\delta}'(r)\right). \quad (138)$$

From the bounds in Lemmas 40 and 43 and the fact that $\xi(s)/\xi(r) \le (1-\hat{\delta}_1)$, we easily conclude that the contribution of $\int_{r+\hat{\delta}}^1$ in (138) to $\frac{d}{dr}\mathcal{A}_k[1](r)$ is $O(1/k^2)$.

Since Lemma 40 implies $|j_k(r)| < c_*$ and $|j_k'(r)| < c_* + c_*/(kr^2)$ for $\frac{1}{2} \ge r \ge \frac{1}{k}$, it follows from the local expansion of $U_1(\mathfrak{s},t)$ and its $\mathfrak{s}$-derivative in a neighborhood of $t=1$ in the first integral in (137) that

$$\left|\frac{d}{dr}\int_r^{r+\hat{\delta}} G_k(r,s)\frac{j_k(s)m_k(s)}{\mathfrak{s}(s)m_k(r)}ds\right| \leqslant \frac{c_*}{k^3r^2}+\frac{c_*}{k^2}$$

and (128) follows.

We now prove (129). We first note that for $r \ge k^{-2/3}$, $s \in (r, r+\hat{\delta})$, from (90), $\partial_r G(r,s) = -1$ at $s=r$ and therefore, from (120), (121), it follows that for $s-r = O(k^{-1}\log k) \ll k^{-1/2}$, $\partial_r G(r,s) \sim -1 < 0$ for $s \in (r, r+\hat{\delta})$. The same is true for $r \in [0, k^{-2/3}]$ since in this regime, $\partial_r G_k(r,s) \sim \partial_\zeta \mathcal{G}_0(\zeta,\eta)$ (see (126)), with $\zeta = r/(\alpha k)$, $\eta = s/(\alpha k)$. Therefore, from (31) and (36), we get

$$-\frac{\partial}{\partial r}\left(\frac{m_{k-1}(r)G_k(r,s)}{m_k(r)}\right) = -\left[(2k+\tau-2)\frac{\sqrt{\Omega(r)}}{\xi(r)} - \frac{F_k'(r)}{F_k(r)}\right]\frac{m_{k-1}(r)G_k(r,s)}{m_k(r)} - \partial_r G_k(r,s)\frac{m_{k-1}(s)}{m_k(r)}. \quad (139)$$

Since the contributions to the integrals from $\int_{r+\hat{\delta}_1}^1$ is $O(\frac{1}{k^2})$, and the first term on the right on (139) is negative for large k, while the second is positive, it follows that

$$\int_r^1 \Omega(s) \left| \frac{\partial}{\partial r} \left(\frac{m_{k-1}(r)}{m_k(r) G_k(r,s)} \right) \right| ds \le \left| \frac{d}{dr} \mathcal{A}_k[1](r) \right|$$
$$+2 \left[(2k + \tau - 2) \frac{\sqrt{\Omega(r)}}{\xi(r)} - \frac{F_k'(r)}{F_k(r)} \right] |\mathcal{A}_k[1](r)| + O\left(\frac{1}{k^2}\right) \le c_* k \tag{140}$$

for $r \in \left[C_2 k^{-1}, 1\right]$. For $r \in \left[0, C_2 k^{-1}\right]$, we note that since the contribution from $\int_{r+\hat{\delta}}^1$ for $\frac{d}{dr}\mathcal{A}_k[1](r)$ is negligible, we have

$$\frac{d}{dr}\mathcal{A}_k[1](r) \sim k\alpha \frac{d}{d\zeta} \int_\zeta^{\zeta + k\alpha\hat{\delta}_1} e^{-Q(\eta)+Q(\zeta)} \left(1 + \frac{a_1}{k}\right) \frac{H(\eta(1-1/k))}{H(\zeta)} \mathcal{G}(\zeta, \eta)$$
$$\sim k\alpha \frac{d}{d\zeta} \int_\zeta^{\zeta + k\alpha\hat{\delta}_1} e^{-\eta+\zeta} \frac{H_0(\eta)}{H_0(\zeta)} \mathcal{G}_0(\zeta, \eta)$$
$$\sim k\alpha \frac{d}{d\zeta} \int_\zeta^{\infty} e^{-\eta+\zeta} \frac{H_0(\eta)}{H_0(\zeta)} \mathcal{G}_0(\zeta, \eta); \tag{141}$$

it follows immediately from Lemma 46 that in this case , $|\frac{d}{dr}\mathcal{A}_k[1](r)| \le c_* k$. Hence the inequality in (140) is valid for all $r \in [0, 1/2]$. □

Lemma 48. *For any* $f \in L^\infty[0, 1]$,

$$(a) \quad For \ \ r \in [0, 1], \quad \|\mathcal{A}_k f\|_\infty \leqslant \left(1 + \frac{c_*}{k^2}\right) \|f\|_\infty, \tag{142}$$

$$(b)\ For \ \ r \in \left[0, \frac{1}{2}\right], \quad \left\| \frac{d}{dr}[\mathcal{A}_k f](r) \right\|_\infty \leqslant c_* k \|f\|_\infty. \tag{143}$$

Proof. Consider the expression for $\mathcal{A}_k f$ from (93). We break up the integral into $\int_r^{r+\delta}$ and $\int_{r+\delta}^1$, where $\delta = C_2 k^{-1} \log k$, with C_2 large enough so that

$$(1 - \delta_1)^{2k-2+\tau} \leqslant \frac{1}{k^{l/2+7/2}}; \quad (1 - \delta_1) := \frac{\mathfrak{s}(r+\delta)}{\mathfrak{s}(r)}.$$

From (36) and Lemma 43, part (2), transforming the integration variable to t, it follows that

$$\left| \int_{r+\delta}^1 \Omega(s) \frac{m_k(s)}{m_k(r)} G_k(r,s) f(s) ds \right| \leqslant \frac{c_*}{k^2} \|f\|_\infty. \tag{144}$$

In $\int_r^{r+\delta}$ (we replace the upper limit $r+\delta$ by 1 if $r + \delta > 1$). Since $\delta_1 = O\left(k^{-1}\log k\right)$ and $t \in (1 - \delta_1, 1)$ then $T_k(\mathfrak{s}, t) \ge 0$ and $G_k(r, s) \geqslant 0$ for $r \in [k^{-2/3}, 1]$. Therefore,

$$\|\mathcal{A}_k f\|_\infty \leqslant \|f\|_\infty \left\{ \left[\int_r^{r+\delta} \frac{\Omega(s) m_{k-1}(s)}{m_k(r)} G_k(r,s) \right] + \frac{c_*}{k^2} \right\}. \tag{145}$$

From (144) we get

$$\left| \int_{r+\delta}^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds \right| \leqslant \frac{c_*}{k^2}.$$

Hence

$$\int_r^{r+\delta} \Omega(s)\frac{m_{k-1}(s)}{m_k(r)}G_k(r,s)ds = \int_r^1 \Omega(s)\frac{m_{k-1}(s)}{m_k(r)}G_k(r,s)ds + O\left(\frac{1}{k^2}\right). \tag{146}$$

Using Lemma 47, (142) (a) follows. For (b) we write

$$\frac{d}{dr}\int_r^1 \Omega(s)\frac{m_{k-1}(s)}{m_k(r)}G_k(r,s)f(s)ds$$
$$= \int_r^1 \Omega(s)\frac{\partial}{\partial r}\left(G_k(r,s)\frac{m_{k-1}(s)}{m_k(r)}\right)f(s)ds. \tag{147}$$

By Lemma 47, the quantity above is bounded by $c_* k\|f\|_\infty$. □

Lemma 49. *For any* $f \in \mathcal{L}_\infty[0,1]$*,*

$$\|\mathcal{H}_k f\|_\infty \leqslant \frac{c_*}{k^2}\|f\|_\infty,$$
$$\left\|\frac{d}{dr}[\mathcal{H}_k f](r)\right\|_\infty \leqslant \frac{c_*}{k^2}\|f\|_\infty.$$

Proof. As before, we choose $\delta = C_2 k^{-1}\log k$ large C_2 independent of k. Using Lemma 43, it follows that

$$\left|\int_{r+\delta}^1 \frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)f(s)ds\right| \leqslant c_*(1-\delta_1)^{2k+2}k^{l/2-5/2}\|f\|_\infty$$
$$\leqslant \frac{c_*}{k^4}\|f\|_\infty, \tag{148}$$
$$\left|\int_{r+\delta}^1 \frac{\partial}{\partial r}\left\{\frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)\right\}f(s)ds\right|$$
$$\leqslant c_*(1-\delta_1)^{2k+2}k^{l/2-3/2}\|f\|_\infty \leqslant \frac{c_*}{k^3}\|f\|_\infty. \tag{149}$$

Now, Lemma 43 implies

$$\left|\int_r^{r+\delta} \frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)f(s)ds\right| \leqslant c_*\frac{\|f\|_\infty}{k^2}\int_0^1 t^{2k+2+\tau}dt$$
$$\leqslant \frac{c_*\|f\|_\infty}{k^3}, \tag{150}$$
$$\left|\int_r^{r+\delta}\frac{\partial}{\partial r}\left\{\frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)\right\}f(s)ds\right| \leqslant \frac{c_*\|f\|_\infty}{k}\int_0^1 t^{2k+2+\tau}dt$$
$$\leqslant \frac{c_*\|f\|_\infty}{k^2}. \tag{151}$$

□

Lemma 50. *There exist* k_0 *and* c_**, independent of* k*, so that for* $k > k_0$*, over the* r*-interval* $(0,1)$*,*

$$\|h_k\|_\infty < c_*. \tag{152}$$

Proof. First we note that for k_0 sufficiently large, $\|h_{k_0}\|_\infty$ exists since g_{k_0} is continuous for $r \in [0, 1]$ and the expression for m_k in (36) shows that $1/m_{k_0}$ is bounded as well for sufficiently large k_0 since $K_{l+1/2}$ has no zeros in the region of interest. Define $r_k = \mathcal{H}_k h_{k+1}$. Note that

$$h_k = \mathcal{A}_k \left(\mathcal{A}_{k-1} h_{k-2} + r_{k-1}\right) + r_k. \tag{153}$$

In $k - k_0$ inductive steps we get

$$h_k = \mathcal{A}_k \mathcal{A}_{k-1} .. \mathcal{A}_{k_0+1} h_{k_0} + \mathcal{H}_k h_{k+1} + \sum_{m=1}^{k-k_0-1} \left(\prod_{j=1}^{m} \mathcal{A}_{k-j+1} \right) \mathcal{H}_{k-m} h_{k-m+1}. \tag{154}$$

