

Dynamics of a massive piston in an ideal gas

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Dynamics of a massive piston in an ideal gas

N. I. Chernov, J. L. Lebowitz, and Ya. G. Sinai

Abstract. This survey is a study of a dynamical system consisting of a massive piston in a cubic container of large size L filled with an ideal gas. The piston has mass $M \sim L^2$ and undergoes elastic collisions with $N \sim L^3$ non-interacting gas particles of mass $m = 1$. It is found that under suitable initial conditions there is a scaling regime with time and space scaled by L in which the motion of the piston and the one-particle distribution of the gas satisfy autonomous coupled equations (hydrodynamic equations) such that in the limit $L \rightarrow \infty$ the mechanical trajectory of the piston converges in probability to the solution of the hydrodynamic equations for a certain period of time. There is also a heuristic discussion of the dynamics of the system on longer intervals of time.

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§1. Introduction

The evolution of a macroscopic system consisting of a gas in a container divided by a massive movable wall (piston) is an old problem in statistical physics with a colourful history. This problem was discussed by Landau and Lifshitz [18] and later by Lebowitz [20], Feynman [9], and Kubo [17]; see also the recent surveys by Lieb [21], Gruber [10], and others [15].

In its simplest form, the model consists of an isolated cylinder filled with gas and divided into two compartments by a large piston which is free to move along the axis of the cylinder; see Fig. 1. Initially, the piston is held fixed by a clamp

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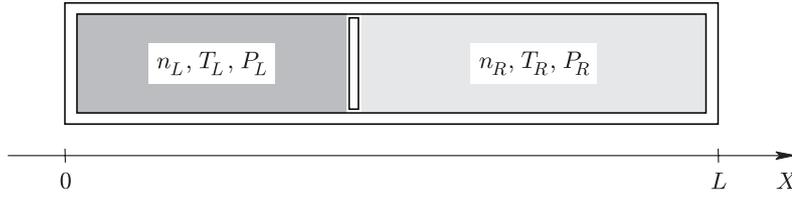


Figure 1. Piston in a cylinder filled with gas

and the gas in each compartment evolves independently and is in equilibrium.¹ We denote the density and temperature in the left and right compartments by n_L, T_L and n_R, T_R , respectively. The gas exerts on the piston a pressure (= force per unit area) given in equilibrium statistical mechanics as a function of the density and temperature: $P_L = P(n_L, T_L)$ and $P_R = P(n_R, T_R)$ on the left and on the right, respectively. At time $t = 0$ the clamp is removed and the piston is released. One now wants to describe the evolution of the system, especially its limit (final) state as $t \rightarrow \infty$.

Starting with $P_L \neq P_R$, the piston moves under the net pressure difference and compresses the gas whose pressure is lower until this pressure builds up and pushes the piston back. Depending on the initial values of n_L, T_L, n_R, T_R and the dynamical characteristics of the gases, the piston may follow a complicated trajectory, moving back and forth, but gradually it comes to rest at a place where the pressure is the same on both sides: $P_L = P_R$. At that time one also expects that the gas in each compartment is again in equilibrium.

However, we observe that the equality $P_L = P_R$ and the fact that the gas in each compartment separately is at equilibrium does not guarantee that $T_L = T_R$. In particular, for dilute gases (which we shall consider from now on), the pressure is related to the density and temperature by $P = nk_B T$, where k_B is Boltzmann's constant, and it is possible that $T_L < T_R$ while $n_L > n_R$, so that the gas in the left compartment is cooler but denser, and in the right one hotter but more dilute (or vice versa). The exact values of the temperatures T_L and T_R , at the time when there is no longer any pressure difference between the left and right and the piston comes to rest, depend on the initial conditions and other characteristics of the system; see [7], [10], for example. Therefore, we have two possibilities. If it happens that $T_L = T_R$ as well as $P_L = P_R$, then the system as a whole is in equilibrium, and we say that it has come to *thermal equilibrium*. On the other hand, if $P_L = P_R$ but $T_L \neq T_R$, then the system is said to be in *mechanical equilibrium*, or *quasi-equilibrium*.

¹For an isolated system of N atoms and total energy E , the equilibrium distribution is defined in statistical mechanics by a uniform probability distribution ρ_{eq} on the energy surface $E = \text{const}$ in the phase space (ρ_{eq} is called a microcanonical ensemble; it remains invariant under the dynamics by the Liouville theorem). One says that the system (gas) is 'in equilibrium' if its states are 'typical' for the measure ρ_{eq} . In this state, a macroscopic gas has (approximately, for large N) a uniform spatial density and a Maxwellian velocity distribution. The latter means that the x, y, z components of the velocity vectors are independent normal random variables $N(0, \sigma^2)$ with the same variance $\sigma^2 = k_B T/m$, where k_B is Boltzmann's constant, T the temperature of the gas (which is a function of E ; see below), and m the mass of an atom.

One may now ask whether the mechanical equilibrium is stable in the sense that it can last forever (assuming that the whole system remains perfectly isolated from the outside world) or whether the gases can find ways to exchange energy through the piston and eventually bring the system to a thermal equilibrium. It was claimed in some textbooks, based on a simplistic interpretation of the laws of thermodynamics (see below), that the mechanical equilibrium can really persist ‘forever’; cf. [10], [11] for some history.

On the other hand, Landau and Lifshitz [18], Feynman [9], and many others argued intuitively that the system should converge from the mechanical equilibrium to a thermal equilibrium. They predicted that the cooler compartment should gradually heat up and the hotter one cool down, while the piston slowly moves from the cooler side to the hotter side with preservation of the pressure balance until the temperatures are equalized and the piston makes its final stop.

The confusion about the evolution of the gas after the establishment of mechanical equilibrium is due to the following: Heat conduction through a wall is normally associated with the internal motion of the molecules of the wall colliding with those of the gas and thus exchanging momentum and energy. However, the piston and the walls in our idealized model are supposed to be rigid, solid, and structureless bodies and the gas atoms bounce off them elastically. This idealization is exactly the reason why the gases in the different compartments could be in equilibrium at different temperatures when the piston was clamped. On the other hand, the unclamped piston interacts with gas atoms as a whole, that is, as one huge and massive molecule. It then makes tiny microscopic movements (vibrations) induced by collisions with atoms on both sides. Hence, some microscopic exchange of momentum and energy does take place. But these microscopic vibrations of the piston are not a part of the macroscopic thermodynamics in which the action of the piston on the gas in each compartment is regarded as an external mechanical force. Under this condition (and assuming that the piston has no entropy of its own) the second law of thermodynamics would predict that the entropy of the gas, as it goes from some initial equilibrium state to a final equilibrium state, could not decrease. When the gas in each compartment is in equilibrium, its thermodynamic entropy is known to be ([18], [3])

$$S_i = N_i[-\log P_i + (1 + 3/2) \log T_i] + f(N_i), \quad i = L, R,$$

where $N_L = n_L V_L$ and $N_R = n_R V_R$ stand for the numbers of atoms in the gases, and the explicit form of $f(N_i)$ is irrelevant since its value does not change in time. Now, if our system does evolve from mechanical equilibrium to thermal equilibrium, keeping the pressure balance $P_L = P_R$ and the total kinetic energy $\frac{3}{2}k_B(N_L T_L + N_R T_R)$ fixed, then one can easily compute (we leave this as an exercise) that the pressure of the gases stays constant in time and the total entropy of the system $S = S_L + S_R$ grows until it reaches its maximum at the point of thermal equilibrium. At the same time the entropy S_R decreases while S_L increases, since T_R falls, T_L rises, and the pressure $P_R = P_L$ remains constant. As already noted, this decrease of the entropy would violate the second law of thermodynamics if the behaviour of the piston were described in the framework of pure mechanics; see further discussions in [21], [7] and critical remarks in [10], [11].

Therefore, the evolution of the system beyond the mechanical equilibrium cannot be described by macroscopic thermodynamics (beyond the statement that any evolution in an isolated macroscopic system does not decrease the total entropy). The actual evolution is a result of microscopic energy transfer between the gases via collisions with the piston. This process is purely microscopic and, in a sense, counterintuitive, as we explain next. Under the collisions with gas atoms on both sides the piston vibrates, that is, it jiggles back and forth. When the piston moves toward the hotter side, the atoms of the hotter gas bounce off the piston with an increased speed and so gain energy, while the atoms of the cooler gas collide with the piston and slow down, hence lose some energy. When the piston moves toward the cooler side it is vice versa. Since, on the average, the hotter gas must cool down and the cooler gas must heat up, one may conclude that the piston's movements toward the cooler side dominate. On the other hand, the piston has to slowly move toward the hotter side to maintain the pressure balance (see above), so its displacements in the direction of the hotter gas actually dominate. It is not quite clear how these seemingly opposite motions manage to coexist. Some physicists joke about a 'conspiracy' between the microscopic vibrations of the piston and the incoming atoms of the gases [11], [12]. In the words of Callen [3], "the movable adiabatic wall presents a unique problem with subtleties."

To understand the mechanism of heat transfer across the piston at mechanical equilibrium ($P_L = P_R$), one usually considers the simplest gas of non-interacting particles, that is, an *ideal gas*. As early as 1959, Lebowitz [20] studied a piston interacting with two infinite reservoirs filled with ideal gases held at different temperatures $T_L \neq T_R$. His piston also interacted with an external potential, for instance, a spring, and it could therefore come to a stationary non-equilibrium state under the influence of the infinite reservoirs. He used an approximation by a Markov process and found the distribution of the piston velocity to be Maxwellian corresponding to some intermediate temperature $T \in (T_L, T_R)$, which led to a systematic heat transfer between the gases. Recently, Gruber and others [12], [11], [25] used kinetic theory to study a freely movable piston of mass M interacting with two infinite ideal gases of atoms of mass $m \ll M$ at equal pressures but different temperatures. They use the expansion of the Boltzmann equation in $\varepsilon = m/M$ to show that a macroscopic heat flux across the piston does occur whenever $T_L \neq T_R$; hence, the system gradually approaches thermal equilibrium. They also found a stationary distribution of the piston velocity, whose average value is given by

$$\langle V \rangle = \frac{\sqrt{2\pi m} (\sqrt{k_B T_R} - \sqrt{k_B T_L})}{4M} + o\left(\frac{m}{M}\right) \quad (1.1)$$

(it is independent of the gas densities). If $T_L < T_R$, then $\langle V \rangle > 0$, confirming our previous observation that the piston moves from the cooler side to the hotter side.

The equation (1.1) shows that the average velocity of the piston is different from zero, albeit just of order $O(m/M)$, despite the perfect pressure balance $P_L = P_R$. However, for a macroscopic-size piston the ratio m/M is so small that the time it takes the piston to cover any noticeable distance is much longer than the age of the universe [11], so such a phenomenon cannot be observed experimentally.

We conclude that the evolution of the system proceeds in two different stages. The first is convergence to mechanical equilibrium, which is relatively fast and can

be computed in principle on the basis of macroscopic equations. The second stage is the transition of the system from mechanical equilibrium to thermal equilibrium. This process is very slow and much less understood.

In addition, for an ideal gas (in which the atoms do not interact) a new problem arises. At time $t = 0$, before the piston is released, the gas atoms move independently of each other, namely, every atom bounces off the walls and the clamped piston, without exchanging momentum or energy with other atoms. Therefore, the velocity distribution does not have to be Maxwellian. A stationary state of the ideal gas can be described by any Poisson process with a uniform spatial density and a symmetric velocity distribution. For example, half the atoms may move toward the piston with unit velocity $v = 1$ and the other half in the opposite direction with velocity $v = -1$, and this state is stationary. However, once the piston is released, the atoms start interacting with each other, indirectly, via collisions with the piston. This provides a way of exchanging momentum and energy between the atoms. One can expect that these interactions will ultimately lead to a true thermal equilibrium, when the velocity distribution becomes Maxwellian, as we explain in §5. However, this process may take even longer than the equilibration of the mean kinetic energies described above.

To consider this new process in its ‘pure’ form, we assume that the system is initially in a homogeneous state, that is, the gas density is constant across the entire cylinder and the velocity distribution is the same in both compartments (but different from Maxwellian). Then there seem to be no forces of any kind that would drive the piston in some direction. In particular, when the piston is initially placed in the middle of the cylinder, then by symmetry there should be no reason for it to move either way! On the other hand, the system is not in equilibrium until the velocity distribution becomes Maxwellian, hence it should find ways to evolve toward equilibrium, thus changing its macroscopic state. We discuss this further in §5.

One can also consider a simpler case in which the container is infinitely long on both sides of the piston and the ideal gases have infinitely many atoms, as in [20], [12], [25]. In that case the problem reduces to the classical Rayleigh gas, that is, to a big massive particle submerged in an ideal gas. In particular, our piston in an infinite cylinder becomes a one-dimensional Rayleigh gas, which we describe in some detail.

Let a heavy free particle (called a *molecule*) of mass M move on a line under elastic collisions with atoms of mass m of an ideal gas with a uniform density n and some velocity distribution $f(v) dv$. We denote by $X(t)$ and $V(t) = \dot{X}(t)$ the position and velocity of the molecule at time t . Even though $f(v)$ need not be Maxwellian, the velocity function $V(t)$ and the coordinate function $X(t)$ can be approximated by certain Gaussian stochastic processes.

Theorem 1.1 (Holley [13]). *Let the density $f(v)$ be symmetric, $f(v) = f(-v)$, and have a finite fourth moment $\int v^4 f(v) dv < \infty$. Then for every finite $t_0 < \infty$ the function $V(t)\sqrt{M}$ on the interval $[0, t_0]$ converges in distribution as $M, n \rightarrow \infty$ and $M/n \rightarrow \text{const}$ to an Ornstein–Uhlenbeck velocity process \mathcal{V}_t , while $X(t)\sqrt{M}$ converges to an Ornstein–Uhlenbeck position process \mathcal{X}_t .*

An Ornstein–Uhlenbeck process (X_t, \mathcal{V}_t) is defined by [24]

$$dX_t = \mathcal{V}_t dt, \quad d\mathcal{V}_t = -a\mathcal{V}_t dt + \sqrt{D} dW_t,$$

where $a > 0$, $D > 0$ are constants and W_t is a Wiener process. The Ornstein–Uhlenbeck position process X_t converges in an appropriate limit (for instance, for $a \rightarrow \infty$ and $a^2/D = \text{const}$) to a Wiener process.

We note that the typical velocity $V(t)$ of the molecule is of order $O(1/\sqrt{M})$, which agrees with the equipartition of energy in the system requiring that the average energies of all particles be equal, that is, $M\langle V^2 \rangle = m\langle v^2 \rangle = m \int v^2 f(v) dv$.

Dürr, Goldstein, and Lebowitz [8] extended the above theorem to arbitrary dimension and to asymmetric velocity distributions. The main technical difficulty in the proof of this theorem comes from the so-called *recollisions*, which occur when an atom collides with the molecule more than once. Recollisions result in intricate autocorrelations in the process $(X(t), V(t))$, which otherwise would be Markovian. The proof consists essentially in estimating the undesirable effect of recollisions and showing that it vanishes in the limit as $M \rightarrow \infty$.

The effect of recollisions becomes crucial, though, if the gas is confined in a finite cylinder. All atoms travel to the walls, bounce off it, and come back to the piston for more and more collisions. The induced autocorrelations build up. There are no standard techniques available to estimate (let alone eliminate) the effect of recollisions in general, but we make partial progress in this direction; see § 4.

In summary, the piston problem raises serious mathematical questions and even leads to confusion in the physical theories. The ‘notorious piston’ (as it is known among physicists) has again attracted much attention recently due to a series of papers [11], [12], [25], [19] where a more extensive mathematical apparatus was developed. At the same time, many new numerical experiments have led to a better theoretical understanding of the underlying dynamics but have also raised some new questions. We emphasize that very few rigorous results are available, even for ideal gases, apart from the Rayleigh-type stochastic approximations in the infinite cylinder mentioned above.

We study the piston in a finite cylinder filled with ideal gases. Our gases are ideal, so we do not need to assume that the velocity distribution of atoms is Maxwellian. Since autocorrelations induced by recollisions present a major difficulty, we specify the initial state in such a way that during a certain interval of time each gas atom collides with the piston at most twice. The main goal of our paper is to describe rigorously the dynamics of the piston during that time interval. We show that, in an appropriate limit, the evolution of the piston and the gas converges to a deterministic process which satisfies a certain closed system of differential equations. Our assumptions here simplify technical considerations but by no means reduce the problem to a triviality. In fact, many intriguing questions still remain open in our context, and we discuss them in the last two sections.

Precise statement of problem and main results. Let us consider a cubic domain Λ_L of size L separated into two parts by a movable wall (piston). Each part of Λ_L contains a gas of non-interacting particles of mass $m = 1$. The particles collide with the outer (fixed) walls of Λ_L and with the moving piston elastically.

The piston has mass $M = M_L$ and moves along the x axis due to collisions with the gas particles on both sides. The size L of the cube is a large parameter of our model, and we are interested in the behaviour as $L \rightarrow \infty$. We assume that M_L is proportional to the area of the piston, that is, $M_L \sim L^2$, and the number of gas particles N is proportional to the volume of the cube Λ_L , that is, $N \sim L^3$, while the particle velocities remain of order one.

The position of the piston at time t is specified by a single coordinate $X = X_L(t)$, $0 \leq X \leq L$, and its velocity is then given by $V = V_L(t) = \dot{X}_L(t)$. Since the components of the particle velocities perpendicular to the x axis play no role in the dynamics, we may assume that each particle has only one coordinate, x , and one component of velocity, v , directed along the x axis.

When a particle with velocity v collides with the piston with velocity V , their velocities v' and V' after the collision are given by

$$V' = (1 - \varepsilon)V + \varepsilon v, \quad (1.2)$$

$$v' = -(1 - \varepsilon)v + (2 - \varepsilon)V, \quad (1.3)$$

where $\varepsilon = 2m/(M + m)$. We assume that $M + m = 2mL^2/a$, where $a > 0$ is a constant, and hence

$$\varepsilon = \frac{2m}{M + m} = \frac{a}{L^2}. \quad (1.4)$$

When a particle collides with a wall at $x = 0$ or $x = L$, its velocity just changes sign.

The evolution of the system is then completely deterministic, but one needs to specify the initial conditions. We assume that the piston starts at the midpoint $X_L(0) = L/2$ with zero velocity $V_L(0) = 0$ (see also § 2). The initial configuration of gas particles and their velocities is chosen at random as a realization of a (two-dimensional) Poisson process on the (x, v) -plane (restricted to $0 \leq x \leq L$) with density $L^2 p_L(x, v)$, where $p_L(x, v)$ is a function satisfying certain conditions (see below), and the factor of L^2 is the cross-sectional area of the container. This means that for any domain $D \subset [0, L] \times \mathbb{R}^1$ the number N_D of gas particles $(x, v) \in D$ at time $t = 0$ has a Poisson distribution with parameter

$$\lambda_D = L^2 \iint_D p_L(x, v) dx dv.$$

For any two non-overlapping domains, say $D_1 \cap D_2 = \emptyset$, the corresponding numbers N_{D_1} and N_{D_2} are statistically independent. We note that the total number N of gas particles is a Poisson random variable, too. The total energy and the total initial momentum are random as well.

Let Ω_L denote the space of all possible configurations of gas particles in Λ_L (that is, all countable subsets of $[0, L] \times \mathbb{R}^1$). For each realization² $\omega \in \Omega_L$ the deterministic piston trajectory will be denoted by $X_L(t, \omega)$ and its velocity by $V_L(t, \omega)$.

²Technically, it is possible that two or more particles collide with the piston simultaneously, and then the dynamics will no longer be defined, but multiple collisions are known to occur with probability zero [13], and so we ignore such anomalies.

Thus, the above model is a mechanical system whose dynamical characteristics $X_L(t, \omega)$ and $V_L(t, \omega)$ depend on the large parameter L and are random (depend on ω) for each L .

To obtain a deterministic description of the dynamics of the piston, one needs to take a limit as $L \rightarrow \infty$ and simultaneously rescale space and time. We introduce new space and time coordinates by

$$y = x/L \quad \text{and} \quad \tau = t/L, \quad (1.5)$$

which corresponds to the Euler scaling for the hydrodynamic limit transition. We call y and τ the *macroscopic* ('slow') variables, as opposed to the original *microscopic* ('fast') variables x and t . Now let

$$Y_L(\tau, \omega) = X_L(\tau L, \omega)/L, \quad W_L(\tau, \omega) = V_L(\tau L, \omega) \quad (1.6)$$

denote the position and velocity of the piston in the macroscopic context. The initial conditions are then $Y_L(0) = X_L(0)/L = 0.5$ and $W(0) = V(0) = 0$.

It is now very natural to assume that the initial density $p_L(x, v)$ agrees with our rescaling,

$$p_L(x, v) = \pi_0(x/L, v), \quad (1.7)$$

where the function $\pi_0(y, v)$ is independent of L . Without loss of generality, we can assume that π_0 is normalized in such a way that

$$\int_0^1 \int_{-\infty}^{\infty} \pi_0(y, v) \, dv \, dy = 1.$$

Then the mean number of particles in the entire container Λ_L is exactly equal to L^3 ,

$$\mathbb{E}(N) = \iint L^2 p_L(x, v) \, dv \, dx = L^3,$$

where $\mathbb{E}(\cdot)$ is the expected value.

Furthermore, we assume that the function $\pi_0(y, v)$ satisfies several technical requirements stated below. The meaning and the purpose of these assumptions will become clear below.

- (P1) *Smoothness.* $\pi_0(y, v)$ is a piecewise C^1 function with uniformly bounded partial derivatives, that is, $|\partial\pi_0/\partial y| \leq D_1$ and $|\partial\pi_0/\partial v| \leq D_1$ for some $D_1 > 0$.
- (P2) *Discontinuity lines.* $\pi_0(y, v)$ may be discontinuous on the line $y = Y_L(0)$ (that is, 'on the piston'). In addition, it may have finitely many ($\leq K_1$) other discontinuity lines in the (y, v) -plane with strictly positive slopes (each line is given by an equation $v = f(y)$, where $f(y)$ is C^1 and $0 < c_1 < f'(y) < c_2 < \infty$).
- (P3) *Density bounds.* Let

$$\pi_0(y, v) > \pi_{\min} > 0 \quad \text{for} \quad v_1 < |v| < v_2 \quad (1.8)$$

for some $0 < v_1 < v_2 < \infty$ and let

$$\sup_{y,v} \pi_0(y, v) = \pi_{\max} < \infty. \tag{1.9}$$

The requirements (1.8) and (1.9) basically mean that $\pi_0(y, v)$ takes values of order one.

(P4) *Velocity ‘cutoff’*. Let

$$\pi_0(y, v) = 0 \quad \text{if} \quad |v| < v_{\min} \quad \text{or} \quad |v| > v_{\max} \tag{1.10}$$

with some $0 < v_{\min} < v_{\max} < \infty$. This means that the speed of the gas particles is bounded above by v_{\max} and below by v_{\min} .

(P5) *Approximate pressure balance*. $\pi_0(y, v)$ must be nearly symmetric about the piston, that is,

$$|\pi_0(y, v) - \pi_0(1 - y, -v)| < \varepsilon_0 \tag{1.11}$$

for all $0 < y < 1$ and some sufficiently small $\varepsilon_0 > 0$.

The requirements (P4) and (P5) are crucial. We shall see that they are made to ensure that the speed $|V_L(t, \omega)|$ of the piston is less than the minimum speed of the gas particles for times $t = O(L)$ with probability close to one. Such assumptions were first made in [19].

We think of $D_1, K_1, c_1, c_2, v_1, v_2, v_{\min}, v_{\max}, \pi_{\min}$, and π_{\max} in (P1)–(P4) as fixed (global) constants and of ε_0 in (P5) as an adjustable small parameter. We assume throughout the paper that ε_0 is small enough, meaning that

$$\varepsilon_0 < \bar{\varepsilon}_0(D_1, K_1, c_1, c_2, v_1, v_2, v_{\min}, v_{\max}, \pi_{\min}, \pi_{\max}).$$

It is important to note that the hydrodynamic limit does *not* require that $\varepsilon_0 \rightarrow 0$. The parameter ε_0 stays positive and fixed as $L \rightarrow \infty$.

Now we state our main result.

Theorem 1.2. *There exist an L -independent function $Y(\tau)$ defined for all $\tau \geq 0$ and a positive $\tau_* \approx 2/v_{\max}$ (actually, $\tau_* \rightarrow 2/v_{\max}$ as $\varepsilon_0 \rightarrow 0$) such that*

$$\sup_{0 \leq \tau \leq \tau_*} |Y_L(\tau, \omega) - Y(\tau)| \rightarrow 0 \tag{1.12}$$

and

$$\sup_{0 \leq \tau \leq \tau_*} |W_L(\tau, \omega) - W(\tau)| \rightarrow 0 \tag{1.13}$$

in probability as $L \rightarrow \infty$. Here $W(\tau) = \dot{Y}(\tau)$.

This theorem establishes the convergence in probability of the random functions $Y_L(\tau, \omega), W(\tau, \omega)$ characterizing the mechanical evolution of the piston to the deterministic functions $Y(\tau), W(\tau)$ in the hydrodynamic limit $L \rightarrow \infty$.

The functions $Y(\tau)$ and $W(\tau)$ satisfy certain (Euler-type) differential equations stated in the next section. Those equations have solutions for all $\tau \geq 0$, but we

can guarantee the convergence (1.12) and (1.13) only for $\tau < \tau_*$. What happens for $\tau > \tau_*$, especially as $\tau \rightarrow \infty$, remains an open problem. Some numerical results and heuristic observations in this direction are presented in §5.

Remarks. The function $Y(\tau)$ is at least C^1 and also piecewise C^2 . On the interval $(0, \tau_*)$ the first derivative $W = \dot{Y}$ (velocity) and the second derivative $A = \ddot{Y}$ (acceleration) remain ε_0 -small, $\sup_{\tau} |W(\tau)| \leq \text{const} \cdot \varepsilon_0$ and $\sup_{\tau} |A(\tau)| \leq \text{const} \cdot \varepsilon_0$; see the next section.

We shall also estimate the rate of convergence in (1.12) and (1.13). Precisely, we claim that there is a $\tau_0 > 0$ ($\tau_0 \approx 1/v_{\max}$) for which

$$|Y_L(\tau, \omega) - Y(\tau)| = O\left(\frac{\log L}{L}\right)$$

if $0 < \tau < \tau_0$ and

$$|Y_L(\tau, \omega) - Y(\tau)| = O\left(\frac{\log L}{L^{1/\tau}}\right)$$

if $\tau_0 < \tau < \tau_*$. The same bounds are valid for $|W_L(\tau, \omega) - W(\tau)|$; see §§3 and 4. These estimates hold with ‘overwhelming’ probability, specifically, for all $\omega \in \Omega_L^* \subset \Omega_L$, where $P(\Omega_L^*) = 1 - O(L^{-\log L})$.

§ 2. Hydrodynamic equations

The equations describing the deterministic function $Y(\tau)$ involve another deterministic function, namely, the scaled density $\pi(y, v, \tau)$ of the gas. Initially we have $\pi(y, v, 0) = \pi_0(y, v)$, and for $\tau > 0$ the density $\pi(y, v, \tau)$ evolves according to the following rules.