We write this abstractly as

$$\mathfrak{h} = \mathfrak{h}^0 + \mathfrak{N}\mathfrak{h}, \tag{155}$$

where

$$\begin{aligned} \mathfrak{h}_k^0 &= \mathcal{A}_k \mathcal{A}_{k-1} .. \mathcal{A}_{k_0+1} h_{k_0}; \\ [\mathfrak{N}\mathfrak{h}]_k &= \mathcal{H}_k h_{k+1} + \sum_{m=1}^{k-k_0-1} \left(\prod_{j=1}^{m} \mathcal{A}_{k-j+1} \right) \mathcal{H}_{k-m} h_{k-m+1}, \end{aligned} \tag{156}$$

and $\mathfrak{N}$ is defined on the space $\mathcal{S}$ of sequences $\mathfrak{h} = \{h_k\}_{k=k_0+1}^{\infty}$ in the norm

$$\|\mathfrak{h}\| = \sup_{k \geqslant k_0+1} \|h_k\|_\infty. \tag{157}$$

Lemmas 48 and 49 imply

$$\begin{aligned} |[\mathfrak{N}\mathfrak{h}]_k| &\leqslant \|\mathfrak{h}\|_\infty \left(\frac{c_*}{k_0^2} + c_* \sum_{m=1}^{k-k_0-1} \left\{ \prod_{j=1}^{m} \left[1 + \frac{c_*}{(k-j+1)^2} \right] \right\} \frac{1}{(k-m)^2} \right) \\ &< \nu \|\mathfrak{h}\|_\infty, \end{aligned} \tag{158}$$

where, if k_0 is large $\nu < 1$ is independent of k. Thus, $\mathfrak{N}$ is contractive and there is a unique solution of (155) in $\mathcal{S}$. □

Lemma 51. *For any $r \in \left[0, \frac{1}{2}\right]$ and for large enough k we have $\|\frac{d}{dr} h_k\|_\infty \leqslant c_* k$.*

Proof. Since by Lemma 50 h_k is bounded, Lemmas 49 and 48 imply

$$|h_k'(r)| \leqslant |\frac{d}{dr}[\mathcal{A}_k h_{k-1}](r)| + |\frac{d}{dr}[\mathcal{H}_k h_{k+1}](r)| \leqslant c_* k.$$

□

Lemma 52. *For all $k \geqslant 1$, $h_k(1) = 1$.*

Proof. In case (**i**), a simple computation shows that

$$\frac{\partial^{2k} g_{n_0-k}}{\partial \mathfrak{s}^{2k}}|_{\mathfrak{s}=0} = i^k h_k(1); \quad (g_{n_0-k} := i^k m_k h_k).$$

(By the differential equation for h_k, all derivatives exist.) Lemma 37 with $j = 2k$ gives

$$i^k = \frac{\partial^{2k}}{\partial \mathfrak{s}^{2k}}|_{\mathfrak{s}=0} g_{n_0-k} = i^k h_k(1),$$

implying the result in case (**i**). In case (**ii**), using Lemma 38, a similar computation shows that

$$i^k = \frac{\partial^{2k+1}}{\partial \mathfrak{s}^{2k+1}}|_{\mathfrak{s}=0} g_{n_0-k} = i^k h_k(1) \ \ (g_{n_0-k} := i^k m_k h_k).$$

□

Definition 53. *Let*

$$\hat{T}_k(\mathfrak{s}, s) = s^{-2k+1-\tau} \int_0^s t^{2k-2+\tau} \mathfrak{s} \frac{\partial}{\partial \mathfrak{s}} T_k(\mathfrak{s}, t) dt, \tag{159}$$

where $T_k(\mathfrak{s}, t)$ is defined in (95).

Lemma 54. *Let $\delta = k^{-1} \log k$ and $S_k(\mathfrak{s}) := \frac{\partial}{\partial \mathfrak{s}} \int_0^1 t^{2k-2} T_k(\mathfrak{s}, t) dt$. If C_2 is large enough, $s \in (0, \delta)$ and $r(\mathfrak{s}) \geqslant k^{-1} C_2$, we have*

$$\hat{T}_k(\mathfrak{s}, s) = \mathfrak{s} S_k(\mathfrak{s}) - \frac{\mathfrak{s} f_1'(\mathfrak{s})}{12} (1-s)^3 + \frac{\mathfrak{s} f_3(\mathfrak{s})}{3kr^3} (1-s)^3 \\ + O\left(\frac{(1-s)^4}{kr^4}, \frac{(1-s)^3}{k^2 r^4}, \frac{(1-s)^2}{k^3 r^4}, \frac{(1-s)}{k^4 r^3}, \frac{(1-s)^3}{kr^2}, \frac{(1-s)^2}{k^2 r^2} \right). \tag{160}$$

Proof. This simply follows by integrating (103) from $t = 1$ to s of T_k and the fact that $\hat{T}_k(\mathfrak{s}, 1) = \mathfrak{s} S_k(\mathfrak{s})$. □

4.13. Proof of Lemma 31. First choose $\epsilon_1 > 0$. From Lemma 49, it follows that

$$\left\| \frac{d}{d\mathfrak{s}} [\mathcal{H}_k h_{k+1}] \right\|_\infty \leqslant \frac{c_*}{k^2} \|h_{k+1}\|_\infty \leqslant \frac{c_*}{k^2},$$

where we applied Lemma 50. Further, we note that

$$\frac{1}{(2k+\tau)(2k+\tau-1)} \frac{d}{d\mathfrak{s}} \mathcal{A}_k h_{k-1}(\mathfrak{s}) \\ = \int_0^1 t^{2k+\tau-2} \frac{\partial T_k}{\partial \mathfrak{s}} (\mathfrak{s}, t) h_{k-1}(\mathfrak{s} t) dt + \int_0^1 t^{2k+\tau-1} T_k(\mathfrak{s}, t) h_{k-1}'(\mathfrak{s} t) dt. \tag{161}$$

We have

$$\int_0^1 t^{2k+\tau-2}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)h_{k-1}(\mathfrak{s}t)dt = h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 dt\ t^{2k+\tau-2}\mathfrak{s}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)$$
$$\times \int_t^1 h'_{k-1}(\mathfrak{s}s)ds = h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 h'_{k-1}(\mathfrak{s}s)\left[\int_0^s t^{2k+\tau-2}\mathfrak{s}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)dt\right]ds$$
$$= h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 h'_{k-1}(\mathfrak{s}s)s^{2k-1+\tau}\hat{T}_k(\mathfrak{s},s)ds = (2k+\tau-1)S_k(\mathfrak{s})$$
$$\times \int_0^1 s^{2k-2+\tau}h_{k-1}(\mathfrak{s}s)ds - \int_0^1 s^{2k-1+\tau}[\hat{T}_k(\mathfrak{s},s) - \hat{T}_k(\mathfrak{s},1)]h'_{k-1}(\mathfrak{s}s)ds. \tag{162}$$

Therefore,

$$\frac{\frac{d}{d\mathfrak{s}}\mathcal{A}_k[h_{k-1}](\mathfrak{s})}{(2k+\tau)(2k+\tau-1)} = \int_0^1 [T_k(\mathfrak{s},s) - \hat{T}_k(\mathfrak{s},s) + \mathfrak{s}S_k(\mathfrak{s})]s^{2k+\tau-1}$$
$$\times h'_{k-1}(\mathfrak{s}s)ds + (2k+\tau-1)S_k(\mathfrak{s})\int_0^1 s^{2k+\tau-2}h_{k-1}(\mathfrak{s}s)ds. \tag{163}$$

We note that

$$(2k+\tau)(2k+\tau-1)S_k(\mathfrak{s}) = \frac{\partial}{\partial \mathfrak{s}}[\mathcal{A}_k[1](\mathfrak{s})] = O\left(\frac{1}{k^3\epsilon_1^2}, \frac{1}{k^2}\right)$$

and that $(2k+\tau-1)\int_0^1 s^{2k+\tau-2}h_{k-1}(\mathfrak{s}s)ds$ has a bound independent of k. Combining (103) with Lemma 54, if k is large so that $k\epsilon_1$ is large, then

$$T_k(\mathfrak{s},s) - [\hat{T}_k(\mathfrak{s},s) - \mathfrak{s}S_k(\mathfrak{s})] = (1-s) + \left(-\frac{kf_1}{4} + \frac{f_2}{r^2}\right)$$
$$\times\left[-\frac{(1-s)^2}{k} + \frac{2}{3}(1-s)^3\right] - \mathfrak{s}\left(-\frac{f_1'}{12} + \frac{f_3}{3kr^3}\right)(1-s)^3$$
$$+O\left(\frac{(1-s)^4}{kr^4}, \frac{(1-s)^3}{k^2r^4}, \frac{(1-s)^2}{k^3r^4}, \frac{(1-s)}{k^4r^3}, \frac{(1-s)^3}{kr^2}, \frac{(1-s)^2}{k^2r^2},\right.$$
$$\left.\times\ \frac{(1-s)^4}{r^3}, \frac{(1-s)^3}{kr^3}, \frac{(1-s)^3}{r}, \frac{(1-s)^2}{kr}\right). \tag{164}$$

From (164), it is clear that $T_k(\mathfrak{s},s) - \hat{T}_k(\mathfrak{s},s) + \mathfrak{s}S_k(\mathfrak{s}) > 0$ if $s \in (1-\delta, 1)$ and $k\epsilon_1$ is sufficiently large. Now, $\mathfrak{s}f_3/\left(3kr^3\right)(1-s)^3 > 0$ exceeds any term following it in (164), except possibly when $1-r$, *i.e.* $\mathfrak{s}$ is small. Thus, if we define

$$M_k = \sup_{r(\mathfrak{s})\in[\epsilon_1,1]} |h_k'(\mathfrak{s})| \tag{165}$$

we get

$$|h_k'(\mathfrak{s})| \leqslant (2k+\tau)(2k+\tau-1)M_{k-1}\left\{\int_{1-\delta_1}^{1} s^{2k+\tau-1}\left[(1-s)+\left(-\frac{kf_1}{4}+\frac{f_2}{r^2}\right)\right.\right.$$
$$\left.\left.\times\ [-\frac{1}{k}(1-s)^2+\frac{2}{3}(1-s)^3]+\mathfrak{s}\frac{f_1'}{12}(1-s)^3\right]ds\right\}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}. \qquad (166)$$

When $(1-r)$ (and thus $\mathfrak{s}$) is small, we can replace the term $\mathfrak{s}f_1'/(12)(1-s)^3$ on the right side of the above equation simply by $(1-s)^3$, which is clearly bigger. From the fact that $\int_{1-\delta}^{1} s^{2k-1}[-k^{-1}(1-s)^2+(2/3)(1-s)^3]ds = O(k^{-5})$, it follows that

$$M_k \leqslant M_{k-1}\left(\frac{2k-1+\tau}{2k+1+\tau}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}\right)+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}. \qquad (167)$$

Let C_3 be large enough and define $k_0(\epsilon_1)=C_3/\epsilon_1$, so that for $k\geqslant k_0$ we have

$$\left(\frac{2k+\tau-1}{2k+\tau+1}+\frac{c_*}{\epsilon_1^2k^3}+\frac{c_*}{k^2}\right)\leqslant\left(\frac{k-1}{k}\right)^{1/2}.$$

Then for $k\geqslant k_0$,

$$M_k \leqslant \left(\frac{k-1}{k}\right)^{1/2} M_{k-1}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}, \qquad (168)$$

implying

$$M_k \leqslant \left(\frac{k_0}{k}\right)^{1/2}$$
$$M_{k_0}+\frac{c_*}{k^{1/2}}\sum_{j=k_0}^{k}\frac{1}{j^{3/2}}+\frac{c_*}{k^1/2}\sum_{j=k_0}\frac{1}{j^{5/2}\epsilon_1^2}\leqslant c_*\frac{k_0^{3/2}}{k^{1/2}}+\frac{c_*}{k^{1/2}k_0^{1/2}}+\frac{c_*}{k^{1/2}k_0^{3/2}\epsilon_1^2}. \qquad (169)$$

The result follows from the definition of M_k and noting that last two terms in (169) are $O(c_*k_0^{3/2}k^{-1/2})$.