(H1) *Free motion.* Inside the container the density satisfies the standard continuity equation for a non-interacting particle system without external forces:

$$\left(\frac{\partial}{\partial \tau} + v \frac{\partial}{\partial y}\right) \pi(y, v, \tau) = 0 \tag{2.1}$$

for all y except $y = 0$, $y = 1$, and $y = Y(\tau)$.

The equation (2.1) has a simple solution

$$\pi(y, v, \tau) = \pi(y - vs, v, \tau - s) \tag{2.2}$$

for $0 < s < \tau$ such that $y - vr \notin \{0, Y(\tau - r), 1\}$ for all $r \in (0, s)$. The equation (2.2) has an advantage over (2.1), namely, it applies to all points (y, v) , including those where the function π is not differentiable.

(H2) *Collisions with the walls.* At the walls $y = 0$ and $y = 1$ we have

$$\pi(0, v, \tau) = \pi(0, -v, \tau), \tag{2.3}$$

$$\pi(1, v, \tau) = \pi(1, -v, \tau). \tag{2.4}$$

(H3) *Collisions with the piston.* At the piston $y = Y(\tau)$ we have

$$\begin{aligned} \pi(Y(\tau) - 0, v, \tau) &= \pi(Y(\tau) - 0, 2W(\tau) - v, \tau) \quad \text{for } v < W(\tau), \\ \pi(Y(\tau) + 0, v, \tau) &= \pi(Y(\tau) + 0, 2W(\tau) - v, \tau) \quad \text{for } v > W(\tau), \end{aligned} \tag{2.5}$$

where v stands for the velocity after the collision and $2W(\tau) - v$ for that before the collision; here

$$W(\tau) = \frac{d}{d\tau}Y(\tau) \tag{2.6}$$

is the (deterministic) velocity of the piston.

It remains to describe the evolution of $W(\tau)$. Suppose that the piston's position at time τ is Y and its velocity is W . The piston is affected by the particles (y, v) hitting it from the right (with $y = Y + 0$ and $v < W$) and from the left (with $y = Y - 0$ and $v > W$).

(H4) *The piston's velocity.* The velocity $W = W(\tau)$ of the piston must satisfy the equation

$$\int_W^\infty (v - W)^2 \pi(Y - 0, v, \tau) dv = \int_{-\infty}^W (v - W)^2 \pi(Y + 0, v, \tau) dv; \tag{2.7}$$

see also the additional requirement (H4') below.

In physical terms, (2.7) is a pressure balance: the piston 'chooses' the velocity W in such a way that the pressure of the incoming particles balances out. The equation (2.7) is instrumental for our deterministic approximation of the piston dynamics.

One can combine the two integrals in (2.7) into one by introducing the density of the particles colliding with the piston ('density on the piston') by

$$q(v, \tau; Y, W) = \begin{cases} \pi(Y + 0, v, \tau) & \text{if } v < W, \\ \pi(Y - 0, v, \tau) & \text{if } v > W. \end{cases} \tag{2.8}$$

Then (2.7) can be rewritten as

$$\int_{-\infty}^\infty (v - W(\tau))^2 \operatorname{sgn}(v - W(\tau)) q(v, \tau; Y(\tau), W(\tau)) dv = 0.$$

We also note that for $\tau > 0$, when (2.5) holds, we have

$$W(\tau) = \frac{\int v \pi(Y - 0, v, \tau) dv}{\int \pi(Y - 0, v, \tau) dv} = \frac{\int v \pi(Y + 0, v, \tau) dv}{\int \pi(Y + 0, v, \tau) dv},$$

that is, the piston's velocity is the average of the nearby particle velocities on each side.

The system of (hydrodynamic) equations given in (H1)–(H4) is closed and, for given appropriate initial conditions, must completely determine the functions $Y(\tau)$, $W(\tau)$, and $\pi(y, v, \tau)$ for $\tau > 0$, as we shall see shortly.

To specify the initial conditions, we set $\pi(y, v, 0) = \pi_0(y, v)$ and $Y(0) = 0.5$. The initial velocity $W(0)$ need not be specified: it comes 'for free' as the solution of the equation (2.7) at time $\tau = 0$. It is easy to check that the initial speed $|W(0)|$ is less than v_{\min} ; in fact, $W(0) \rightarrow 0$ as $\varepsilon_0 \rightarrow 0$ in (P5).

We first determine conditions under which the equation (2.7) has a solution W . Let

$$v_{\sup}^-(\tau) = \sup\{v : \pi(Y - 0, v, \tau) > 0\}$$

(with the convention that the supremum of an empty set is $-\infty$) and

$$v_{\inf}^+(\tau) = \inf\{v : \pi(Y + 0, v, \tau) > 0\}$$

(similarly, the infimum of an empty set is $+\infty$).

Lemma 2.1. *There are three cases:*

- (a) *If $v_{\text{sup}}^- > v_{\text{inf}}^+$ or $v_{\text{sup}}^- = v_{\text{inf}}^+ \in \mathbb{R}^1$, then (2.7) has a unique solution $W \in [v_{\text{inf}}^+, v_{\text{sup}}^-]$.*
- (b) *If $v_{\text{sup}}^- < v_{\text{inf}}^+$, then the solutions of (2.7) fill the entire interval $[v_{\text{sup}}^-, v_{\text{inf}}^+]$.*
- (c) *If $v_{\text{sup}}^- = v_{\text{inf}}^+ = \infty$ or $v_{\text{sup}}^- = v_{\text{inf}}^+ = -\infty$, then (2.7) has no real solutions.*

Proof. In the case (a), the difference between the left-hand side and the right-hand side of (2.7) is a continuous and strictly monotonically decreasing function of W . It is positive for $W < v_{\text{inf}}^+$ and negative for $W > v_{\text{sup}}^-$. The rest of the proof goes by direct inspection. \square

Under our assumptions (P1)–(P4) it is easy to show (we do not elaborate) that for every $\tau > 0$ the density $\pi(y, v, \tau)$ has compact support in the (y, v) -plane, that is, $\pi(y, v, \tau) \equiv 0$ for $|v| > v_{\text{max}}(\tau)$. Therefore, the ‘no solution’ case (c) never occurs. The multiple solution case (b) is very unlikely, but not impossible. If this happens, then the velocity $W(\tau)$ must be determined uniquely by an additional requirement:

- (H4') *If $W(\tau - 0) \in [v_{\text{sup}}^-, v_{\text{inf}}^+]$, we define $W(\tau)$ by continuity, $W(\tau) = W(\tau - 0)$. If $W(\tau - 0) < v_{\text{sup}}^-$ or $W(\tau - 0) > v_{\text{inf}}^+$, we set $W(\tau) = v_{\text{sup}}^-$ or $W(\tau) = v_{\text{inf}}^+$, respectively.*

This completes the definition of $W(\tau)$ started by (H4).

The velocity $W(\tau)$ is continuous for generic piecewise smooth densities $\pi(y, v, \tau)$, but in some cases the continuity of $W(\tau)$ might be violated. However, the following simple lemma is helpful.

Lemma 2.2. *Suppose that for every $\tau \in [a, b]$ the density $\pi(y, v, \tau)$ is piecewise C^1 and has finitely many C^1 smooth discontinuity lines in the (y, v) -plane with positive slopes, as required of $\pi_0(y, v)$ in § 1. Then $W(\tau)$ is continuous and piecewise differentiable on the interval $[a, b]$.*

We now pause to make a few remarks. The piston mass is never used in our equations, because its macroscopic mass is zero. Indeed, for the mechanical system described in § 1 the piston mass is $\sim L^2$, while the total mass of the gas particles is $\sim L^3$, and hence the relative mass of the piston vanishes as $L \rightarrow \infty$. Let us consider now the total (macroscopic) mass of the gas

$$\mathcal{M}_{\text{tot}}(\tau) = \int_0^1 \int \pi(y, v, \tau) dv dy,$$

the masses in the left and right compartments separately,

$$\mathcal{M}_L(\tau) = \int_0^{Y(\tau)} \int \pi(y, v, \tau) dv dy,$$

$$\mathcal{M}_R(\tau) = \int_{Y(\tau)}^1 \int \pi(y, v, \tau) dv dy,$$

and the total kinetic energy

$$2\mathcal{E}_{\text{tot}}(\tau) = \int_0^1 \int v^2 \pi(y, v, \tau) dv dy.$$

The following lemma is left as a (simple) exercise.

Lemma 2.3. *The quantities \mathcal{M}_{tot} , \mathcal{M}_L , \mathcal{M}_R , and \mathcal{E}_{tot} remain constant in τ .*

The main equation (2.7) also preserves the total momentum $\iint v \pi(y, v, \tau) dv dy$, but this quantity changes due to collisions with the walls.

Remark. Previously [19], Lebowitz, Piasecki, and Sinai studied the piston dynamics under essentially the same initial conditions as our (P1)–(P5). They argued heuristically that the piston dynamics could be approximated by certain deterministic equations in the original (microscopic) variables x and t . In fact, the present article grew as a continuation of [19]. The deterministic equations found in [19] correspond to our (2.2)–(2.6) with an obvious transformation back to the variables x, t , but our main equation (2.7) has a different counterpart in the context of [19], which is

$$\frac{d}{dt}V(t) = a \left[\int_V^\infty (v - V(t))^2 \pi(Y - 0, v, t) dv - \int_{-\infty}^V (v - V(t))^2 \pi(Y + 0, v, t) dv \right]. \quad (2.9)$$

Here $X = X(t)$ and $V = V(t) = \dot{X}(t)$ stand for the deterministic position and velocity of the piston and $\pi(x, v, t)$ for the density of the gas (the constant a appeared in (1.4)). We refer to [19] for more details and for a heuristic derivation of (2.9). Since (2.9) is a differential equation, unlike our (2.7), the initial velocity $V(0)$ must be specified separately, and it is customary to set $V(0) = 0$. The equation (2.9) can be reduced to (2.7) in the limit as $L \rightarrow \infty$ as follows. One can show (we omit the details) that (2.9) is a dissipative equation whose solution with any (small enough) initial condition $V(0)$ converges to the solution of (2.7) during a t -time interval of length $\sim \log L$. That interval has length $\sim L^{-1} \log L$ on the τ axis, and thus it vanishes as $L \rightarrow \infty$; this is why we replace (2.9) with (2.7) and ignore the initial condition $V(0)$ when working with the thermodynamic variables τ and y . For the same reason, it is convenient to reset the initial value of the piston velocity in the mechanical model of §1 from $V(0) = 0$ to $V(0) = W(0)$; see Theorem 3.5 below. The equation (2.9) will not be used further in this paper.

Let us now describe the solution of the hydrodynamic equations (H1)–(H4) in more detail. We assume that for some $\tau > 0$ the gas density $\pi(y, v, \tau)$ satisfies the following requirements, which are similar to the conditions (P1)–(P5) imposed on the initial function $\pi_0(y, v)$ in §1.

- (P1') *Smoothness.* $\pi(y, v, \tau)$ is a piecewise C^1 function with uniformly bounded partial derivatives, that is, $|\partial\pi/\partial y| \leq D'_1$ and $|\partial\pi/\partial v| \leq D'_1$ for some $D'_1 > 0$.
- (P2') *Discontinuity lines.* $\pi(y, v, \tau)$ has finitely many ($\leq K'_1$) discontinuity lines in the (y, v) -plane with strictly positive slopes (each line is given by an equation $v = f(y)$, where $f(y)$ is C^1 and $0 < c'_1 < f'(y) < c'_2 < \infty$).
- (P3') *Density bounds.* Let

$$\pi(y, v, \tau) > \pi'_{\min} > 0 \quad \text{for} \quad v'_1 < |v| < v'_2 \quad (2.10)$$

for some $0 < v'_1 < v'_2 < \infty$, and let

$$\sup_{y, v} \pi(y, v, \tau) = \pi'_{\max} < \infty. \quad (2.11)$$

(P4') *Velocity 'cutoff'*. Let

$$\pi(y, v, \tau) = 0 \quad \text{for} \quad |v| \leq v'_{\min} \quad \text{or} \quad |v| \geq v'_{\max} \tag{2.12}$$

with some $0 < v'_{\min} < v'_{\max} < \infty$.

Lastly, we want to assume as in (P5) that $\pi(y, v, \tau)$ is nearly symmetric about the piston. However, this assumption requires a little extra work, since the piston does not have to stay at the middle point $Y(0) = 0.5$. For every $Y \in (0, 1)$ we denote by h_Y the unique homeomorphism of $[0, 1]$ such that $h_Y(0) = 1, h_Y(1) = 0, h_Y(Y) = Y$, and h_Y is linear on the subintervals $[0, Y]$ and $[Y, 1]$. Next, we consider $[0, Y] \times \mathbb{R}^1$ as a manifold in which the points $(0, v)$ and $(0, -v)$ are identified for all $v > 0$, as are the points (Y, v) and $(Y, -v)$ for $v > 0$. Similarly, let $[Y, 1] \times \mathbb{R}^1$ be a manifold in which one identifies $(1, v)$ with $(1, -v)$ and (Y, v) with $(Y, -v)$ for all $v > 0$. We denote by d_Y the distance on each of these two manifolds induced by the Euclidean metric $(dy^2 + dv^2)^{1/2}$. The reason why we need this special distance will be clear later, in the proof of Proposition 2.10.

(P5') *Approximate pressure balance*. We require that

$$|Y(\tau) - 0.5| < \varepsilon'_0 \tag{2.13}$$

and that for any point (y, v) with $0 \leq y \leq 1$ and $v'_{\min} \leq |v| \leq v'_{\max}$ there is another point (y_*, v_*) 'across the piston' (that is, a point such that $(y - Y)(y_* - Y) < 0$, where $Y = Y(\tau)$) satisfying

$$d_Y((y_*, v_*), (h_Y(y), -v)) < \varepsilon'_0 \tag{2.14}$$

and

$$|\pi(y, v, \tau) - \pi(y_*, v_*, \tau)| < \varepsilon'_0 \tag{2.15}$$

for some sufficiently small $\varepsilon'_0 > 0$. In addition, we require that

$$\varepsilon'_0 < C'_0 \varepsilon_0 \tag{2.16}$$

with some constant $C'_0 > 0$.

Actually, the map $(y, v) \mapsto (y_*, v_*)$ in (P5'), which we denote by R_τ , is one-to-one and will be explicitly constructed below in the proof of Proposition 2.10.

Again we think of $D'_1, K'_1, c'_1, c'_2, v'_1, v'_2, v'_{\min}, v'_{\max}, \pi'_{\min}, \pi'_{\max}$, and now also of C'_0 , as global constants. They must be bounded on the time interval on which we consider the dynamics (and v'_{\min} and π'_{\min} must be bounded away from zero); hence, we may treat all these constants as being independent of τ . By (2.16), ε'_0 is a small adjustable parameter, just like ε_0 in (P5).

Now we derive rather elementary but important consequences of the above assumptions. Since the density $\pi(y, v, \tau)$ vanishes for $|v| < v'_{\min}$, so does the function $q(v, \tau; Y, W)$ defined by (2.8). Moreover, for any $|W| < v'_{\min}$ the function $q(v, \tau; Y, W)$ is independent of W and can be redefined by

$$q(v, \tau; Y) = \begin{cases} \pi(Y + 0, v, \tau) & \text{for } v < 0, \\ \pi(Y - 0, v, \tau) & \text{for } v > 0. \end{cases} \tag{2.17}$$

The equation (2.7) can also be simplified, namely, the factor $\text{sgn}(v - W)$ can be replaced by $\text{sgn } v$. Then expanding the squares in (2.7) reduces it to a simple quadratic equation for W :

$$\mathcal{Q}_0 W^2 - 2\mathcal{Q}_1 W + \mathcal{Q}_2 = 0, \tag{2.18}$$

where

$$\mathcal{Q}_0 = \int \text{sgn } v \cdot q(v, \tau; Y) \, dv, \tag{2.19}$$

$$\mathcal{Q}_1 = \int v \text{sgn } v \cdot q(v, \tau; Y) \, dv, \tag{2.20}$$

$$\mathcal{Q}_2 = \int v^2 \text{sgn } v \cdot q(v, \tau; Y) \, dv \tag{2.21}$$

with $Y = Y(\tau)$. The integrals $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ have the following physical meaning:

$$\begin{aligned} m\mathcal{Q}_0 &= m_L - m_R, \\ m\mathcal{Q}_1 &= p_L - p_R, \\ m\mathcal{Q}_2 &= 2(e_L - e_R), \end{aligned}$$

where m_L, p_L, e_L represent the total mass, momentum, and energy of the incoming gas particles (per unit length) on the left-hand side of the piston, and m_R, p_R, e_R stand for those on the right-hand side of it. The value \mathcal{Q}_2 also represents the net pressure exerted on the piston by the gas if the piston did not move. Of course, if $\mathcal{Q}_2(\tau) = 0$, then we must have $W(\tau) = 0$, which agrees with (2.18). The next lemma follows easily from (P1')–(P5'). It means that the function $q(v, \tau; Y(\tau))$ is nearly symmetric in v about $v = 0$.

Lemma 2.4. *For any smooth function $f(v)$ defined for $v > 0$*

$$\left| \int_0^\infty f(v)q(v, \tau; Y(\tau)) \, dv - \int_{-\infty}^0 f(-v)q(v, \tau; Y(\tau)) \, dv \right| \leq C_f \varepsilon_0,$$

where the factor $C_f > 0$ depends on f but not on ε_0 .

Convention. We refer to constants that do not depend on our small adjustable parameter ε_0 in (P5) and (P5') as *global constants* (such as C_f in the above lemma). All the constants in (P1)–(P5) and (P1')–(P5') are global, except ε_0 itself and the related ε'_0 . In many cases we denote diverse global constants by $C_i, i \geq 0$, or just by C .

Lemma 2.4 implies that \mathcal{Q}_0 and \mathcal{Q}_2 are small, more precisely,

$$\max\{|\mathcal{Q}_0|, |\mathcal{Q}_2|\} \leq C\varepsilon_0, \tag{2.22}$$

where $C > 0$ is a global constant. At the same time, the assumption (P3') guarantees that

$$\mathcal{Q}_1 \geq \mathcal{Q}_{1,\min} > 0, \tag{2.23}$$

where $\mathcal{Q}_{1,\min}$ is another global constant.

If ε_0 is small enough, then there is a unique root W of the quadratic polynomial (2.18) on the interval $(-v'_{\min}, v'_{\min})$, and W corresponds to the unique solution of (2.7). Since the absolute value of W is less than that of the other root of (2.18), W can be expressed by

$$W = \frac{Q_1 - \sqrt{Q_1^2 - Q_0 Q_2}}{Q_0}, \tag{2.24}$$

where the sign before the radical is $-$ rather than $+$. Of course, (2.24) applies whenever $Q_0 \neq 0$, while for $Q_0 = 0$ we simply have

$$W = \frac{Q_2}{2Q_1}. \tag{2.25}$$

Corollary 2.5. *If ε_0 is small enough, then*

$$|W(\tau)| \leq B\varepsilon_0 < v'_{\min}/3 \tag{2.26}$$

with some global constant $B > 0$.

Proof. This follows immediately from the equations (2.22)–(2.25). \square

We now make an important remark.

Remark (on extension). Let us consider the density of the incoming gas particles on the left-hand side of the piston, that is, $\pi(y, v, \tau)$ for $y = Y(\tau) - 0$ and $v > v'_{\min}$. This function ‘terminates’ on the piston; in other words, it has a discontinuity in y at $y = Y(\tau)$. But it can be naturally extended smoothly ‘across the piston’, that is, for $y > Y(\tau)$, if one ignores the interaction with the piston of the gas coming from the left compartment at times $s \in (\tau - \delta, \tau)$ and applies the rule (H1) instead as if the gas had ‘passed through the piston’. This defines a smooth extension of $\pi(y, v, \tau)$ from the region $y \leq Y(\tau)$ to the region $Y(\tau) < y < Y(\tau) + O(\delta)$ for all $v \geq v'_{\min}$. This extension allows us to differentiate the function $q(v, \tau; Y)$ defined by (2.17) with respect to Y for any $v \geq v'_{\min}$. A similar extension can be made for the density $\pi(y, v, \tau)$, from the region $y \geq Y(\tau)$ to the region $Y(\tau) - O(\delta) < y < Y(\tau)$ for all $v \leq -v'_{\min}$, and hence $q(v, \tau; Y)$ becomes differentiable with respect to Y for $v \leq -v'_{\min}$. We note that our extension can be defined unambiguously, because we need it only for $|v| \geq v'_{\min}$, while the piston’s velocity remains less than v'_{\min} .

Now the quantities Q_0, Q_1 , and Q_2 defined by (2.19)–(2.21) become differentiable in Y for each fixed τ , and the assumptions (P1')–(P4') easily imply that

$$|dQ_i/dY| \leq C_1, \quad i = 0, 1, 2, \tag{2.27}$$

where $C_1 > 0$ is a global constant.

Corollary 2.6. *The piston acceleration $A(\tau) = dW(\tau)/d\tau$ satisfies*

$$|A(\tau)| \leq C\varepsilon_0 \tag{2.28}$$

with a global constant $C > 0$.

Proof. We differentiate the quadratic equation (2.18) with respect to τ and obtain

$$A(\tau) = \frac{(dQ_0/d\tau)W^2 - 2(dQ_1/d\tau)W + (dQ_2/d\tau)}{2(Q_1 - Q_0W)}.$$

Clearly, the denominator is bounded away from zero, and the numerator has an upper bound of order ε_0 , because $|dQ_i/d\tau| = |(dQ_i/dY)W| \leq \text{const} \cdot \varepsilon_0$ by (2.27) and (2.26). \square

More importantly, we can now derive the existence and uniqueness of a solution of the hydrodynamic equations (H1)–(H4) as long as the conditions (P1')–(P5') continue holding.

Lemma 2.7. *If the hydrodynamic equations (H1)–(H4) have a solution on an interval $0 \leq \tau \leq T$ and the conditions (P1')–(P5') hold on this interval, then the solution is unique at $\tau = T$ and can be extended immediately beyond the point $\tau = T$.*

Proof. The only differential equation in our system (H1)–(H4) is (2.6), in which $W(\tau)$ is the root of the quadratic equation (2.18) given by (2.24), (2.25). Due to the above extension remark we can think of W as an implicit function of Y , that is, effectively $W = F(Y, \tau)$. Then the differential equation (2.6) takes the canonical form

$$\frac{d}{d\tau}Y(\tau) = F(Y(\tau), \tau). \tag{2.29}$$

For this equation to have a unique solution, it is sufficient that $F(Y, \tau)$ have a bounded partial derivative with respect to Y .

Since W is a root of the quadratic equation (2.18), we can differentiate (2.18) with respect to Y and obtain

$$\frac{\partial F(Y, \tau)}{\partial Y} = \frac{(dQ_0/dY)W^2 - 2(dQ_1/dY)W + (dQ_2/dY)}{2(Q_1 - Q_0W)}.$$

We already know that the denominator is bounded away from zero. It follows from (2.27) that the numerator stays bounded above, and hence

$$\left| \frac{\partial F(Y, \tau)}{\partial Y} \right| \leq \kappa \tag{2.30}$$

with a global constant $\kappa > 0$. \square

Next we consider the evolution of a point (y, v) in the domain

$$\mathcal{G} := \{(y, v) : 0 \leq y \leq 1\}$$

under the rules (H1)–(H3), that is, as it moves freely with constant velocity and collides elastically with the walls and the piston. We denote by (y_τ, v_τ) its position and velocity at time $\tau \geq 0$. Then (H1) translates into $\dot{y}_\tau = v_\tau$ and $\dot{v}_\tau = 0$ whenever $y_\tau \notin \{0, 1, Y(\tau)\}$, (H2) becomes $(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, -v_{\tau-0})$ whenever $y_{\tau-0} \in \{0, 1\}$, and (H3) gives

$$(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, 2W(\tau) - v_{\tau-0}) \tag{2.31}$$

whenever $y_{\tau-0} = Y(\tau)$. We note that (2.31) corresponds to a special case of the mechanical collision rules (1.2)–(1.3) with $\varepsilon = 0$ (equivalently, $m = 0$). Hence, the point (y, v) moves in \mathcal{G} as if it were a gas particle with zero mass.

The motion of points in (y, v) is described by a one-parameter family of transformations $\mathcal{F}^\tau : \mathcal{G} \rightarrow \mathcal{G}$ defined by $\mathcal{F}^\tau(y_0, v_0) = (y_\tau, v_\tau)$ for $\tau > 0$. We also write $\mathcal{F}^{-\tau}(y_\tau, v_\tau) = (y_0, v_0)$. According to (H1)–(H3), the density $\pi(y, v, \tau)$ satisfies a simple equation of the form

$$\pi(y_\tau, v_\tau, \tau) = \pi(\mathcal{F}^{-\tau}(y_\tau, v_\tau), 0) = \pi_0(y_0, v_0) \tag{2.32}$$

for all $\tau \geq 0$. It is also easy to see that for each $\tau > 0$ the map \mathcal{F}^τ is one-to-one and preserves area, that is, $\det |D\mathcal{F}^\tau(y, v)| = 1$.

Because of (P4), the initial density $\pi_0(y, v)$ can be positive only in the region

$$\mathcal{G}^+ := \{(y, v) : 0 \leq y \leq 1, v_{\min} \leq |v| \leq v_{\max}\}.$$

Hence, we restrict ourselves to the points $(y, v) \in \mathcal{G}^+$ only. At any time $\tau > 0$, the images of those points are confined to the region $\mathcal{G}^+(\tau) := \mathcal{F}^\tau(\mathcal{G}^+)$. In particular, $\pi(y, v, \tau) = 0$ for $(y, v) \notin \mathcal{G}^+(\tau)$.

We now make an important observation. If a point (y_τ, v_τ) collides with a piston whose velocity is slow, $|W(\tau)| \ll |v_\tau|$, then they cannot recollide too soon: the point must travel to a wall, bounce off it, and then travel back to the piston before hitting it again. This is quantified in the following lemma.

Lemma 2.8. *Let a point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ collide with the piston, that is, $y_\tau = Y(\tau)$. Then during the interval $(\tau, \tau + \Delta)$ with*

$$\Delta = \frac{1 - 2\mathcal{B}\varepsilon_0\tau}{v'_{\max} + 3\mathcal{B}\varepsilon_0}$$

it cannot recollide with the piston, that is, $y_s \neq Y(s)$ for $s \in (\tau, \tau + \Delta)$ under the assumption that (P1')–(P5') hold during this interval.