4.14. Proof of Lemma 33. From Lemma 31 and the definition of k_0, it follows that

$$|h_k'(\epsilon_1)| \leqslant \frac{C_4C_3^{3/2}}{k^{1/2}\epsilon_1^{3/2}}$$

for $k\geqslant C_3{\epsilon_1}^{-1}=k_0$. Using $h_k(1)=1$, it follows that for $k\geq C_3/r$,

$$|h_k(r)-1| \leqslant \int_r^1 |h_k'(r')|dr' \leqslant \frac{C_4C_3^{3/2}}{\frac{1}{2}(kr)^{1/2}}.$$

Additionally, if[7] $\alpha kr \geqslant \left(\frac{C_4C_3^{3/2}\alpha^{1/2}}{\frac{1}{2}\epsilon}\right)^2 = L_\epsilon$ then $|h_k(r)-1|\leqslant\epsilon$.

[7] It is to be noted that for small enough ϵ the inequality $\alpha kr \geq L_\epsilon$ always implies $k\geq C_3/r$.

4.15. Proof of Lemma 35. For $\zeta \in [0, L_\epsilon]$, using the *a priori* boundedness of h_k in k and Lemma 51, we note that both $\tilde{h}_k(\zeta) := h_k(r(\zeta))$ and $(\tilde{h}_k)_\zeta$ are bounded independently of k. Hence the sequence $\{\tilde{h}_k\}_{k\geqslant 2}$ is bounded and equicontinuous. By Ascoli-Arzelà's theorem, there exists a subsequence $\tilde{h}_{k_j}(\zeta)$ converging to a continuous function $\tilde{h}$. The first part of the result is proved. We first prove that $|\tilde{h}(\zeta)-1| \leq 4\epsilon$. Now, from Lemma 33,

$$|\tilde{h}_k(\zeta) - 1| \leqslant \epsilon \quad \text{for } \zeta \in [L_\epsilon, \alpha k] \text{ for sufficiently large } k. \tag{170}$$

Let $\tilde{h}_{k,j}$ be a subsequence that converges to $\tilde{h}$ for $\zeta \in [0, L_\epsilon]$. Let ζ_m, ζ_M be a minimum, and a maximum point of $\tilde{h}$ on $[0, L_\epsilon]$ and the corresponding minimum and maximum values are denoted by m and M respectively. Continuity at the endpoint $\zeta = L_\epsilon$ implies that $M \geq 1-\epsilon$, $m \leq 1+\epsilon$. If both $M-1-\epsilon < 0$ and $m-1+\epsilon > 0$, there is nothing to prove because in that case it is clear that $|\tilde{h}(\zeta)-1| \leq 2\epsilon$. Now, consider the possibility that **(i)**: $M > 1+\epsilon$. In a similar manner, we will also consider the possibility **(ii)**: $m < 1-\epsilon$. Consider **(i)** first. Since at the end point of the interval, $\tilde{h}(L_\epsilon) < 1+\epsilon$, from continuity there exists an interval $[a,b] \subset [\zeta_M, L_\epsilon]$ of nonzero length for which

$$\tilde{h}(\eta) \leq \frac{1}{2}(M+1+\epsilon) < M \text{ for } \eta \in [a,b]. \tag{171}$$

For some $\hat{L} > L_\epsilon$, independent of k (to be determined shortly), we write

$$\begin{aligned}\left[\mathcal{A}_k^0 f\right](\zeta) &= \left(\int_\zeta^{\hat{L}} + \int_{\hat{L}}^{k\alpha\epsilon_1}\right) K(\zeta,\eta) f(\eta(1-k^{-1}))d\eta \\ &\qquad \text{with } K(\zeta,\eta) := e^{-Q(\eta)+Q(\zeta)}\left(1+\frac{a_1}{k}\right)\frac{H(\eta(1-k^{-1})}{H(\zeta)}\mathcal{G}(\zeta,\eta)d\eta \\ &=: [\mathcal{A}_k^{00} f](\zeta) + [\mathcal{A}_k^{01} f](\zeta).\end{aligned} \tag{172}$$

For fixed ζ and η we have

$$\lim_{k\to\infty} K(\zeta,\eta) = K_0(\zeta,\eta) = e^{-\eta+\zeta}\frac{H_0(\eta)}{H_0(\zeta)}\mathcal{G}_0(\zeta,\eta). \tag{173}$$

On our interval we have $\eta \geqslant \zeta$. Thus $\mathcal{G}_0 \geqslant 0$ (see (126)); $\mathcal{G}_0$ can vanish only if $\eta = \zeta$. Furthermore, by (171) we have $\zeta_M \notin [a,b]$. We can then define

$$J = \frac{3\sup_{[0,L_\epsilon]}|\tilde{h}|}{(b-a)K_m}, \text{ where } K_m = \min_{\eta\in[a,b]} K_0(\zeta_M,\eta) > 0.$$

Note that $Q(\eta) \sim \eta$ for large k and, aside from the exponential term, K is algebraically bounded. We can thus choose $\hat{L} > L_\epsilon$ large enough independently of k, so that

$$|[\mathcal{A}_k^{01} f](\zeta)| \leqslant \epsilon J^{-1}\|f\|_{\infty,[L_\epsilon,k\alpha\epsilon_1]}. \tag{174}$$

There is a subsequence of $\tilde{h}_{k_j}$ that converges uniformly on $\in [0,\hat{L}]$; for simplicity, we will use the same notation $\tilde{h}_{k,j}$ for the subsequence. It is clear that the limit is $\tilde{h}(\zeta)$ if $\zeta \in [0, L_\epsilon]$. We keep the notation $\tilde{h}$ for the limit on $[0,\hat{L}]$. We note that (170) implies

$$|\tilde{h}(\zeta) - 1| \leqslant \epsilon \quad \text{for } \zeta \in [L_\epsilon, \hat{L}]. \tag{175}$$

Now choose a small $\epsilon_2 > 0$. It is clear that in the interval $[L_\epsilon, \hat{L}]$, $\tilde{h}(\zeta) \le 1 + \epsilon < M$. For sufficiently large k_j, using continuity of $\tilde{h}(\zeta)$, we have

$$\begin{aligned}
[\mathcal{A}^{00}_{k,j}\tilde{h}(\zeta_M)] &\le \int_{\eta\in[\zeta_M,\hat{L}]\setminus[a,b]} K(\zeta_M,\eta)\tilde{h}(\eta)d\eta + \int_a^b K(\zeta,\eta)\tilde{h}(\eta)d\eta + M\epsilon_2 \\
&\leqslant M\int_{\eta\in[\zeta_M,\hat{L}]\setminus[a,b]} K(\zeta_M,\eta)d\eta + \frac{1}{2}(M+1+\epsilon)\int_a^b K(\zeta_M,\eta)d\eta + \epsilon_2 M \\
&= M\int_0^{\hat{L}} K(\zeta_M,\eta)d\eta - \frac{1}{2}(M-1-\epsilon)\int_a^b K(\zeta_M,\eta)d\eta + M\epsilon_2 \\
&\leqslant M\mathcal{A}^{00}_{k_j}[1](\zeta_M) - \frac{(b-a)}{3}(M-1-\epsilon)K_m + M\epsilon_2.
\end{aligned}$$

Since $\mathcal{A}_{k_j}[1] = \mathcal{A}^{00}_{k_j}[1] + \mathcal{A}^{01}_{k_j}[1] + \mathcal{A}^{1}_{k_j}[1]$ (see (112) and (172)) Lemmas 47, 44 and (174) imply that for large k_j we have

$$[\mathcal{A}^{00}_{k_j}[1]](\zeta_M) \leqslant 1 + \frac{\epsilon}{J} + \epsilon_2.$$

Hence, for large k_j we have

$$[\mathcal{A}^{00}_{k_j}\tilde{h}](\zeta_M) \leqslant M\left(1 + \frac{\epsilon}{J} + 2\epsilon_2\right) - \frac{K_m}{3}(M-1-\epsilon)(b-a). \tag{176}$$

Now, there exists N so that if $j \geqslant N$, $\|\tilde{h}_{k_j} - \tilde{h}\|_{\infty,[0,\hat{L}]} < \epsilon_2$ and $\Lambda_j = \mathcal{A}_{k_{j+1}}...\mathcal{A}_{k_j+1}$ satisfies

$$\|\Lambda_j - I\|_\infty \leqslant \epsilon_2$$

while

$$r_{j+1} := B_{k_{j+1}} + \sum_{m=1}^{k_{j+1}-k_j-1}\prod_{l=1}^{m}\mathcal{A}_{k_{j+1}-l+1}B_{k_{j+1}-m},$$

where $B_l = \mathcal{H}_l h_{l+1}$, satisfies the estimate

$$|r_{j+1}| < \epsilon_2.$$

Therefore, from

$$\tilde{h}_{k_{j+1}} = \Lambda_j\mathcal{A}_{k_j}\tilde{h}_{k_j} + r_{j+1}$$

it follows that

$$\tilde{h}_{k_{j+1}}(\zeta_M) \geqslant \tilde{h}(\zeta_M) - \epsilon_2 = M - \epsilon_2.$$

On the other hand, at $\zeta = \zeta_M$ we have

$$\Lambda_j\mathcal{A}_{k_j}\tilde{h}_{k_j} + r_{j+1} \leqslant (1+\epsilon_2)\left[M(1+\frac{\epsilon}{J}+2\epsilon_2) + \epsilon_2 - \frac{K_m}{3}(M-1-\epsilon)(b-a)\right] + \epsilon_2. \tag{177}$$

Thus,

$$M - \epsilon_2 \leqslant (1+\epsilon_2)\left[M(1+\frac{\epsilon}{J}+2\epsilon_2)+\epsilon_2 - \frac{K_m}{3}(M-1-\epsilon)(b-a)\right]+\epsilon_2.$$

This is true for any ϵ_2, hence as $\epsilon_2 \downarrow 0$. Thus,

$$M \leqslant \left[M\left(1+\frac{\epsilon}{J}\right) - \frac{K_m}{3}(M-1-\epsilon)(b-a)\right].$$

However, from the definition of J, this implies $M-1-\epsilon \leqslant \epsilon$. We note that for **(ii)**, we repeat the above argument for $-\tilde{h}$, which has a maximum at ζ_m, to conclude that either $(-m)-(-1+\epsilon) \leq 0$ or $(-m)-(-1+\epsilon) = 1-\epsilon-m \leqslant \epsilon$. Therefore,

$$1-2\epsilon \leqslant m \leqslant M \leqslant 1+2\epsilon,$$

implying that $|\tilde{h}-1| \leq 4\epsilon$.