Proof. The point's speed after the collision is at least $v'_{\min} - 2\mathcal{B}\varepsilon_0$ and at most $v'_{\max} + 2\mathcal{B}\varepsilon_0$. The piston cannot ‘catch up’ with the point, since $|W(\tau)| < v'_{\min} - 2\mathcal{B}\varepsilon_0$ by (2.26). So the point travels to the wall, bounces off it, and travels back to the piston, and all that takes the time

$$\Delta \geq 2D/(v'_{\max} + 3\mathcal{B}\varepsilon_0),$$

where $D = \min\{Y(\tau), 1 - Y(\tau)\} \geq 0.5 - \mathcal{B}\varepsilon_0\tau$. \square

Therefore, as long as (P1')–(P4') are satisfied, the collisions of each moving point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ with the piston occur at well-separated moments of time, which allows us to effectively count them. For $(x, v) \in \mathcal{G}^+$

$$N(y, v, \tau) = \#\{s \in (0, \tau) : y_s = Y(s), v_s \neq W(s)\}$$

is the number of collisions of the point (y, v) with the piston during the interval $(0, \tau)$. For each $\tau > 0$ we partition the region $\mathcal{G}^+(\tau)$ into the subregions

$$\mathcal{G}_n^+(\tau) := \{\mathcal{F}^\tau(y, v) : (y, v) \in \mathcal{G}^+ \text{ and } N(y, v, \tau) = n\},$$

so that $\mathcal{G}_n^+(\tau)$ is occupied by the points that have experienced exactly n collisions with the piston during the interval $(0, \tau)$.

For each $n \geq 1$ we define $\tau_n > 0$ to be the first time when some point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ experiences its $(n + 1)$ st collision with the piston, that is,

$$\tau_n = \sup\{\tau > 0 : \mathcal{G}_{n+1}^+(\tau) = \emptyset\}.$$

In particular, $\tau_1 > 0$ is the earliest time when some point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ experiences its first recollision with the piston. Hence, no recollisions occur on the interval $[0, \tau_1)$, and we call it the *zero-recollision interval*. Similarly, at most one recollision with the piston is possible for any point on the interval (τ_1, τ_2) , and we call it the *one-recollision interval*.

The time τ_* in Theorem 1.2 is the earliest time when some point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ either experiences its third collision with the piston, or has its second collision with the piston under the condition that the first collision occurred after τ_1 . Hence, $\tau_* \leq \tau_2$, and τ_* is actually very close to τ_2 ; see the next lemma.

Lemma 2.9. *Let (P1')–(P5') hold on the interval $(0, n/v_{\max} + \delta)$ for some $n \geq 1$ and $\delta > 0$. Then*

$$|\tau_k - k/v_{\max}| \leq C\varepsilon_0$$

for all sufficiently small ε_0 and for all $1 \leq k \leq n$, where $C > 0$ is a global constant that may depend on n . Moreover,

$$|\tau_* - 2/v_{\max}| \leq C\varepsilon_0.$$

Proof. The necessary lower bounds for τ_k follow from Lemma 2.8. The necessary upper bounds are just as easy to obtain; we omit the details. \square

It is clear at this point that the hydrodynamic equations (H1)–(H4) have a unique and ‘well-behaved’ solution as long as the conditions (P1')–(P5') continue holding with some small ε_0 . Our next goal is to show that this is indeed the case.

Proposition 2.10. *Let $T > 0$. If the initial density $\pi_0(y, v)$ satisfies (P1)–(P5) and the ε_0 in (P5) is small enough (for the given T), then the conditions (P1')–(P5') hold on the interval $0 < \tau < T$.*

We note that the corresponding global constants in (P1')–(P5') depend on T as specified below.

Proof. The main idea is to show that the restrictions on $\pi(y, v, \tau)$ imposed by (P1)–(P5) at $\tau = 0$ ‘deteriorate’ very slowly as time goes on, and hence the conditions (P1')–(P5') continue holding (‘propagate’) with the respective global constants slowly changing in time.

We first note that as long as (P1')–(P5') hold, the number of collisions grows at most linearly in τ , that is, on any interval $(0, \tau)$ on which (P1')–(P5') hold, every moving point $(y_s, v_s) \in \mathcal{G}^+(s)$ experiences at most $\tau v'_{\max} + 1$ collisions with the piston if ε_0 is small enough; see Lemmas 2.8 and 2.9. Next, we examine the conditions (P1')–(P5') individually and show that each of them should hold up to time T provided that the others do.

We start with (P1). Due to (H1) we have

$$\frac{\partial\pi(y, v, \tau + s)}{\partial y} = \frac{\partial\pi(y - sv, v, \tau)}{\partial y}$$

and

$$\frac{\partial\pi(y, v, \tau + s)}{\partial v} = \frac{\partial\pi(y - sv, v, \tau)}{\partial v} - s \frac{\partial\pi(y - sv, v, \tau)}{\partial y}$$

for all $s > 0$ such that the moving point located at (y, v) at time $\tau + s$ did not experience collisions with the piston during the interval $(\tau, \tau + s)$. Thus, between collisions with the walls and the piston, the partial derivatives of $\pi(y, v, \tau)$ can grow with τ at most linearly. Collisions with the walls can change only the sign of the derivatives of π and not their absolute values.

Let us now consider the effect of interactions with the piston. We estimate the partial derivatives of $\pi(y, v, \tau)$ at a point (y, v) after a collision with the piston at some earlier time $s \in (0, \tau)$. For simplicity, we assume that there are no other collisions of the moving point (y, v) with the piston or the walls on the interval (s, τ) . Then s satisfies the equation

$$Y(s) = y - (\tau - s)v. \tag{2.33}$$

By (H3) and (H1) we have

$$\begin{aligned} \pi(y, v, \tau) &= \pi(y - (\tau - s)v, v, s + 0) = \pi(y - (\tau - s)v, 2W - v, s - 0) \\ &= \pi(y - (\tau - s)v - (s - s_0)(2W - v), 2W - v, s_0), \end{aligned}$$

where $s_0 < s$ is any earlier time (which we consider fixed) and $W = W(s)$ is the piston velocity at the time of collision. Let $y_0 = y - (\tau - s)v - (s - s_0)(2W - v)$ and $v_0 = 2W - v$. Then

$$\begin{aligned} \frac{\partial\pi(y, v, \tau)}{\partial y} &= \frac{\partial\pi(y_0, v_0, s_0)}{\partial y} \left[1 + v \frac{ds}{dy} - 2(s - s_0) \frac{dW}{dy} - (2W - v) \frac{ds}{dy} \right] \\ &\quad + \frac{\partial\pi(y_0, v_0, s_0)}{\partial v} \cdot 2 \frac{dW}{dy}. \end{aligned}$$

Differentiating (2.33) with respect to y gives

$$\frac{dY}{ds} \cdot \frac{ds}{dy} = 1 + v \frac{ds}{dy},$$

and hence

$$\frac{ds}{dy} = \frac{1}{W - v}.$$

Moreover,

$$\frac{dW}{dy} = \frac{dW}{ds} \cdot \frac{ds}{dy} = \frac{A}{W - v},$$

where $A = A(s)$ is the piston acceleration at the time of collision. As long as (P1')–(P5') hold, we have $W = O(\varepsilon_0)$ and $A = O(\varepsilon_0)$; hence, $ds/dy = -v^{-1} + O(\varepsilon_0)$ and

$$\frac{\partial \pi(y, v, \tau)}{\partial y} = \frac{\partial \pi(y_0, v_0, s_0)}{\partial y} + O(\varepsilon_0).$$

In other words, the piston (due to its low speed and acceleration) acts almost as a wall, which changes only the sign of $\partial \pi / \partial y$. A similar calculation (we omit it) holds for the partial derivative with respect to v .

Thus, as long as (P1')–(P5') hold, the density $\pi(y, v, \tau)$ remains piecewise C^1 , and its partial derivatives can grow at most linearly with τ .

Next, we check the condition (P2'). We begin with three special discontinuity lines that do not appear explicitly in (P2). They are created immediately by the reflections at the walls and the piston at time $\tau = 0$, since the initial density $\pi(y, v, 0)$ does not have to satisfy (H2)–(H3). Those discontinuity lines are $y = 0$, $y = 0.5$, and $y = 1$ at $\tau = 0$, and their images for $\tau > 0$ are the slanted lines

$$y = v\tau, \quad y = 0.5 + v\tau, \quad y = 1 + v\tau, \tag{2.34}$$

respectively; see Fig. 2. Their slope at any time τ is positive and constant, $dy/dv = \tau$. It is not bounded away from zero as $\tau \rightarrow 0$, so we have a technical violation of (P2') for small τ , but it will be clear immediately below why this should not bother us.

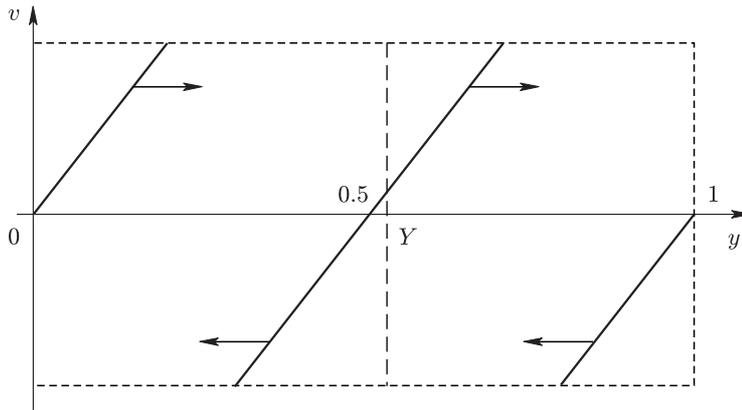


Figure 2. Slanted discontinuity lines. The dashed vertical line shows the piston position

The singularity lines (2.34) exist only in the region $v_{\min} \leq |v| \leq v_{\max}$ (elsewhere $p \equiv 0$), and hence they cannot intersect the piston $y = Y(\tau)$ for small τ . It takes some time, at least

$$\tau^* = \frac{0.5}{v_{\max} + \mathcal{B}\varepsilon_0} > 0,$$

before any of these singularity lines ‘reaches’ the piston and its effect has to be reckoned with. At that time the slopes of those lines are bounded away from zero, $dy/dv \geq \tau^* > 0$, and hence (P2') holds.

We now consider the evolution of all discontinuity curves of the function $\pi(y, v, \tau)$ as τ increases. Let a discontinuity curve of $\pi(y, v, s)$ at time s be given by an equation $y = g_s(v)$, and thus its slope is $h_s(v) = dg_s(v)/dv$. Since the curve and its slope change in time, the function g_s and its derivative h_s depend on s . According to (2.2), we have $g_{s+\tau}(v) = g_s(v) + v\tau$ between collisions with the piston and the walls, and therefore

$$\frac{dg_s(v)}{ds} = v \quad \text{and} \quad \frac{dh_s(v)}{ds} = 1. \tag{2.35}$$

Hence, between collisions with the piston, the slope of discontinuity curves grows linearly with τ (in particular, the slope remains positive).

Now let the curve $y = g_s(v)$ cross the piston at some point

$$g_s(v) = Y(s) \tag{2.36}$$

(this equation makes v a function of s). After the collision with the piston, this point transforms to $(Y, 2W - v)$, according to the rule (H3), where $W = W(s)$ stands for the piston velocity. If $\tau > s$ is some fixed time, then the image of our point at time τ is $(Y + (\tau - s)(2W - v), 2W - v)$. These points form a curve in the (y, v) -plane, parameterized by s (the collision time). This is a discontinuity curve for the density $\pi(y, v, \tau)$ at time τ . Let $y_s = Y + (\tau - s)(2W - v)$ and $v_s = 2W - v$ be the coordinates of a point on that curve. To compute the slope dy_s/dv_s , we first differentiate y_s and v_s with respect to the parameter s ,

$$\frac{dy_s}{ds} = W + (\tau - s) \left[2 \frac{dW}{ds} - \frac{dv}{ds} \right] - (2W - v) = v - W + (\tau - s) \left[2A - \frac{dv}{ds} \right]$$

and

$$\frac{dv_s}{ds} = 2 \frac{dW}{ds} - \frac{dv}{ds} = 2A - \frac{dv}{ds},$$

where $A = A(s)$ is the piston acceleration (at the collision time s). Differentiating (2.36) with respect to s and using (2.35) gives

$$\frac{dg_s(v)}{ds} \cdot \frac{dv}{ds} + v = W,$$

and hence

$$\frac{dv}{ds} = \frac{W - v}{h_s(v)}.$$

Therefore, the slope of our singularity curve at time τ is

$$\frac{dy}{dv}(\tau) = \frac{(v - W)[h_s(v) + \tau - s] + 2Ah_s(v)(\tau - s)}{v - W + 2Ah_s(v)}. \tag{2.37}$$

As long as (P1')–(P5') hold, we have $W = O(\varepsilon_0)$ and $A = O(\varepsilon_0)$, and therefore

$$\frac{dy}{dv}(\tau) = h_s(v) + \tau - s + O(\varepsilon_0). \tag{2.38}$$

Hence, every collision with the piston adds only an $O(\varepsilon_0)$ correction to the linear growth of the slopes of discontinuity curves.

Next we check the conditions (P3')–(P5') by using the following lemma.

Lemma 2.11. *Let (P1')–(P5') hold on an interval $(0, \tau)$. Then for every point $(y, v) \in \mathcal{G}^+(\tau)$ there is another point $(y_0, v_0) \in \mathcal{G}^+$ such that $\pi(y, v, \tau) = \pi(y_0, v_0, 0)$ and*

$$||v| - |v_0|| = 2(v'_{\max}\tau + 1)\mathcal{B}\varepsilon_0.$$

Proof. We set $(y_0, v_0) = \mathcal{F}^{-\tau}(y, v)$ and use (2.32). At each collision of the point (y_0, v_0) with the piston, the speed $|v|$ changes by $2|W| \leq 2\mathcal{B}\varepsilon_0$ according to (2.31) and (2.36), and the number of collisions is bounded by $v'_{\max}\tau + 1$. \square

Lemma 2.11 implies immediately that (P3') and (P4') continue holding with global constants v'_1, v'_2, v'_{\min} , and v'_{\max} changing slowly with time: they change at most by $CT\varepsilon_0$ on the interval $(0, T)$ with a global constant $C > 0$. In particular, v'_1 and v'_{\min} remain positive provided that ε_0 is small enough. The constants π'_{\min} and π'_{\max} do not change at all.

To check (P5'), we explicitly construct the map $R_\tau : (y, v) \mapsto (y_*, v_*)$ in (2.14) and (2.15); it is defined here by $R_\tau = \mathcal{F}^\tau \circ R_0 \circ \mathcal{F}^{-\tau}$, where $R_0(y, v) = (1 - y, -v)$ is a simple reflection across the piston at time $\tau = 0$. Now (2.13) follows from (2.26), and (2.15) follows from (P5).

Lastly, we derive (2.14) from Lemma 2.11. Let (y, v) be a moving point at time τ and let $(y_0, v_0) = \mathcal{F}^{-\tau}(y, v) \in \mathcal{G}^+$ be the pre-image at time zero. Let us compare the evolution of the point (y_0, v_0) and of its mirror image $R_0(y_0, v_0) = (1 - y_0, -v_0) \in \mathcal{G}^+$ across the piston during the interval $(0, \tau)$. Due to (2.13) and (2.26), these two points experience collisions with the walls and the piston at moments that differ at most by $O(\varepsilon_0)$. Their velocities also differ at most by $O(\varepsilon_0)$, and hence their positions at time T are almost symmetric about the piston, up to $O(\varepsilon_0)$. This implies (2.14).

We note that the numbers of collisions experienced by the above two moving points by a given time T can be different, because one point may have just collided with the piston or a wall, while the other may be about to collide with it. To take care of this case, we introduced the special distance d_Y in (P5'). \square

We summarize our main results in the following theorem.

Theorem 2.12. *Let $T > 0$ be given. If the initial density $\pi_0(y, v)$ satisfies (P1)–(P5) with a sufficiently small ε_0 , then:*

- (a) *a solution of our hydrodynamic equations (H1)–(H4) exists and is unique on the interval $(0, T)$;*
- (b) *the density $\pi(y, v, \tau)$ satisfies (P1')–(P5') for all $0 < \tau < T$;*
- (c) *the piston velocity and acceleration remain small, $|W(\tau)| = O(\varepsilon_0)$ and $|A(\tau)| = O(\varepsilon_0)$, and the piston position remains close to the midpoint 0.5 in the sense that $|Y(\tau) - 0.5| = O(\varepsilon_0)$ for all $0 < \tau < T$;*
- (d) *$|\tau_k - k/v_{\max}| = O(\varepsilon_0)$ for all $1 \leq k < Tv_{\max}$, and if $Tv_{\max} > 2$, then also $|\tau_* - \tau_2| = O(\varepsilon_0)$.*

Corollary 2.13. *If $\varepsilon_0 = 0$, and hence the initial density $\pi_0(y, v)$ is completely symmetric about the piston, then the solution is trivial, with $Y(\tau) \equiv 0.5$ and $W(\tau) \equiv 0$ for all $\tau > 0$.*

Lastly, we demonstrate the reason for our assumption that all the discontinuity curves of the initial density $\pi(y, v)$ must have positive slopes. It would be quite

tempting to let $\pi(y, v)$ have more general discontinuity lines, for instance, allow it to be smooth for $v_{\min} < |v| < v_{\max}$ and abruptly drop to 0 at $v = v_{\min}$ and $v = v_{\max}$. The following example shows why this is not acceptable.

Example. Suppose that the initial density $\pi_0(y, v)$ has a horizontal discontinuity line $v = v_0$ (say, $v_0 = v_{\min}$ or $v_0 = v_{\max}$). After one interaction with the piston the image of this discontinuity line can oscillate up and down, due to fluctuations of the piston acceleration (Fig. 3). As time goes on, this oscillating curve ‘travels’ to the wall and comes back to the piston, experiencing some distortions on its way caused by the differences in velocities of its points (Fig. 3). When this curve comes back to the piston again, it may well have ‘turning points’ at which its tangent line is vertical, or may even contain vertical segments of positive length. This produces undesirable singularities or even discontinuities of the piston velocity and acceleration. The same phenomena can also occur when a discontinuity line of the initial density $\pi_0(y, v)$ has a negative slope.

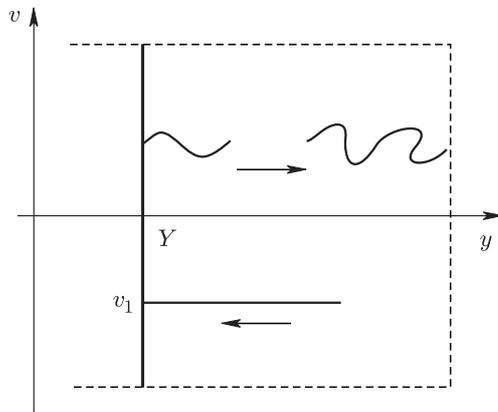


Figure 3. A horizontal discontinuity line (bottom) comes off the piston as an oscillating curve (top)

§ 3. Dynamics before the first recollision

In this section we begin to study the mechanical model of the piston in the ideal gas described in § 1. We claim that the random trajectory of the piston described by the functions $Y_L(\tau, \omega) = X_L(\tau L, \omega)/L$ and $W_L(\tau, \omega) = V_L(\tau L, \omega)$ (cf. (1.6)) converges in probability as $L \rightarrow \infty$ to the solution $Y(\tau)$, $W(\tau)$ of the hydrodynamic equations (this solution was found in the previous section) on the zero-recollision interval $(0, \tau_1)$.

Convention. For brevity, we suppress the dependence on L and ω in our expressions if it causes no confusion. For example, we write $X(t)$ and $V(t)$ instead of $X_L(t, \omega)$ and $V_L(t, \omega)$, respectively, and so on.

We work here with the microscopic time t . First, we define the ‘microscopic’ gas density for all $t \geq 0$ and denote it by $p(x, v, t)$. For $t = 0$ the density is initialized by

$p(x, v, 0) = \pi_0(x/L, v)$; see (1.7). For $t > 0$ its evolution is defined by rules similar to (H1)–(H3): the free motion between collisions,

$$p(x, v, t) = p(x - vs, v, t - s) \tag{3.1}$$

for $s > 0$ such that $x - vr \notin \{0, X(t - r), L\}$ for all $r \in (0, s)$; reflections at the walls,

$$p(0, v, t) = p(0, -v, t) \quad \text{and} \quad p(L, v, t) = p(L, -v, t); \tag{3.2}$$

and elastic collisions with the piston,

$$p(X(t) \pm 0, v, t) = p(X(t) \pm 0, 2V(t) - v, t). \tag{3.3}$$

Since the last equation involves the random functions $X(t)$ and $V(t)$, the density $p(x, v, t)$ depends on ω , that is, it is now a random function.

The evolution of the density $p(x, v, t)$ can be conveniently described with the help of a one-parameter family of transformations F^t similar to \mathcal{F}^τ defined in §2. Let (x, v) be a point in the domain

$$G := \{(x, v) : 0 \leq x \leq L\}.$$

Its trajectory (x_t, v_t) for $t > 0$ is determined by the free motion $\dot{x}_t = v_t, \dot{v}_t = 0$ inside the container whenever $x_t \notin \{0, X(t), L\}$, reflections $(x_{t+0}, v_{t+0}) = (x_{t-0}, -v_{t-0})$ at the walls whenever $x_{t-0} \in \{0, L\}$, and collisions

$$(x_{t+0}, v_{t+0}) = (x_{t-0}, 2V(t) - v_{t-0}) \tag{3.4}$$

with the piston whenever $x_{t-0} = X(t)$. The family of transformations F^t is defined by $F^t(x_0, v_0) = (x_t, v_t)$ for $t > 0$. We also write $F^{-t}(x_t, v_t) = (x_0, v_0)$. We now simply have

$$p(x_t, v_t, t) = p(F^{-t}(x_t, v_t), 0) = p(x_0, v_0, 0). \tag{3.5}$$

Note that for each $t > 0$ the map F^t is a bijection of G and preserves area, that is,

$$\det |DF^t(x, v)| = 1. \tag{3.6}$$

We emphasize that the transformation F^t , like the density $p(x, v, t)$, is random, that is, depends on ω .

Remark. The piston velocity $V(t)$ is a piecewise constant function updated at the moments of collision with gas atoms by the rules (1.2)–(1.3). If t is such a collision time, then the value $V(t)$ in equation (3.3) must be replaced by the average $(V(t - 0) + V(t + 0))/2$ of the one-sided limits. This modification is important, since it makes the rule (3.3) equivalent to (1.2)–(1.3) when (x_t, v_t) represents an actual gas particle of mass m . Otherwise it corresponds to the motion of a particle of zero mass, and we may call it a *virtual particle*.

Because of (P4), the initial density $p(x, v, 0)$ can be positive only in the region

$$G^+ := \{(x, v) : 0 \leq x \leq L, v_{\min} \leq |v| \leq v_{\max}\},$$

and therefore the set G^+ contains all the gas particles at time $t = 0$. For any $t > 0$ the region $G^+(t) := F^t(G^+)$ contains all the actual gas particles at time t , and $p(x, v, t) = 0$ for all $(x, v) \notin G^+(t)$.

For each point $(x, v) \in G$ and $t > 0$ we define the number of collisions with the piston during the interval $(0, t)$,

$$N(x, v, t) = \#\{s \in (0, t) : x_s = X(s), v_s \neq V(s)\}.$$

Then we partition the region G into the subregions

$$G_n(t) := \{F^t(x, v) : (x, v) \in G \text{ and } N(x, v, t) = n\}$$

and set $G_n^+(t) := G^+(t) \cap G_n(t)$. The region $G_n^+(t)$ is occupied by the points that have experienced exactly n collisions with the piston during the time interval $(0, t)$.

We emphasize that our transformations F^t and the regions $G_n(t)$ and $G_n^+(t)$ depend on ω , that is, are random. We note, however, that they are completely determined by the trajectory of the piston, that is, by the function $X(s)$, $0 < s < t$.

The family of transformations $\mathcal{F}^\tau : \mathcal{G} \rightarrow \mathcal{G}$ introduced in §2 induces a (deterministic) family $\tilde{F}^t : G \rightarrow G$ defined as follows: if $\mathcal{F}^\tau(y, v) = (y_\tau, v_\tau)$, then we set $\tilde{F}^{\tau L}(yL, v) := (y_\tau L, v)$. The transformations \tilde{F}^t define a deterministic density function on G by $\tilde{p}(x, v, t) := p(\tilde{F}^t(x, v), 0)$; this is related to the density $\pi(y, v, \tau)$ studied in §2 by $\tilde{p}(x, v, t) = \pi(x/L, v, t/L)$. We also set $\tilde{G}^+(t) := \mathcal{F}^t(G^+)$.

The gas particles in $G_0(t)$ determine a Poisson process, as the following lemma shows. Let $\omega \in \Omega$ and $t > 0$. We fix the trajectory of the piston $X(s)$, $0 < s < t$. This completely specifies the region $G_0(t)$ and the density $p(x, v, t)$.

Lemma 3.1. *The conditional distribution of the gas particles in $G_0(t)$ (given the trajectory $X(s)$, $0 < s < t$, of the piston) is Poisson with the density function $L^2 p(x, v, t)$.*

Proof. Let $D \subset G_0(t)$ be any domain. Then its pre-image $F^{-(t-s)}(D)$ remains at a positive distance from the piston $X(s)$ for all $s \in (0, t)$. Hence, the particles starting out from the region $F^{-t}(D)$ and ending up in the region D cannot affect the piston during the time interval $(0, t)$. Therefore, the number of particles in D at time t , which is equal to the number of particles in $F^{-t}(D)$ at time 0, is independent of the piston trajectory. Thus, it is a Poisson random variable with parameter

$$\lambda_D(t) = L^2 \iint_{F^{-1}(D)} p(x, v, 0) dx dv = L^2 \iint_D p(x, v, t) dx dv.$$

The equality of the above integrals follows from (3.5) and (3.6). \square

Remark. For any domain $D \subset G_0(t)$ its pre-image $F^{-t}(D)$ is actually independent of ω . Indeed, let F_0^t be another family of transformations on G determined by the free motion on the entire interval $0 < x < L$ and by elastic reflections only at the walls $x = 0$ and $x = L$ (as if the piston did not exist). Then we have $F^{-t}(D) = F_0^{-t}(D)$ for any domain $D \subset G_0(t)$.

For $n \geq 1$, we define T_n to be the earliest time the piston interacts with points from $G_n^+(t)$ (thus creating the region $G_{n+1}^+(t)$), or, equivalently,

$$T_n = \sup_{t>0} \{G_{n+1}^+(t) = \emptyset\}. \tag{3.7}$$

The times $T_n = T_n(\omega)$ are random analogues of the times τ_n introduced in §2. In particular, T_1 is the time of the first recollision in the system (by an actual or a virtual particle).

Lemma 3.2. *For all $\omega \in \Omega$*

$$T_1 \leq T_{1,\max} := L/v_{\max}. \tag{3.8}$$

Proof. The fastest particles $(x, v) \in G^+$ that collide with the piston at time 0 move with speed v_{\max} and recollide with the piston at a time t which satisfies $v_{\max}t + |X(t) - L/2| = L$. For all such t we have

$$T_1 \leq t = \frac{L - |X(t) - L/2|}{v_{\max}}.$$

This proves the lemma. \square

During the time interval $(0, T_1)$ the piston interacts with particles in $G_0^+(t)$. We denote by

$$\begin{aligned} \mathcal{X}_0(t) = & \{(x, v) : x = X(t) + 0, -v_{\max} < v < -v_{\min}\} \\ & \cup \{(x, v) : x = X(t) - 0, v_{\min} < v < v_{\max}\} \end{aligned}$$

the union of the two immediate one-sided neighbourhoods of the piston which contain all ‘incoming’ particles that are about to collide with the piston.