5. Appendix

5.1. Short proof of the regularity of the unitary propagator.

Theorem 5. *Assume that $H_1 = H + V(x,t)$, where H is time independent and self-adjoint, and $V(\cdot,t)$ is in $L^\infty(\mathbb{R}^n)$ for every t and is differentiable in time, with integrable derivative. Consider the Schrödinger problem*

$$i\psi_t = H_1\psi;\quad \psi(x,0) \in D(H). \tag{178}$$

Then there exists a strongly differentiable unitary propagator on $L^2(\mathbb{R}^n)$ $U(t)$ so that $\psi(x,t) = U(t)\psi_0 \in D(H)$ for all t and $\psi(x,t)$ solves (178).

Proof. We note that it is enough to prove this property on a finite interval $[0,\epsilon]$, since the problem can be restarted at $t=\epsilon$. Let $y = \psi - e^{-t}\psi_0$. Then y satisfies the inhomogeneous Schrödinger equation

$$iy_t = y_0 e^{-t} + Hy + Vy;\ y_0 := i\psi_0 + H\psi_0 + V\psi_0,\quad y(0)=0. \tag{179}$$

We transform this equation into an integral equation, *formally for now*. Straightforward calculations show that

$$i(e^{iHt}y)_t = e^{iHt}e^{-t}y_0 + e^{iHt}Vy \tag{180}$$

or (still formally)

$$\begin{aligned} ie^{iHt}y &= \left(\int_0^t e^{(iH-1)s}ds\right)y_0 + \int_0^t e^{iHs}V(s)y(s)ds \\ &= (iH-1)^{-1}(e^{iHt-t}-1)y_0 + \int_0^t e^{iHs}V(s)y(s)ds \end{aligned} \tag{181}$$

or, equivalently,

$$iy = (iH-1)^{-1}(e^{-t}-e^{-iHt})y_0 + e^{-iHt}\int_0^t e^{iHs}V(s)y(s)ds. \tag{182}$$

It is clear that (182) is contractive in the norm $\sup_{t\in[0,\epsilon]}\|\cdot\|_{L^2(\mathbb{R}^3)}$ for small ϵ, and has a unique solution. Clearly, the first term on the right side of (182) is differentiable in time and the derivative is continuous since e^{-iHt} is; let u_0 denote this derivative.

We now write a *formal* equation for $u = y_t$. We have

$$iu = u_0 + \int_0^t e^{-iHs}V'(t-s)\left(\int_0^{t-s} u(s')ds'\right)ds + \int_0^t e^{-iHs}V(t-s)u(t-s)ds. \tag{183}$$

This equation is also contractive, and has a unique solution, in the same space. Thus both sides of (183) are integrable in time. By integration and appropriate changes of variables and order of integration, we see that $\int_0^t u(s)ds$ satisfies the same equation as y, which has a unique solution. Thus $y = \int_0^t u(s)ds$ is strongly differentiable. Since both y and $e^{iHt}y$ are strongly differentiable (the latter by inspection from (181)), $y \in D(H)$ for all t and is strongly differentiable. It is clear that $\psi \in D(H)$ and easy to check that it is differentiable and satisfies (178). □

5.2. Laplace transform of the Schrödinger equation. We look more generally at equations of the form

$$i\psi_t = H\psi + V(t,x)\psi, \tag{184}$$

where H is self-adjoint and time independent, and $V(x,t)$ is bounded on $\mathbb{R}^3$ and differentiable and bounded in t, and $\psi(x,0) \in D(H)$. The conditions on V can be relaxed. (For the purpose of this paper, H would be taken to be H_C.)

Proposition 55. *Under the assumptions above, the Laplace transform $\hat{\psi}(p,\cdot)$ of $\psi(t,\cdot)$ exists for* Re $p > 0$*; it is in $D(H)$ and satisfies*

$$(p+iH)\hat{\psi} = \psi_0 - i\widehat{V\psi}. \tag{185}$$

Proof. We take the unitary propagator of the time-independent problem, $U = e^{-iHt}$ and apply $U^*(t) = U^{-1}(t)$ to both sides of (184). Since (cf. § 1.2) U^{-1} is strongly differentiable, with derivative $iU^{-1}H$, and ψ is $t-$differentiable in L^2, $U^{-1}\psi$ is differentiable and we get

$$(U^{-1}\psi)_t = iU^{-1}H\psi + U^{-1}\psi_t = -iU^{-1}V\psi. \tag{186}$$

Since $U^{-1}V\psi$ is continuous in t, we can integrate both sides and get, after multiplication by U and using the fact that $U^{-1}(t) = U(-t)$,

$$\psi = U\psi_0 - iU\int_0^t U^{-1}V\psi(s)ds = U\psi_0 - i\int_0^t U(t-s)(V\psi)(s)ds = U\psi_0 - iU*(V\psi), \tag{187}$$

where $*$ is the usual Laplace convolution. Taking the Laplace transform (which clearly exists) in (187) and using standard functional calculus we get

$$\hat{\psi} = (p+iH)^{-1}\psi_0 - i(p+iH)^{-1}\widehat{V\psi}, \tag{188}$$

and thus $\hat{\psi}$ is a $D(H)$ solution of (185). □

Now, from Eq. (7), it follows that $\hat{y}$ satisfies (9). Furthermore, using (188) and the fact that y^0 and Ω_j are compactly supported, we see that $\hat{y}$ also satisfies

$$\hat{y}(p,\cdot) = \mathfrak{R}_0 \chi_{\mathsf{B}} \hat{y}^0(p,\cdot) - \mathfrak{R}_0 \chi_{\mathsf{B}} \left[\sum_{j\in\mathbb{Z}} \Omega_j \hat{y}(\hat{p} - ij\omega, \cdot) \right], \tag{189}$$

where $\mathfrak{R}_0 = (H_C - ip)^{-1}$.

5.3. Analyticity of $(I - \mathfrak{C}_{\mathrm{l,m}})^{-1}$ *in* X. This is standard, and can be seen directly from analytic functional calculus. We provide a self-contained argument, for completeness. We write $\mathfrak{C}_X$ to emphasize the $X-$ dependence of $\mathfrak{C}$, and for simplicity of notation we drop the (l, m) subscript. We have

$$\begin{aligned}(I - \mathfrak{C}_{X_1})^{-1} - (I - \mathfrak{C}_{X'})^{-1} &= (I - \mathfrak{C}_{X'})^{-1}(\mathfrak{C}_{X_1} - \mathfrak{C}_{X'})(I - \mathfrak{C}_{X_1})^{-1} \text{ and} \\ &\quad (I - \mathfrak{C}_{X'})^{-1}\left[I + (\mathfrak{C}_{X_1} - \mathfrak{C}_{X'})(I - \mathfrak{C}_{X_1})^{-1}\right] \\ &= (I - \mathfrak{C}_{X_1})^{-1}.\end{aligned} \tag{190}$$

We fix X_1 and let $X' \to X_1$. Since $(I - \mathfrak{C}_{X_1})^{-1}$ is bounded, then $\|(\mathfrak{C}_{X_1} - \mathfrak{C}_{X'})(I - \mathfrak{C}_{X_1})^{-1}\| \to 0$ as $X' \to X_1$ and

$$I + (\mathfrak{C}_{X_1} - \mathfrak{C}_{X'})(I - \mathfrak{C}_{X_1})^{-1} \tag{191}$$

is invertible when X_1 and X' are close enough and $[I + (\mathfrak{C}_{X_1} - \mathfrak{C}_{X'})(I - \mathfrak{C}_{X_1})^{-1}]^{-1} \to I$ in operator norm as $X' \to X_1$. Thus

$$(I - \mathfrak{C}_{X'})^{-1} \to (I - \mathfrak{C}_{X_1})^{-1} \tag{192}$$

in operator norm, as $X_1 \to X'$. Now diferentiability in X follows from (190).

5.4. Coulomb Green's function representation. The retarded Green's functions $G = G_+$ is defined as the solution of the equation,

$$\mathcal{A}_0 G(x, x'; k) = \delta(x - x') \tag{193}$$

in distributions, satisfying the radiation condition

$$G(x, x'; k) \sim F(\theta, \phi) e^{ikr} r^{-1-i\nu}; \text{ as } \ r \to \infty, \tag{194}$$

where

$$k = \sqrt{ip} \ \ (\operatorname{Im} k > 0 \ \text{ if} \operatorname{Re} p > 0), \ \ \nu = \frac{b}{2k}. \tag{195}$$

Equivalently, G is the $\mathbb{R}^3 \setminus \{0\}$ solution of (193) with zero right hand side, satisfying (194) and $|x - x'| G(x, x'; k) \to (4\pi)^{-1}$ as $x - x' \to 0$.

Proposition 56.

$$\mathfrak{R}_0 \chi_{\mathsf{B}} g = \int_{\mathsf{B}} G(x, x'; k) g(x') dx'. \tag{196}$$

Proof. The function

$$f := \int_{\mathsf{B}} G(x, x'; k) g(x') dx' \tag{197}$$

solves, as can be checked, the equation

$$\mathcal{A}_0 f = \chi_{\mathsf{B}} g \tag{198}$$

with the radiation condition (194). Such a solution is unique since the difference of two solutions satisfies the equation $\mathcal{A}_0 f = 0$ (with the radiation condition (194)). Multiplying by $G(x, x'; k)$, integrating over a volume and passing to the limit where the volume approaches $\mathbb{R}^3$, we see that $f \equiv 0$. □

Symmetries of the Coulomb potential $-b/r$ allow for a closed form of G (cf. [26]–where the sign is chosen differently) in terms of Whittaker functions $\mathfrak{W}$ and $\mathfrak{M}$,

$$G(x; x'; k) = \frac{\Gamma(1 - i\nu)}{4\pi i k |x - x'|} \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \mathfrak{W}_{i\nu, \frac{1}{2}}(-ik\xi) \mathfrak{M}_{i\nu, \frac{1}{2}}(-ik\eta), \tag{199}$$

where $\operatorname{Im} k > 0$, $2k\nu = b$ and

$$\xi = |x| + |x'| + |x - x'|, \quad \eta = |x| + |x'| - |x - x'|. \tag{200}$$

The Whittaker functions are defined in terms of the Kummer functions M and U by the relations, see [1], Chap. 13,

$$\begin{aligned} \mathfrak{M}_{\kappa,\mu}(z) &= e^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} M\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right), \ -\pi < \arg z \leqslant \pi, \\ \mathfrak{W}_{\kappa,\mu}(z) &= e^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} U\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right), \ -\pi < \arg z \leqslant \pi. \end{aligned} \tag{201}$$

The following integral representation follows from [1], Chap. 13, for the values we are interested in, $z_1 = -ik\xi$, $z_2 = -ik\eta$, $a = 1 - i\nu$, $b = 2$ (a different "b" than the one in our Coulomb potential)

$$\mathfrak{M}_{i\nu; \frac{1}{2}}(z) = \frac{e^{-\frac{1}{2}z} z J(z)}{\Gamma(1 - i\nu)\Gamma(1 + i\nu)}; \ W_{i\nu; \frac{1}{2}}(z) = \frac{e^{-\frac{1}{2}z} z I(z)}{\Gamma(1 - i\nu)}, \tag{202}$$

where I and J are as defined in (51) and the expression is valid in the regions where the integrals converge (in particular, $|\operatorname{Im} \nu| < 1$). For other values of ν of interest, the integrals can be replaced by appropriate contour integrals. For instance J would be replaced by

$$\left(1 - e^{-2\pi\nu}\right)^{-1} \oint_C e^{zt} t^{-i\nu} (1 - t)^{i\nu} dt,$$

where C is a smooth simple curve encircling [0, 1], as it can be checked by calculating the jump across the cut of the integrand. It follows from these integral representations that the Green's function is analytic at any (small) p, $\operatorname{Re} p \neq 0$. Substituting (202) into (199), we obtain (49).