To study the piston dynamics on the zero-recollision interval, it is convenient to assume that the piston is slow enough and interacts only with the original particles that started in G^+ at time 0. A subinterval $(0, S_1) \subset (0, T_1)$ where the piston satisfies these requirements is said to be a ‘slow’ interval.

Definition. We define $(0, S_1) \subset (0, T_1)$ to be the maximal time interval on which

- (a) $|V(t)| < v_{\min}$;
- (b) $\mathcal{X}_0(t) \subset G_0(t)$.

By the condition (b), all the incoming particles that are about to hit the piston at time t have started out in G^+ at time zero and have not interacted with the piston during the time interval $(0, t)$.

Almost all our considerations in this section are restricted to the slow interval $(0, S_1)$. But at the end of the section we shall see that for typical $\omega \in \Omega$ the slow interval coincides with the whole interval $(0, T_1)$, that is, $S_1 = T_1$.

For each $t \in (0, S_1)$ we define the density $q(v, t)$ of colliding particles ‘on the piston’ by

$$q(v, t) = \begin{cases} p(X(t) - 0, v, t) & \text{for } v > 0, \\ p(X(t) + 0, v, t) & \text{for } v < 0 \end{cases} \tag{3.9}$$

(cf. (2.17)). Next, we define

$$Q_0(t) = \int \operatorname{sgn} v \cdot q(v, t) \, dv, \tag{3.10}$$

$$Q_1(t) = \int v \operatorname{sgn} v \cdot q(v, t) \, dv, \tag{3.11}$$

$$Q_2(t) = \int v^2 \operatorname{sgn} v \cdot q(v, t) \, dv, \tag{3.12}$$

in a way similar to (2.19)–(2.21) in §2.

Since the restriction of $p(x, v, t)$ to the domain $G_0(t)$ coincides with $\tilde{p}(x, v, t) = \pi(x/L, v, t/L)$, the conditions (P1')–(P2') imply the following assertion.

Lemma 3.3. *The density $p(x, v, t)$ restricted to the region $G_0(t)$ is piecewise C^1 smooth in the (x, v) -plane, and $|\partial p(x, v, t)/\partial x| \leq D'_1/L$. The discontinuity lines of $p(x, v, t)$ within the region $G_0(t)$ have slope of order $O(1/L)$ (for large L , they are almost parallel to the x axis).*

By the above lemma the quantities $Q_i, i = 0, 1, 2$, are smooth as functions of the piston position X , and

$$\left| \frac{\partial Q_i}{\partial X} \right| \leq \frac{\operatorname{const}}{L}, \tag{3.13}$$

where const is a global constant.

The following theorem gives a key technical estimate of this section.

Theorem 3.4. *For sufficiently large L there is a set $\Omega_0^* \subset \Omega$ of initial configurations of gas particles such that:*

(i) *for some constant $c > 0$*

$$P(\Omega_0^*) > 1 - L^{-c \log \log L},$$

(ii) *for each configuration $\omega \in \Omega_0^*$ and for each time interval*

$$(t, t + \Delta t) \subset (0, S_1)$$

such that

$$\frac{1}{L^2} < \Delta t < \frac{1}{L^{2/3} \log L}, \tag{3.14}$$

the change of the velocity of the piston during $(t, t + \Delta t)$ satisfies

$$V(t + \Delta) - V(t) = \mathcal{D}(t)\Delta t + \chi, \tag{3.15}$$

where

$$\mathcal{D}(t)/a = Q_0(t)V^2(t) - 2Q_1(t)V(t) + Q_2(t) \tag{3.16}$$

and

$$|\chi| \leq C \frac{\log L \sqrt{\Delta t}}{L} \tag{3.17}$$

with some constant $C > 0$.

Remark. The function $\mathcal{D}(t)/a$ in (3.16) is the random analogue of the quadratic polynomial (2.18). The term $\mathcal{D}(t) \Delta t$ in (3.15) is the main (‘deterministic’) component of the dynamics of the piston velocity. The term χ represents random fluctuations.

Proof. The set Ω_0^* consists of all configurations that satisfy certain requirements. We start with *preliminary requirements*.

Let us consider the discrete set of times $t_i = i/L^2$, where $i = 0, 1, \dots, I$ and $I = [T_{1,\max} L^2]$, and partition the domain G into the strips $S_j := \{(x, v) : j/L^2 \leq x < (j+1)/L^2\}$, where $j = 0, 1, \dots, L^3 - 1$. For each i and j we denote by $N_{i,j}$ the number of gas particles in the region $S_j \cap G_0(t_i)$ at time t_i . Our preliminary requirements are that

$$N_{i,j} \leq \log L \tag{3.18}$$

for all $0 \leq i \leq I$ and $0 \leq j < L^3$. We observe that $N_{i,j}$ is the number of gas particles in the region $F^{-t_i}(S_j \cap G_0(t_i))$ at time 0, and

$$F^{-t_i}(S_j \cap G_0(t_i)) \subset F_0^{-t_i}(S_j).$$

Thus, $N_{i,j}$ does not exceed the number of gas particles in $F_0^{-t_i}(S_j)$ at time 0; we denote the latter by $\tilde{N}_{i,j}$. Further, $\tilde{N}_{i,j}$ is a Poisson random variable whose parameter $\lambda_{i,j}$ is bounded:

$$\lambda_{i,j} \leq L^2 \pi_{\max} |S_j| \leq 2\pi_{\max}(v_{\max} - v_{\min}).$$

According to Corollary A.4, for each i, j our requirement (3.18) fails with probability $< L^{-d \log \log L}$ for some $d > 0$. The total number of pairs i, j is equal to $L^3 I = L^5 T_{1,\max} \leq L^6 / 2v_{\max}$. Hence, all our preliminary requirements hold with probability $> 1 - L^{-c' \log \log L}$ for some global constant $c' > 0$.

We now turn to the proof of (3.14)–(3.16). Let $i = [L^2 t]$ and $t_1 = (i+1)/L^2$, and note that $t_1 - t \leq L^{-2}$. One can easily derive from our preliminary requirements that the number of gas particles colliding with the piston on the time interval (t, t_1) is less than $\text{const} \cdot \log L$. Hence, the piston velocity V does not change by more than $\text{const} \cdot \log L / L^2$ during this interval. This amount is less than the bound for χ in (3.17) for all Δt satisfying (3.14). Therefore, we can ignore the interval (t, t_1) and assume that $t = t_1$. We note that the quantities Q_0, Q_1, Q_2 in (3.16) also change as the result of the substitution $t = t_1$, but only by an amount $< \text{const}/L^3$, in view of Lemma 3.3. This change can obviously be ignored, too. So, we suppose that $t = i/L^2$ for some $i = 0, 1, \dots, I$.

Now we state our *main requirements*. We again partition the x axis into intervals $S_j := \{j/L^2 \leq x < (j+1)/L^2\}$, where $j = 0, 1, \dots, L^3 - 1$. For each j we set $x_j = j/L^2$. For each integer p , $|p| \leq v_{\min} L^2$, we set $v_p = p/L^2$ and for each integer $1 \leq q \leq L^2$ we set $d_q = q/L^2$. For each triple (j, p, q) we now define two trapezoidal domains in the (x, v) -plane (see Fig. 4 below),

$$D_{j,p,q}^- = \left\{ (x, v) : \frac{v - v_p}{x - x_j} < -\frac{1}{d_q}, v_{\min} < v < v_{\max} \right\}, \tag{3.19}$$

$$D_{j,p,q}^+ = \left\{ (x, v) : \frac{v - v_p}{x - x_j} < -\frac{1}{d_q}, -v_{\max} < v < -v_{\min} \right\} \tag{3.20}$$

(here $-1/d_q$ is the slope of the oblique side of these trapezoids), and two strips

$$U_{j,p,q}^- = \{(x, v) : |x - x_j + d_q(v - v_p)| < 10v_{\max}/L^2, v_{\min} < v < v_{\max}\},$$

$$U_{j,p,q}^+ = \{(x, v) : |x - x_j + d_q(v - v_p)| < 10v_{\max}/L^2, -v_{\max} < v < -v_{\min}\}.$$

We note that $U_{j,p,q}^\pm$ are neighbourhoods of the oblique sides of the trapezoids $D_{j,p,q}^\pm$.

Let us consider all times $t_i = i/L^2$ for $i = 0, 1, \dots, I$. We denote by $N_{i,j,p,q}^\pm$ the number of particles in the region $F_0^{-t_i}(D_{j,p,q}^\pm)$ at time 0 and by $M_{i,j,p,q}^\pm$ the number of particles in the region $F_0^{-t_i}(U_{j,p,q}^\pm)$ at time 0. These are Poisson random variables. The parameter of the variable $N_{i,j,p,q}^\pm$ is

$$\lambda_{i,j,p,q}^\pm = \mathbf{E}(N_{i,j,p,q}^\pm) = L^2 \int_{F_0^{-t_i}(D_{j,p,q}^\pm)} p(x, v, 0) dx dv.$$

One can verify directly that the $\lambda_{i,j,p,q}^\pm$ are uniformly bounded below by a positive constant (even for the smallest d_p , that is, for $d_q = 1/L^2$), due to the assumption (1.8) on the initial density. The parameters of the variables $M_{i,j,p,q}^\pm$ are uniformly bounded above by

$$\mathbf{E}(M_{i,j,p,q}^\pm) < \pi_{\max}|U_{i,j,p,q}^\pm| < 20 v_{\max}(v_{\max} - v_{\min})\pi_{\max}.$$

Our main requirements are that

$$|N_{i,j,p,q}^\pm - \lambda_{i,j,p,q}^\pm| \leq \log L \sqrt{\lambda_{i,j,p,q}^\pm} \tag{3.21}$$

and

$$M_{i,j,p,q}^\pm \leq \log L. \tag{3.22}$$

By Lemma A.3 and Corollary A.4, the probability that at least one of these requirements fails is less than $L^{-d \log \log L}$ with some constant $d > 0$. The total number of quadruples (i, j, p, q) does not exceed $L^9 T_{1,\max} \leq L^{A''}$ with some fixed $A'' > 0$. Therefore, all our main requirements hold with probability $> 1 - L^{-c'' \log \log L}$ for some constant $c'' > 0$.

In addition, let

$$Z_{i,j,p,q}^\pm = \sum_{(x,v) \in F_0^{-t_i}(D_{j,p,q}^\pm)} v$$

(taken at time 0). This is an ‘integrated’ Poisson random variable, as defined in the Appendix. (Technically, we require there that the domain must be on one side of the x axis, and now it may happen that the region $F_0^{-t_i}(D_{j,p,q}^\pm)$ crosses the wall $x = 0$ or $x = L$ and then lies on both sides on the x axis; in that case we must replace t_i by a nearby time $t_{i'} < t_i$ in such a way that $F_0^{-t_{i'}}(D_{j,p,q}^\pm)$ lies entirely on one side of the x axis and define $Z_{i,j,p,q}^\pm$ at time $t_i - t_{i'}$ rather than at 0;

some obvious modifications are then needed, but we omit the details.) The estimates obtained in the Appendix yield

$$E(Z_{i,j,p,q}^\pm) = L^2 \int_{F_0^{-t_i}(D_{j,p,q}^\pm)} vp(x, v, 0) dx dv, \tag{3.23}$$

$$D(Z_{i,j,p,q}^\pm) = L^2 \int_{F_0^{-t_i}(D_{j,p,q}^\pm)} v^2 p(x, v, 0) dx dv. \tag{3.24}$$

Our last main requirement is that

$$|Z_{i,j,p,q}^\pm - E(Z_{i,j,p,q}^\pm)| \leq \log L \sqrt{D(Z_{i,j,p,q}^\pm)} \tag{3.25}$$

for all i, j, p, q . The probability of failure for these requirements is estimated exactly as above, by using Lemma A.6.

We now turn to the estimation of the piston velocity on the time interval $(t, t + \Delta t)$. We recall that $t = t_i$ for some $i = 0, 1, \dots, I$.

Velocity decomposition scheme. Here we obtain a general formula for the piston velocity; this formula is used below in the proof of several theorems. The laws of elastic collisions imply [19] that

$$V(t + \Delta t) = (1 - \varepsilon)^k V(t) + \varepsilon \sum_{j=1}^k (1 - \varepsilon)^{k-j} v_j. \tag{3.26}$$

Here k is the number of particles colliding with the piston during the time interval $(t, t + \Delta t)$, and the v_j are their velocities enumerated in the order in which the particles collide. The equation (3.26) can easily be verified by induction on k .

We modify the formula (3.26) as follows:

$$V(t + \Delta t) = (1 - \varepsilon k) V(t) + \varepsilon \sum_{j=1}^k v_j + \chi^{(1)} + \chi^{(2)}, \tag{3.27}$$

where

$$\begin{aligned} \chi^{(1)} &= V(t)[(1 - \varepsilon)^k - 1 + \varepsilon k], \\ \chi^{(2)} &= \varepsilon \sum_{j=1}^k v_j [(1 - \varepsilon)^{k-j} - 1]. \end{aligned}$$

Let us assume that the fluctuations of the velocity $V(s)$ on the interval $(t, t + \Delta t)$ are bounded,

$$\sup_{s \in (t, t + \Delta t)} |V(s) - V(t)| \leq \delta V. \tag{3.28}$$

We consider two regions in the (x, v) -plane,

$$D_1 = \left\{ (x, v) : \frac{v - V(t) - (\text{sgn } v)\delta V}{x - X(t)} < -\frac{1}{\Delta t}, v_{\min} < |v| < v_{\max} \right\}, \tag{3.29}$$

$$D_2 = \left\{ (x, v) : \frac{v - V(t) + (\text{sgn } v)\delta V}{x - X(t)} < -\frac{1}{\Delta t}, v_{\min} < |v| < v_{\max} \right\}. \tag{3.30}$$

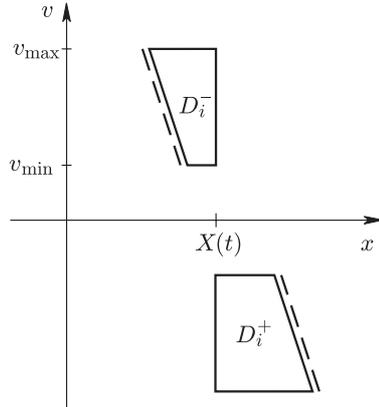


Figure 4. Region D_1 is bounded by solid lines. Region D_2 is bounded by a dashed line

Each of them is a union of two trapezoids $D_i = D_i^+ \cup D_i^-$, $i = 1, 2$, where D_i^- stands for the upper and D_i^+ for the lower trapezoid; see Fig. 4.