5.5. Dependence of A in Eq. (56) on Z, p. We now seek to determine the asymptotics of A in (56) in the resolvent $\chi_B \mathfrak{R}_\beta \chi_B$ in terms of $\lambda = \sqrt{-ip}$ and $Z = \exp[i\pi b/(2\lambda)]$ for $X = (\sqrt{p}, Z) \in \overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$ for sufficiently small ϵ.

Recall the expression A in (56). Note that since

$$\alpha = \sqrt{\lambda^2 - ic} \sim e^{-i\pi/4} c^{1/2} \left(1 + O(\lambda^2)\right) \equiv \alpha_0 + \lambda^2 \alpha_1 + \cdots , \tag{203}$$

$$\kappa_1 = \frac{b}{2\alpha} = \frac{b e^{i\pi/4}}{\sqrt{c}} \left[1 + O(\lambda^2)\right] \equiv \kappa_{1,0} + \lambda^2 \kappa_{1,2} \ + \ , \tag{204}$$

each of $m_1(a)$ and $w_1(a)$ is analytic in λ for small λ, with the expansion

$$m_1(a) = \frac{1}{r} \mathfrak{M}_{\kappa_1, l+1/2}(2\alpha a) \sim m_{1,0}(a) + \lambda m_{1,1}(a) + \cdots , \tag{205}$$

$$w_1(a) = \frac{1}{a} \mathfrak{W}_{\kappa_1, l+1/2}(2\alpha a) \sim w_{1,0}(a) + \lambda w_{1,1}(a) + \cdots . \tag{206}$$

The asymptotics in this case is also differentiable with respect to a and we get similar expressions as above for $m_1'(a)$ and $w_1'(a)$. It follows that the expression for f_0 in (55) also possesses a regular series expansion in λ:

$$f_0(a) = f_{0,0}(a) + \lambda f_{0,1}(a) + \cdots . \tag{207}$$

To simplify A as in (56) for small λ, we now consider the asymptotics of $w_2(a)$ and $w_2'(a)$ for small λ.

5.6. Asymptotics of $w_2(a)$, $w_2'(a)$ for small λ. Since $w_2(a) = \frac{1}{a} \mathfrak{W}_{\kappa, l+1/2}(2\lambda a)$, with $\kappa = b/(2\lambda)$, it follows from formula (13.1.33) and analytic continuation to larger values of κ of (13.2.5) of [1], p. 505 and the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin[\pi x]$ that

$$w_2(a) = -\frac{e^{-i\pi(l-\kappa)} e^{-\lambda a} (2\lambda a)^{(l+1)} \Gamma(\kappa - l)}{2\pi i a} H(2\lambda a; \kappa, l),$$
$$\text{where} \ \ H(z; \kappa, l) = \int_C e^{-zt} t^{l-\kappa} (1+t)^{l+\kappa} dt, \tag{208}$$

where the contour C starts at ∞e^{i0}, circles around the origin once counter-clockwise to the right of $t = -1$ and goes to $\infty e^{i2\pi}$. In defining the integrand, we choose arg $t \in [0, 2\pi]$, $\arg(1+t) \in (-\pi, \pi]$ so that there is no branch cut on the real axis between -1 and 0.

It follows from (208) that

$$w_2'(a) = \left(-\lambda + \frac{l}{a}\right) w_2(a) + \frac{e^{-i\pi(l-\kappa)} e^{-\lambda a} (2\lambda)^{(l+2)} a^l \Gamma(\kappa - l)}{2\pi i} H_1(2\lambda a; \kappa, l),$$
$$\text{where} \ \ H_1(z; \kappa, l) = \int_C e^{-zt} t^{l-\kappa+1} (1+t)^{l+\kappa} dt. \tag{209}$$

We now seek to determine $H(2\lambda a; b/(2\lambda), l)$ and $H_1\ (2\lambda a, b/(2\lambda), l)$ asymptotically for small λ. For that purpose it is convenient to define

$$\epsilon_2 = 2\lambda \left(\frac{a}{b}\right)^{1/2} \ , \tau = \epsilon_2 t \ , \quad P(\tau; \epsilon_2) = -\frac{1}{\epsilon_2} \log\left(1 + \frac{\epsilon_2}{\tau}\right) + \tau, \tag{210}$$

where we use the principal branch of log in defining $P(\tau;\epsilon_2)$ above. Then, noting that in the definition of log τ and log $(\tau+\epsilon_2)$, $\arg\tau\in[0,2\pi)$ and $\arg(\tau+\alpha)\in(-\pi,\pi]$, we have

$$\log\tau-\log(\tau+\epsilon_2)=-\log\left(1+\frac{\epsilon_2}{\tau}\right)$$

for τ in the upper-half plane, while for τ in the lower-half plane, we have

$$\log\tau-\log(\tau+\epsilon_2)=i2\pi-\log\left(1+\frac{\epsilon_2}{\tau}\right).$$

It is readily checked that

$$H\left(2\lambda a;\frac{b}{2\lambda},l\right)=\frac{b^{l+1/2}}{2^{2l+1}\lambda^{2l+1}a^{l+1/2}}\left\{\int_{C_1}\tau^l(\tau+\epsilon_2)^l\exp\left[-\sqrt{ab}P(\tau;\epsilon_2)\right]d\tau\right.$$
$$\left.+\exp\left(-\frac{i\pi b}{\lambda}\right)\int_{C_2}\tau^l(\tau+\epsilon_2)^l\exp\left[-\sqrt{ab}P(\tau;\epsilon_2)\right]d\tau\right\},\tag{211}$$

$$H_1\left(2\lambda a;\frac{b}{2\lambda},l\right)=\frac{b^{l+1}}{2^{2l+2}\lambda^{2l+2}a^{l+1}}\left\{\int_{C_1}\tau^{l+1}(\tau+\epsilon_2)^l\exp\left[-\sqrt{ab}P(\tau;\epsilon_2)\right]d\tau\right.$$
$$\left.+\exp\left(-\frac{i\pi b}{\lambda}\right)\int_{C_2}\tau^{l+1}(\tau+\epsilon_2)^l\exp\left[-\sqrt{ab}P(\tau;\epsilon_2)\right]d\tau\right\}.\tag{212}$$

Here C_1 is a contour in the upper-half complex τ-plane from $+\infty$ to $-\epsilon_2$ along a steepest descent line, passing through the saddle point $\tau_{s,1}=i(1+o(1))$, where $P'(\tau_{s,1};\epsilon_2)=0$. The contour C_2 is the steepest descent line in the lower-half τ-plane from $\tau=-\epsilon_2$ to $+\infty$ through the saddle point $\tau_{s,2},=-i(1+o(1))$ where $P'(\tau_{s,2};\epsilon_2)=0$. We rewrite w_2 and w_2' as

$$w_2(a)=\frac{(-1)^{l+1}e^{-\lambda a}b^{l+1/2}\Gamma(\kappa-l)}{2^{l+1}\sqrt{a}\lambda^l}\left[ZM_1(\sqrt{ab},\epsilon_2)+Z^{-1}M_2(2\sqrt{ab},\epsilon_2)\right],\tag{213}$$

where

$$M_1(\zeta,\epsilon_2)=\frac{1}{\pi i}\int_{C_1}e^{-\zeta P(\tau;\epsilon_2)}\tau^l(\tau+\epsilon_2)^l d\tau,$$
$$M_2(\zeta,\epsilon_2)=\frac{1}{\pi i}\int_{C_2}e^{-\zeta P(\tau;\epsilon_2)}\tau^l(\tau+\epsilon_2)^l d\tau,\tag{214}$$

$$w_2'(a)=\left(-\lambda+\frac{l}{a}\right)w_2(a)+\frac{(-1)^l e^{-\lambda a}b^{l+1}\Gamma(\kappa-l)}{2^{l+1}a\lambda^l}$$
$$\times\left[ZM_3(\sqrt{ab})+Z^{-1}M_4(\sqrt{ab})\right],\tag{215}$$

where

$$M_3(\zeta,\epsilon_2)=\frac{1}{\pi i}\int_{C_1}e^{-\zeta P(\tau;\epsilon_2)}\tau^{l+1}(\tau+\epsilon_2)^l d\tau\tag{216}$$
$$\text{and}\quad M_4(\zeta,\epsilon_2)=\frac{1}{\pi i}\int_{C_2}e^{-\zeta P(\tau;\epsilon_2)}\tau^{l+1}(\tau+\epsilon_2)^l d\tau.$$

It follows that, with $\epsilon_2 = 2\lambda\sqrt{a/b}$, we have

$$\frac{w_2'(a)}{w_2(a)} = -\lambda + \frac{l}{a} - \left(\frac{b^{1/2}}{a^{1/2}}\right)\left(\frac{Z^2 M_3(\sqrt{ab}, \epsilon_2) + M_4(\sqrt{ab}, \epsilon_2)}{Z^2 M_1(\sqrt{ab}, \epsilon_2) + M_2(\sqrt{ab}, \epsilon_2)}\right). \tag{217}$$

5.6.1. Analyticity in ϵ_2

Proposition 57. *The functions* $M_i(\sqrt{ab}, \cdot)$, $i = 1, ..., 4$, *are analytic near zero.*

Proof. We look at M_1, the others being similar. We can make a change of variable

$$q = P(\tau; \epsilon_2) - P(\tau_{s,1}; \epsilon_2), \tag{218}$$

where the function q is real on the steepest descent contour and changes monotonically from ∞ to 0, as we move from $+\infty$ to $\tau = \tau_{s,1}$, and then increases monotonically again from 0 to ∞ as we move along the steepest descent path from $\tau = \tau_{s,1}$ to $\tau = -\epsilon_2$. We denote the two branches of the inverse function $\tau(q)$ in (218) by $\tau_1(q)$ and $\tau_2(q)$. Noting that

$$\frac{dP(\tau; \epsilon_2)}{d\tau} = \frac{1}{\tau(\tau + \epsilon_2)} + 1,$$

we have

$$M_1(\zeta, \epsilon_2) = e^{-\zeta\tau_{s,1}}\left(\int_0^\infty e^{-\zeta q}\left(\frac{\tau_2^{l+1}(\tau_2 + \epsilon_2)^{l+1}}{\tau_2^2 + 1 + \epsilon_2\tau_2}\right)dq - \int_0^\infty e^{-\zeta q}\left(\frac{\tau_1^{l+1}(\tau_1 + \epsilon_2)^{l+1}}{\tau_1^2 + 1 + \epsilon_2\tau_1}\right)dq\right). \tag{219}$$

It is easy to check that $(\tau_i - \tau_{s_1})^2$ is analytic for small ϵ_2, regular in q and nonzero at $\epsilon_2 = 0$ for all q.

Furthermore, the integrands in (219) are clearly bounded by an L^1 function uniformly in ϵ_2 (see (210) and (218)), ensuring ϵ_2-analyticity of the integrals. □

Returning to the original variable τ we get

$$\begin{aligned}\frac{1}{\pi i}M_1(\sqrt{ab}, 0) &= \frac{1}{\pi i}\int_{C_{1,0}} e^{-\sqrt{ab}\left(-\frac{1}{\tau}+\tau\right)}\tau^{2l}d\tau \\ &= \int_0^\pi \exp\left[i(2l+1)\theta - 2\sqrt{ab}\sin\theta\right]d\theta \\ &= J_{2l+1}\left(2\sqrt{ab}\right) - i\left[Y_{2l+1}\left(2\sqrt{ab}\right) + \mathcal{G}_{2l+1}\left(2\sqrt{ab}\right)\right]\end{aligned} \tag{220}$$

and

$$\begin{aligned}\frac{1}{\pi i}M_2(\sqrt{ab}, 0) &= \frac{1}{\pi i}\int_{C_{2,0}} e^{-\sqrt{ab}\left(-\frac{1}{\tau}+\tau\right)}\tau^{2l}d\tau \\ &= \int_{-\pi}^0 \exp\left[i(2l+1)\theta - 2\sqrt{ab}\sin\theta\right]d\theta \\ &= J_{2l+1}\left(2\sqrt{ab}\right) + i\left[Y_{2l+1}\left(2\sqrt{ab}\right) + \mathcal{G}_{2l+1}\left(2\sqrt{ab}\right)\right],\end{aligned} \tag{221}$$

where J_{2l+1} and Y_{2l+1} are the usual Bessel functions of order $2l+1$ and

$$\mathcal{G}_{2l+1}(\nu) \equiv \frac{1}{\pi}\int_0^\infty \left\{\exp[(2l+1)t] + (-1)^{2l+1}\exp[-(2l+1)t]\right\} e^{-\nu\sinh t}dt$$
$$= \frac{2}{\pi}\int_0^\infty \sinh\left((2l+1)t\right) e^{-\nu\sinh t}dt, \tag{222}$$

$$\tau_{s,1} = i\sqrt{1-\frac{\epsilon_2^2}{4}} - \frac{\epsilon_2}{2} = i + \text{Series in } \epsilon_2. \tag{223}$$

Thus, asymptotically, to the leading order in λ, we have with $\nu = 2\sqrt{ab}$,

$$-\sqrt{\frac{a}{b}}\frac{w_2'(a)}{w_2(a)}$$
$$= \frac{\left[Z^2\left(J_{2l+2}(\nu) - iY_{2l+2}(\nu) - i\mathcal{G}_{2l+2}(\nu)\right) + \left(J_{2l+2}(\nu) + iY_{2l+2}(\nu) + i\mathcal{G}_{2l+2}(\nu)\right)\right]}{\left[Z^2\left(J_{2l+1}(\nu) - iY_{2l+1}(\nu) - i\mathcal{G}_{2l+1}(\nu)\right) + \left(J_{2l+1}(\nu) + iY_{2l+1}(\nu) + i\mathcal{G}_{2l+1}(\nu)\right)\right]}$$
$$\times (1 + O(\lambda)). \tag{224}$$

The discussion on $w_2'(a)/w(a)$ shows that

$$A = \frac{f_0(a)w_2'(a) - f_0'(a)w_2(a)}{m_1'(a)w_2(a) - m_1(a)w_2'(a)} \tag{225}$$

is an analytic function of the extended parameter set X for $X = \left(\sqrt{p}, Z\right) \in \overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$ as long as the denominator for A is nonvanishing as $\lambda \to 0$.

We can prove it is nonvanishing by simplifying the leading order expression in λ for $w_2'(a)/w_2(a)$, defined as $w_{2,0}'(a)/w_{2,0}(a)$ under the further assumption that a and c (as in the definition of β) are sufficiently large.

5.6.2. Further simplification for large a. For large a, there is additional simplification since

$$J_{2l+1}\left(2\sqrt{ab}\right) \pm iY_{2l+1}\left(2\sqrt{ab}\right) \sim \left(\frac{(-1)^{l+1}}{\pi^{1/2}a^{1/4}b^{1/4}}\right)$$
$$\times \exp\left[\pm i\left(2\sqrt{ab} + \frac{\pi}{4}\right)\right], \tag{226}$$

$$J_{2l+2}\left(2\sqrt{ab}\right) \pm iY_{2l+2}\left(2\sqrt{ab}\right) \sim \left(\frac{(-1)^{l+1}}{\pi^{1/2}a^{1/4}b^{1/4}}\right)$$
$$\times \exp\left[\pm i\left(2\sqrt{ab} - \frac{\pi}{4}\right)\right], \tag{227}$$

and from Watson's Lemma, we get

$$\mathcal{G}_{2l+1}\left(2\sqrt{ab}\right) = O(1/a), \quad \mathcal{G}_{2l+2}\left(2\sqrt{ab}\right) = O(1/\sqrt{a}). \tag{228}$$

It follows that for large a,

$$\frac{w_{2,0}'(a)}{w_{2,0}(a)} \sim \frac{r_2}{a}\left(\frac{n_1 - Z^2}{Z^2 + n_1}\right)\left(1 + O(a^{-1/2})\right), \tag{229}$$

where

$$n_1 = ie^{4i\sqrt{ba}}, \quad Z = \exp\left[\frac{i\pi b}{2\lambda}\right], \quad r_2 = i\sqrt{ab}. \tag{230}$$

5.6.3. Nonvanishing of the denominator of A in (225) Now, defining

$$m = m_1(a), \quad m' = m_1'(a), \quad f = f_0(a), \quad f' = f_0'(a), \tag{231}$$

we have to the leading order in λ, for large a,

$$\begin{aligned} -A &= \frac{r_2\left[\dfrac{n_1 - Z^2 + O(\lambda)}{n_1 + Z^2 + O(\lambda, a^{-1/2})}\right] - af'}{r_2 m\left[\dfrac{n_1 - Z^2 + O(\lambda)}{n_1 + Z^2 + O(\lambda, a^{-1/2})} - \dfrac{3}{4r_2}\right] - am'} \\ &= \frac{f\left[4r_2(n_1 - Z^2) + O(\lambda)\right] - 4af'\left[n_1 + Z^2 + O(\lambda)\right]}{m\left[4r_2(n_1 - Z^2) + O(\lambda)\right] - 4am'\left[n_1 + Z^2 + O(\lambda)\right]}. \end{aligned} \tag{232}$$

The denominator of A is

$$D = -m\left\{Z^2\left(4a\frac{m'}{m} + 4r_2 + O(\lambda)\right) + n_1\left(4a\frac{m'}{m} - 4r_2 + O(\lambda)\right)\right\}. \tag{233}$$

We note that

$$\begin{aligned} m &\equiv m_1(a) = \frac{1}{a}\mathfrak{M}_{\frac{b}{2\alpha}, l+1/2}(2\alpha a) = e^{-\alpha a}(2\alpha)^{l+1}a^l M\left(l+1-\frac{b}{2\alpha}, 2l+2, 2\alpha a\right) \\ &\sim (2\lambda)^{l+1}a^l\frac{e^{\alpha a}}{\Gamma\left(l+1-\frac{b}{2\alpha}\right)}\left[1 + O\left(\alpha a)^{-1}\right)\right] \text{ for } \alpha \text{ large,} \end{aligned} \tag{234}$$

and for large α in the fourth quadrant

$$m' \equiv m_1'(a) \sim (2\alpha)^{l+1}a^l\alpha\frac{e^{\alpha a}}{\Gamma\left(l+1-\frac{b}{2\alpha}\right)}\left[1 + O(\alpha a)^{-1}\right], \tag{235}$$

$$\alpha = \sqrt{\lambda^2 - ic} \to c^{1/2}\exp\left[-i\frac{\pi}{4}\right] \text{ as } \lambda \to 0. \tag{236}$$

Therefore, D can be zero for large enough c (*i.e.* large β) only if

$$\begin{aligned} &Z^2\left(2\sqrt{2}ac^{1/2}(1-i)\left[1 + O\left((ca)^{-1}\right)\right] + 4i\sqrt{ba}\right) \\ &\quad = -n_1\left(2\sqrt{2}ac^{1/2}(1-i)\left[1 + O\left((ca)^{-1}\right)\right] - 4i\sqrt{ba}\right) + O(\lambda). \end{aligned}$$

Taking the absolute square of both sides, we obtain,

$$\begin{aligned} &|Z|^2\left([2a\sqrt{2c}]^2\left(1 + O(c^{-1}a^{-1}\right) + [4\sqrt{ab} - 2a\sqrt{2c}\left(1 + O(c^{-1}a^{-1}\right)]^2\right) \\ &\quad = \left([2a\sqrt{2c}]^2\left(1 + O(c^{-1}a^{-1}\right) + [4\sqrt{ab} + 2a\sqrt{2c}(1 + O(c^{-1}a^{-1})]^2\right) + O(\lambda). \end{aligned}$$

This is impossible, since $|Z| \leq 1$. This means that for large enough a and c (that is, β large), D cannot be zero. It means that the resolvent is well-defined as $p = 0$ is approached from the closure of $\mathbb{H}$.

Note 58. Note that the denominator of A in (232) vanishes at points in the region $|Z| > 1$, where, as a result, the resolvent $\mathfrak{R}_\beta$ has poles. From the relation between Z and p, it follows that $p = 0$ is an accumulation point of a sequence of poles in the left half plane approaching zero tangentially to $i\mathbb{R}$.

5.7. Stationary phase analysis needed to calculate the ionization rate. We know that the solution $\hat{y}(is, x)$ is analytic in the extended parameter $\left(\sqrt{is}, Z)\right)$, where

$$Z = \exp\left[i\pi b/\left(2\sqrt{s}\right)\right]. \tag{237}$$

So, for $X = \left(\sqrt{is}, Z\right) \in \overline{\mathbb{D}_\epsilon^+} \times \overline{\mathbb{D}}$,

$$\hat{y}(is, x) = \sum_{l=0}^{\infty} s^{l/2} F_l(Z). \tag{238}$$

Consider

$$G(s) \equiv \sum_{l=4}^{\infty} s^{l/2} F_l\left(\exp\left[\frac{i\pi b}{2\sqrt{s}}\right]\right). \tag{239}$$

It is clear that $G(s)$ is a C^1 function of s in $[-a, a]$. Integration by parts gives

$$\int_{-a}^{a} G(s) e^{ist} ds = O(t^{-1}). \tag{240}$$

Now note that

$$F_l\left(\exp\left[\frac{i\pi b}{2\sqrt{s}}\right]\right) = \sum_{j\geq 0} D_{j,l} \exp\left[i\frac{\pi bj}{2\sqrt{s}}\right] \tag{241}$$

with $D_{j,l}$ decreasing exponentially with j, because of analyticity of $F_l(Z)$ for $|Z| \leq 1$. For $0 \leq l \leq 3$, it follows there exists constants c and C independent of j so that

$$\sum_{l=0}^{3} |D_{j,l}| \leq Ce^{-cj}. \tag{242}$$

It follows that for large t, we have

$$|\sum_{l=1}^{3} \sum_{j=[\sqrt{t}]+1}^{\infty} D_{j,l} \int_{-a}^{a} \exp\left[\frac{ibj}{2\sqrt{s}}\right] e^{ist} s^{l/2} ds| \leq C_1 e^{-c\sqrt{t}}. \tag{243}$$

Further, for large t,

$$\left|\sum_{l=0}^{3} D_{0,l} \int_{-a}^{a} e^{ist} s^{l/2} ds\right| \leq \frac{C_2}{t}. \tag{244}$$

Therefore,

$$\int_{-a}^{a} \sum_{l=0}^{3} \left(s^{l/2} F_l(Z)\right) e^{ist} ds = \sum_{0\le l\le 3} \int_{-a}^{a} s^{l/2} F_l(Z) e^{ist)} ds \tag{245}$$

$$\sim \sum_{0\le l\le 3} \sum_{j=1}^{[\sqrt{t}]} D_{j,l} \int_{-a}^{a} s^{l/2} \exp\left[i \left\{ st + j \frac{\pi b}{2\sqrt{s}} \right\} \right] ds$$

$$+O\left(\frac{1}{t}\right).$$

We first evaluate the terms of the form

$$\int_{-a}^{a} s^{l/2} e^{its+id_j/\sqrt{s}} ds \tag{246}$$

for large t, where

$$d_j = \frac{j\pi b}{2}.$$

The contribution from $\int_{-a}^{0}$ is obviously small, at most $O(1/t)$, uniformly for all t, since the integrand vanishes exponentially as $s \to 0^-$. So we only consider, for $1 \le j \le \left[\sqrt{t}\right]$,

$$\int_{0}^{a} s^{l/2} e^{its+id_j s^{-1/2}} ds. \tag{247}$$

We have a point of stationary phase at $s = s_{0,j}$, where

$$s_{0,j} = \left(\frac{d_j}{2t}\right)^{2/3}. \tag{248}$$

Note that $s_{0,j} \ll 1$ for t large since j is restricted to $j \le \sqrt{t}$. It is then convenient to rescale $s = s_{0,j} q$, to obtain

$$s_{0,j}^{1+l/2} \int_{0}^{\frac{a}{s_{0,j}}} \exp\left[i\nu_j \left(q + 2q^{-1/2}\right)\right] q^{l/2} dq, \quad \text{where } \nu_j = \frac{2^{-2/3}}{d_j^{2/3}} t^{1/3}. \tag{249}$$

Using standard stationary phase arguments we obtain that, for large t, and hence large ν_j,

$$|s_{0,j}^{1+l/2} \int_{0}^{a/s_{0,j}} \exp\left[i\nu_j \left(q + 2q^{-1/2}\right)\right] q^{l/2} dq - \frac{\sqrt{2\pi} s_{0,j}^{l+1/2} e^{i\nu_j}}{\sqrt{\nu_j}} e^{-i\pi/4}|$$

$$\le C \frac{s_{0,j}^{1+l/2}}{\nu_j}. \tag{250}$$

For large t, the dominant contribution comes from the term with $l = 0$ and so

$$\left| \int_{-a}^{a} \hat{y}(is,x) e^{ist} ds - \sum_{j=0}^{[\sqrt{t}]} D_{j,0} \frac{\sqrt{2\pi} s_{0,j} e^{i\nu_j}}{\sqrt{\nu_j}} e^{-i\pi/4} \right| \le C t^{-1} \sum_{j=0}^{[\sqrt{t}]} e^{-cj} \le C_1 t^{-1}. \tag{251}$$

The sum over j is clearly convergent because of the exponential decay of $D_{j,0}$; hence $[\sqrt{t}]$ in the upper limit can be replaced by ∞. From the definition of $s_{0,j}$ and ν_j, it follows that

$$\int_{-a}^{a} \hat{y}(is,x)e^{ist}ds = O\left(t^{-5/6}\right). \tag{252}$$

At all other singular points, $p = in\omega$, $n \in \mathbb{Z}$, the behavior is similar, and a similar calculation gives a $e^{in\omega t}t^{-5/6}$ contribution. Since $Y \in \mathcal{H}$, there is sufficient decay in n to ensure that the sum over all such contributions is convergent.

5.8. Calculation of j_k. Substituting the explicit expressions for $m_k(r)$ and $m_{(k-1)}(r)$, it may be checked that in both cases, $\tau = 0$ and $\tau = 1$, corresponding to (**i**) and (**ii**) respectively

$$j_k = k^2\alpha^2\mathfrak{s}(r)j_k^{(2)} + k\alpha^2\mathfrak{s}(r)j_k^{(1)} + j_k^{(0)} \quad , \quad \text{where} \tag{253}$$

$$j_k^{(2)} = \frac{4\Omega}{\alpha^2\mathfrak{s}^2}\left(1 - \frac{H(\alpha k)H(\zeta - \zeta/k)}{H(\alpha(k-1))H(\zeta)}\right) + \frac{H''(\zeta)}{H(\zeta)} - \frac{l(l+1)}{\zeta^2} - \frac{4\sqrt{\Omega(r)}}{\alpha\mathfrak{s}(r)}\frac{H'(\zeta)}{H(\zeta)},$$

$$j_k^{(1)} = -\frac{2(1-2\tau)\Omega}{\alpha^2\mathfrak{s}^2}\left(1 - \frac{H(\alpha k)H(\zeta - \zeta/k))}{H(\alpha(k-1))H(\zeta)}\right) + \frac{b}{\alpha\zeta}$$
$$+\left[-\frac{2\tau\sqrt{\Omega}}{\alpha\mathfrak{s}} + \frac{\omega\mathfrak{s}}{2\sqrt{\Omega}\alpha} - \frac{\Omega'}{2\alpha\Omega}\right]\frac{H'(\zeta)}{H(\zeta)},$$

$$j_k^{(0)} = \frac{5\mathfrak{s}\Omega'^2}{16\Omega^2} - \frac{\omega\mathfrak{s}^2\Omega'}{4\Omega^{3/2}} - \frac{\mathfrak{s}\Omega''}{4\Omega} - \frac{(1+2\tau)}{4}\omega\mathfrak{s} + \frac{\omega^2\mathfrak{s}^3}{16\Omega} - (\omega n_0 - ip_1)\mathfrak{s},$$

where $\mathfrak{s}(r) = \int_r^1 \sqrt{\Omega(s)}ds$, $\zeta = k\alpha r$ and $j_k := \mathfrak{s}[\mathcal{L}_k m_k - \Omega m_{k-1}]/m_k$. Recall that $H(\zeta)$ satisfies

$$H'' = 2\left(1 - \frac{\omega}{2k\alpha^2} + \frac{\Omega'(0)(1+2\zeta)}{4k\alpha\Omega(0)} + \frac{\tau}{2k}\right)H' + \left(\frac{l(l+1)}{\zeta^2} - \frac{b}{\alpha\zeta k}\right)H, \tag{254}$$

where

$$\alpha = 2\frac{\sqrt{\Omega(0)}}{\mathfrak{s}(0)}, \tag{255}$$

and that $H(\zeta)$ has the following asymptotic behavior:

$$H(\zeta) \sim 1 + \frac{l(l+1)}{2\zeta} + \frac{b}{2k\alpha}\log\zeta + O\left(\frac{\log\zeta}{k\zeta}, \frac{1}{\zeta^2}\right), \quad (\zeta, k \to \infty, \ , \ \zeta \le k\alpha). \tag{256}$$

Now, we claim that for any $r \in (0,1)$, $|j_k^{(2)} + k^{-1}j_k^{(1)}| \le Ck^{-2}$. In the regime $r \ll 1$, we use Taylor expansion:

$$\Omega = \Omega(0) + \Omega'(0)\frac{\zeta}{k\alpha} + O\left(\frac{1}{k^2}\right) \quad , \quad \mathfrak{s} = \mathfrak{s}(0) - \sqrt{\Omega(0)}\frac{\zeta}{k\alpha} + O\left(\frac{1}{k^2}\right) \tag{257}$$

and substitute $r = \zeta/(k\alpha)$ in $j_k^{(2)} + k^{-1}j_k^{(1)}$; we then use $\alpha = 2\sqrt{\Omega(0)}/\mathfrak{s}(0)$, (254) and the asymptotic behavior (256) to evaluate $H(\alpha k)$ and $H(\alpha(k-1))$ to find $j_k^{(2)} +$

$k^{-1}j_k^{(1)} \sim k^{-2}g(\zeta)$ for some bounded differentiable function $g(\zeta)$, with asymptotic behavior $g(\zeta) \sim \text{const.}/\zeta$ for large ζ. When r is not small, we use the asymptotic behavior (256) to evaluate all terms involving the function H and to find the same inequality $|j_k^{(2)} + k^{-1}j_k^{(1)}| \le Ck^{-2}$.

Therefore, $j_k(r) = O(1)$ in all regimes. Further, it is easily checked that in the regime $k \gg \zeta \gg 1$, $j_k(r) = O\left(1, \zeta^{-1}\right) = O(1/(kr), 1)$. Since the asymptotics is differentiable (since H satisfies a second order differential equation), it follows $j_k'(r) = O(k^{-1}r^{-2}, 1)$. When r is not small, using (256), it is readily checked that $j_k' = O(1)$.

5.9. Generalizations. In fact, the same asymptotic arguments hold more generally if

$$V(t,x) = \sum_{j=-M}^{M} e^{ij\omega t}\Omega_j(r)$$

with $\Omega_j(r)$ satisfying the conditions we used for Ω. We substitute for $r = O(1)$,

$$g_{n_0-k}(r) = \frac{c_*}{\Gamma(2k/M+1)} \exp\left[k\log f_0(r) + \sum_{j=1}^{M} k^{1-j/M} f_j(r)\right],$$

and calculate the error term R_k as before. By requiring that the $O(k^{2-2j/M})$ terms vanish for $j = 0, .., M$, we obtain $(M+1)$ first order differential equations for f_j. To leading order

$$f_0(r) = \left[\int_r^1 \sqrt{\Omega_{-M}(s)}ds\right]^{2/M}.$$

The expressions for $f_j(r)$ for $j \geqslant 1$ are more complicated and involve arbitrary constants to be determined from the information for small k at $r = 1$. Again because of the presence of $r^{-2}l(l+1)$ in $\mathcal{L}_k$, the remainder is $O(r^{-2})$, which is $O(k^2)$ when $r = O(k^{-1})$. We write

$$g_{n_0-k}(r) \sim c_* \exp\left[k\log f_0(r) + \sum_{j=1}^{M} k^{1-j/M} f_j(r)\right] \frac{H(\alpha kr)}{\Gamma(2k/M+1)}. \qquad (258)$$

Then, if $\zeta = O(1)$, we find to leading order $H(\zeta) \sim H_0(\zeta)$, where

$$H_0'' - 2H_0' - \frac{l(l+1)}{\zeta^2}H_0 = 0,$$

where now $\alpha = 2\sqrt{\Omega_{-M}(0)}/\mathfrak{s}(0)$ and $\mathfrak{s}(r) = \int_r^1 \sqrt{\Omega_{-M}(s)}$. As for $M = 1$, we have to require $H_0(\zeta) \sim 1$ as $\zeta \to \infty$. This leads to

$$H_0(\zeta) = \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+\frac{1}{2}}(\zeta).$$

For nonzero g_{n_0-k}, the constant multiple in (258) is expected to be nonzero. On the other hand, the asymptotic behavior as $\zeta \downarrow 0$, $H_0(\zeta) \sim c_*\zeta^{-l}$ implies that the behavior at $r = 0$ of g_{n_0-k}/r is not acceptable unless every g_n vanishes identically.

*

The analysis is likely to extend to systems with H_C replaced by

$$H_W = -\Delta - b/r + W(r),$$

where b may be zero and $W(r) = O(r^{-1-\epsilon})$ for large r and is in $L^\infty(\mathbb{R}^3)$. Under these assumptions, $W(r)$ does not participate in the asymptotics, to the orders relevant to the proofs.

5.10. Further remarks on the asymptotics.

Remark 59. A weaker statement than Theorem 4 suffices to complete the proof of Theorem 1. For instance, it suffices to show that for sufficiently large j, $|R_{k,j}| < 1$, where

$$r^{l+1} v_{n_0-k_j}(r) = i^{k_j} r^l m_{k_j}(r)[1 + R_{k_j}(r)].$$

Remark 60. Stronger results than those in Proposition 36 hold. Noting that for any integer $q \geqslant 0$ we have

$$\|\mathcal{A}_{k_j+q} \ldots \mathcal{A}_{k_j+2}\mathcal{A}_{k_j+1}\mathcal{A}_{k_j}[\tilde{h} - 1]\|_\infty \leqslant \prod_{q'=0}^{\infty} \left(1 + \frac{c_*}{(k_j + q')^2}\right) \|\tilde{h} - 1\|_\infty,$$

while

$$\mathcal{A}_{k_j+q} \ldots \mathcal{A}_{k_j+2}\mathcal{A}_{k_j+1}\mathcal{A}_{k_j}[1] = 1 + O(k_j^{-1}),$$

and the fact that $\|\mathcal{H}_{k_j+q}\tilde{h}\|_\infty \leqslant c_* k_j^{-2}$, it follows that the sequence $\tilde{h}_k$, satisfying

$$\tilde{h}_k = \mathcal{A}_k \tilde{h}_{k-1} + \mathcal{H}_k \tilde{h}_{k+1},$$

has the property $\lim_{k\to\infty} \tilde{h}_k = 1$. Indeed, this is in accordance with the heuristic arguments presented in § 5.9. While these results completely justify the formal asymptotics, they are not needed in the proofs and we omit the details.

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References

1. Abramowitz, M., Stegun, I.A.: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Wiley-Interscience, 1984
2. Agmon, S.: Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola. Norm. Sup. Pisa, Ser. IV **2**, 151–218 (1975)
3. Agmon, S.: Analyticity properties in scattering and spectral theory for schrodinger operators with long-range radial potentials. Duke Math. J. **68**(2), 337–399 (1992)
4. Belissard, J.: Stability and instability in quantum mechanics. In: *Trends and Developments in the Eighties*, Albeverio, S., Blanchard, Ph. (eds.) Singapore: World Scientific, 1985, pp. 1–106
5. Bourgain, J.: On long-time behaviour of solutions of linear Schrödinger equations with smooth time-dependent potential. In: *Geometric Aspects of Functional Analysis*, Lecture Notes in Math. **1807**, Berlin: Springer, 2003, pp. 99–113
6. Bourgain, J.: Growth of Sobolev norms in linear Schrödinger equatios with quasi-periodic potential. Commun. Math. Phys. **204**(1), 207–240 (1999)
7. Bourgain, J.: On growth of Sobolev norms in linear Schrödinger equations with smooth time-dependent potential. J. Anal Math. **77**, 315–348 (1999)
8. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal **3**(2), 107–156 (1993)
9. Buchholz, H.: *The Confluent Hypergeometric Function*. Berlin-Heidelberg-NewYork: Springer-Verlag, 1969
10. Costin, O., Costin, R.D., Lebowitz, J.L.: *Transition to the Continuum of a Particle in Time-Periodic Potentials*, Advances in Differential Equations and Mathematical Physics, AMS Contemporary Mathematics **327** ed. Karpeshina, Yu., Stolz, C., Weikard, R., Zeng, Y. Providence, RI: Amer. Math. Soc., 2003, pp. 75–86
11. Costin, O., Lebowitz, J.L., Rokhlenko, A.: Exact results for the ionization of a model quantum system. J. Phys. A: Math. Gen. **33**, 1–9 (2000)
12. Costin, O., Costin, R.D., Lebowitz, J.L., Rokhlenko, A.: Evolution of a model quantum system under time periodic forcing: conditions for complete ionization. Commun. Math. Phys. **221**(1), 1–26 (2001)
13. Costin, O., Rokhlenko, A., Lebowitz, J.L.: *On the Complete Ionization of a Periodically Perturbed Quantum System*. CRM Proceedings and Lecture Notes **27**, Providence, RI: Amer. Math. Soc., 2001, pp. 51–61
14. Costin, O., Soffer, A.: Resonance theory for Schrödinger operators. Commun. Math. Phys. **224**, 133–152 (2001)
15. Costin, O., Costin, R.D., Lebowitz, J.L.: Time asymptotics of the Schrödinger wave function in time-periodic potentials. J. Stat. Phys. **116**(1–4), 283–310 (2004)
16. Costin, O., Lebowitz, J.L., Stucchio, C.: Ionization in a one-dimensional dipole model. Rev. Math. Phys. **7**, 835–872 (2008)
17. Treves, F.: *Basic Linear Partial Differential Equations*. London-New York: Academic Press, 1975
18. Costin, O., Lebowitz, J.L., Stucchio, C., Tanveer, S.: Exact results for ionization of model atomic systems. J. Math Phys. **51**, 015211 (2010)
19. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: *Schrödinger Operators*. Berlin-Heidelberg-NewYork: Springer-Verlag, 1987
20. Galtbayar, A., Jensen, A., Yajima, K.: Local time-decay of solutions to Schrödinger equations with time-periodic potentials. J. Stat. Phys. **116**(1–4), 231–282 (2004)
21. Goldberg, M.: Strichartz estimates for the Schrödinger equation with time-periodic Ln/2 potentials. J. Funct. Anal. **256**(3), 718–746 (2009)
22. Hislop, P.D., Sigal, I.M.: *Introduction to Spectral Theory with Applications to Schrödinger Operators*. Applied Mathematical Sciences **113**, Berlin-Heidelberg-NewYork: Springer, 1996
23. Hörmander, L.: *Linear Partial Differential Operators*. Berlin-Heidelberg-NewYork: Springer, 1963
24. Howland, J.S.: Stationary scattering theory for time dependent Hamiltonians. Math. Ann. **207**, 315–335 (1974)
25. Jauslin, H.R., Lebowitz, J.L.: Spectral and stability aspects of quantum Chaos. Chaos **1**, 114–121 (1991)
26. Hostler, L., Pratt, R.H.: Coulomb's Green's function in closed form. Phys. Rev. Lett. **10**(11), 469–470 (1963)
27. Jensen, A.: High energy resolvent estimates for generalized many-body Schrodinger operators. Publ. RIMS, Kyoto U. **25**, 155–167 (1989)
28. Kato, T.: *Perturbation Theory for Linear Operators*. Berlin-Heidelberg-NewYork: Springer Verlag, 1995
29. Koch, P.M., van Leeuven, K.A.H.: The importance of resonances in microwave "Ionization" of excited hydrogen atoms. Phys. Repts. **255**, 289–403 (1995)
30. Miller, P.D., Soffer, A., Weinstein, M.I.: Metastability of breather modes of time dependent potentials. Nonlinearity **13**, 507–568 (2000)
31. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics*. New York: Academic Press, 1972

32. Möller, J.S., Skibsted, E.: Spectral theory of time-periodic many-body systems. Adv. Math. **188**(1), 137–221 (2004)
33. Möller, J.S.: Two-body short-range systems in a time-periodic electric field. Duke Math. J. **105**(1), 135–166 (2000)
34. Rodnianski, I., Tao, T.: *Long-time Decay Estimates for Schrödinger Equations on Manifolds*. Ann. of Math. Stud. **163**, Princeton, NJ: Princeton Univ. Press, 2007
35. Rokhlenko, A., Costin, O., Lebowitz, J.L.: Decay versus survival of a local state subjected to harmonic forcing: exact results. J. Phys. A: Mathematical and General **35**, 8943 (2002)
36. Schlag, W., Rodnianski, I.: Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math **3**, 451–513 (2004)
37. Herbst, I., Möller, J.S., Skibsted, E.: Asymptotic completeness for N-body Stark Hamiltonians. Commun. Math. Phys. **174**(3), 509–535 (1996)
38. Merzbacher, E.: *Quantum Mechanics*. 3rd ed., New York: Wiley, 1998
39. Simon, B.: Schrödinger operators in the twentieth century. J. Math. Phys. **41**, 3523 (2000)
40. Slater, L.J.: *Confluent hypergeometric functions*. Cambridge: Cambridge University Press, 1960
41. Soffer, A., Weinstein, M.I.: Nonautonomous Hamiltonians. J. Stat. Phys. **93**, 359–391 (1998)
42. Wasow, W.: *Asymptotic Expansions for Ordinary Differential Equations*. New York: Interscience Publishers, 1968
43. Yajima, K.: Resonances for the AC-Stark effect. Commun. Math. Phys. **87**(3), 331–352 (1982/83)
44. Graffi, S., Yajima, K.: Exterior complex scaling and the AC-Stark effect in a Coulomb field. Commun. Math. Phys. **89**(2), 277–301 (1983)
45. Yajima, K.: Scattering theory for Schrödinger equations with potentials periodic in time. J. Math. Soc. Japan **29**, 729 (1977)
46. Yajima, K.: Existence of solutions of Schrödinger evolution equations. Commun. Math. Phys. **110**, 415 (1987)

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