We note that $D_1 \subset D_2$. The estimate (3.28) implies that all the particles in the region D_1 necessarily collide with the piston during the time interval $(t, t + \Delta t)$. Moreover, the trajectory of every point $(x, v) \in D_1$ hits the piston within a time less than Δt , and hence $D_1 \subset G_0(t)$ by the condition (b) in the definition of the slow interval $(0, S_1)$. The estimate (3.28) also implies that all the particles actually colliding with the piston during the interval $(t, t + \Delta t)$ are contained in D_2 (though it is not necessarily true that $D_2 \subset G_0(t)$).

We now obtain an upper bound for k . Since the velocities of the gas particles and the piston are bounded by v_{\max} , all the particles colliding with the piston during the interval $(t, t + \Delta t)$ are contained in the region $S' \cap G_0(t)$, where

$$S' = \{(x, v) : |x - X(t)| < 2v_{\max}\Delta t\}.$$

Hence, k does not exceed the number of particles in the region $F_0^{-t}(S')$ at time zero. Our preliminary requirements imply that

$$k \leq 2v_{\max}\Delta t L^2 \log L. \tag{3.31}$$

This allows us to bound the quantities $\chi^{(1)}$ and $\chi^{(2)}$ for large L as follows:

$$|\chi^{(u)}| \leq v_{\max}\varepsilon^2 k^2 \leq 4a^2 v_{\max}^3 (\log L)^2 (\Delta t)^2 \tag{3.32}$$

for $u = 1, 2$. It also follows from (3.27) that

$$\delta V \leq 5av_{\max}\Delta t \log L. \tag{3.33}$$

We denote by k_r^\pm the number of particles in the regions $D_r^\pm \cap G_0(t)$ for $r = 1, 2$ at time $t = t_i$. We also denote by k^- the number of particles colliding with the piston ‘on the left’, and by k^+ that ‘on the right’ (of course, $k^- + k^+ = k$). Clearly,

$$k_1^\pm \leq k^\pm \leq k_2^\pm. \tag{3.34}$$

Since $D_1 \subset G_0(t)$, k_1^\pm is equal to the number of particles in the region $F_0^{-t}(D_1^\pm)$ at time 0. Similarly, k_2^\pm is equal to the number of particles in $F_0^{-t}(D_2^\pm \cap G_0(t))$ at time 0.

The trapezoids D_r^\pm , $r = 1, 2$, can be well approximated by some trapezoids $D_{j,p,q}^\pm$ defined earlier in our main requirements. In fact, the horizontal sides $v = \pm v_{\min}$ and $v = \pm v_{\max}$ are common for all trapezoids, and only the vertical side $x = X(t)$ and the oblique side need an approximation. The symmetric difference between D_r^\pm and $D_{j,p,q}^\pm$ lies inside some strips S_j and $U_{j,p,q}^\pm$ (also defined above). Then our main requirements guarantee that

$$k_1^\pm \leq \lambda_1^\pm - \log L \sqrt{\lambda_1^\pm} \quad \text{and} \quad k_2^\pm \leq \lambda_2^\pm + \log L \sqrt{\lambda_2^\pm},$$

where

$$\begin{aligned} \lambda_1^\pm &= L^2 \int_{F_0^{-t}(D_1^\pm)} p(x, v, 0) \, dx \, dv = L^2 \int_{D_1^\pm} p(x, v, t) \, dx \, dv, \\ \lambda_2^\pm &= L^2 \int_{F_0^{-t}(D_2^\pm \cap G_0(t))} p(x, v, 0) \, dx \, dv \leq \lambda_1^\pm + L^2 \pi_{\max} |D_2^\pm \setminus D_1^\pm|; \end{aligned}$$

here we used the boundedness of the density (1.9). The area of the domain $D_2^\pm \setminus D_1^\pm$ is bounded:

$$|D_2^\pm \setminus D_1^\pm| < 4v_{\max} \delta V \Delta t.$$

Moreover, (1.9) implies that

$$\lambda_1^\pm \leq L^2 \pi_{\max} |D_1^\pm| \leq 2\pi_{\max} v_{\max}^2 L^2 \Delta t.$$

Combining the above estimates gives

$$|k^\pm - \lambda_1^\pm| \leq \text{const} \cdot (L \log L \sqrt{\Delta t} + L^2 \delta V \Delta t). \tag{3.35}$$

We now turn to the quantity

$$Z = \sum_{j=1}^k v,$$

which is also involved in the main equation (3.27). We can again decompose $Z = Z^- + Z^+$, where Z^- and Z^+ stand for the sums of velocities of the particles colliding with the piston on the left and on the right, respectively. We set

$$Z_1^\pm = \sum_{(x,v) \in D_1^\pm} v$$

(taken at time t). An analysis similar to that above and the requirement (3.25), together with the formulae (3.23)–(3.24), give

$$|Z^\pm - \mathbb{E}(Z_1^\pm)| \leq \text{const} \cdot (L \log L \sqrt{\Delta t} + L^2 \delta V \Delta t). \tag{3.36}$$

We now combine (3.27) with the subsequent estimates and obtain

$$V(t + \Delta) - V(t) = -\varepsilon\lambda_1 V(t) + \varepsilon E(Z_1) + \chi^{(3)}, \tag{3.37}$$

where

$$\begin{aligned} \lambda_1 &= \lambda_1^+ + \lambda_1^- = L^2 \int_{D_1} p(x, v, t) dx dv, \\ E(Z_1) &= E(Z_1^+) + E(Z_1^-) = L^2 \int_{D_1} v p(x, v, t) dx dv, \\ |\chi^{(3)}| &\leq \text{const} \cdot [L^{-1} \log L \sqrt{\Delta t} + \delta V \Delta t + (\log L)^2 (\Delta t)^2]. \end{aligned}$$

Using Lemma 3.3 (as we already did in deriving (3.13)) to bound possible fluctuations of the density $p(x, v, t)$ within D_1 gives

$$\int_{D_1} p(x, v, t) dx dv = (Q_1(t) - Q_0(t))V(t)\Delta t + \chi^{(4)}, \tag{3.38}$$

$$\int_{D_1} v p(x, v, t) dx dv = (Q_2(t) - Q_1(t))V(t)\Delta t + \chi^{(5)}, \tag{3.39}$$

where

$$|\chi^{(u)}| \leq \text{const} \cdot [(\Delta t)^2/L + \delta V \Delta t] \tag{3.40}$$

for $u = 4, 5$.

Therefore, we obtain

$$V(t + \Delta t) - V(t) = \mathcal{D}(t)\Delta t + \chi, \tag{3.41}$$

where

$$|\chi| \leq \text{const} \cdot [L^{-1} \log L \sqrt{\Delta t} + \delta V \Delta t + (\log L)^2 (\Delta t)^2]. \tag{3.42}$$

By using (3.14) and (3.33), it is easy to incorporate the second and third terms in (3.42) into the first. Theorem 3.4 is proved. \square

For the next theorem we rewrite (P5) in terms of the microscopic coordinates,

$$|p(x, v, 0) - p(L - x, -v, 0)| < \varepsilon_0. \tag{3.43}$$

We also assume that the initial velocity $V(0)$ of the piston is set to be $W(0)$ rather than 0; see the remark in §2 and another remark below.

Theorem 3.5. *If $\varepsilon_0 > 0$ is small enough, then for all sufficiently large L , for each configuration $\omega \in \Omega_0^*$, and for all $t \in (0, S_1)$*

(i) *there is a constant $B > 0$ such that*

$$|V(t)| < B\varepsilon_0, \tag{3.44}$$

(ii) *there is a constant $C_0 > 0$ such that*

$$|V(t) - V_0(t)| < \frac{C_0(\log L)^{3/2}}{L^{2/3}}, \tag{3.45}$$

where $V_0(t)$ is defined by

$$V_0(t) = \frac{Q_1(t) - \sqrt{Q_1^2(t) - Q_0(t)Q_2(t)}}{Q_0(t)} \tag{3.46}$$

whenever $Q_0(t) \neq 0$ and by

$$V_0(t) = \frac{Q_2(t)}{2Q_1(t)} \tag{3.47}$$

otherwise.

Proof. We start with the proof of (i) and obtain (ii) as a ‘side result’.

Assume that (i) is false, and let $t_* < S_1$ be the first time (3.44) fails, that is, let

$$|V(t_* + 0)| \geq B\varepsilon_0. \tag{3.48}$$

Since (3.44) holds for $t < t_*$, the piston’s position satisfies

$$|X(t) - L/2| \leq B\varepsilon_0 t \tag{3.49}$$

for all $t < t_*$. We assume for the moment that the piston did not move at all, in other words, $X(s) = L/2$ for all $0 < s < t$. Then by (3.43) the density $q(v, t)$ of the gas ‘on the piston’ would be almost symmetric, that is,

$$|q(v, t) - q(-v, t)| < \varepsilon_0.$$

Hence, we would have

$$|Q_i(t)| < C_1\varepsilon_0, \quad i = 0, 2, \tag{3.50}$$

with $C_1 = v_{\max}^3$. We emphasize that C_1 does not depend on the choice of B in (3.44). Below we introduce some other constants C_i , $i \geq 2$, none depending on B .

If the piston actually moves and covers the distance $X(t) - L/2$, then (3.13) and (3.50) imply that

$$|Q_i(t)| < C_1\varepsilon_0 + C_2L^{-1}|X(t) - L/2|, \quad i = 0, 2, \tag{3.51}$$

with some constant $C_2 > 0$. At the same time, $Q_1(t)$ stays bounded above and below by positive constants for all $t < t_*$,

$$0 < Q_{1,\min} \leq Q_1(t) \leq Q_{1,\max} < \infty, \tag{3.52}$$

where $Q_{1,\min}$ and $Q_{1,\max}$ are constants determined by π_{\min} and π_{\max} in (P3).

By (3.49), (3.51), and (3.8) we have

$$|Q_i(t)| < (C_1 + Bv_{\max}^{-1})\varepsilon_0, \quad i = 0, 2, \tag{3.53}$$

for all $t < t_*$. This and (3.52) imply that the quadratic polynomial (3.16) has two real roots, and the smaller one (the root closer to zero) is given by (3.46)–(3.47), cf. (2.24) and (2.25) in §2. Due to (2.22) and (3.52) we have

$$|V_0(t)| < C_3\varepsilon_0 + C_4L^{-1}|X(t) - L/2| \tag{3.54}$$

for all $t < t_*$ and for some constants $C_3, C_4 > 0$ (independent of the choice of B).

We note that (3.54) and (3.8) imply the boundedness of $V_0(t)$:

$$|V_0(t)| \leq \text{const} = C_3\varepsilon_0 + C_4v_{\max}^{-1} \tag{3.55}$$

for all $t < t_*$.

Next, we must estimate the derivative $dV_0(t)/dt$. (Note that the function $V_0(t)$ defined by (3.46)–(3.47) is continuous and piecewise differentiable, and it should not be confused with the piston velocity $V(t)$, which is piecewise constant and hence not even continuous). By (3.13),

$$\left| \frac{dQ_i(t)}{dt} \right| = \left| \frac{dQ_i(t)}{dX} \cdot V(t) \right| \leq \frac{\text{const} \cdot \varepsilon_0}{L} \tag{3.56}$$

for $i = 0, 1, 2$. Differentiating the quadratic equation

$$Q_0(t)V_0^2(t) - 2Q_1(t)V_0(t) + Q_2(t) = 0$$

with respect to t gives

$$\frac{dV_0(t)}{dt} = \frac{(dQ_0/dt)V_0^2 - 2(dQ_1/dt)V_0 + (dQ_2/dt)}{2(Q_1 - Q_0V_0)}. \tag{3.57}$$

In view of (2.22), (3.52), (3.55), and (3.56) we have

$$\left| \frac{dV_0(t)}{dt} \right| \leq \frac{E_0\varepsilon_0}{L}, \tag{3.58}$$

where $E_0 > 0$ is a constant.

Let us now consider the quantity \mathcal{D} defined by (3.16) as a function of V (with fixed Q_i , $i = 0, 1, 2$). Its derivative is

$$\frac{\partial \mathcal{D}}{\partial V} = 2a[Q_0V - Q_1].$$

By (3.52) and (3.53), there are positive constants $0 < E_1 < E_2$ such that for all $t < t_*$

$$-E_2 < \frac{\partial \mathcal{D}}{\partial V} < -E_1.$$

Therefore, by the mean value theorem, for all $t < t_*$

$$-E_2 < \frac{\mathcal{D}(t)}{V(t) - V_0(t)} < -E_1. \tag{3.59}$$

Let us now prove (3.45) for all $t < t_*$ with some constant $C_0 > 0$ independent of the choice of B in (3.44). We recall that the initial velocity of the piston was set to be $V(0) = W(0)$ and that $W(0) = V_0(0)$; see §2. Arguing by contradiction, let $t \in (0, t_*)$ be the first time (3.45) fails. We denote by

$$\Delta_0 = \frac{1}{L^{3/2} \log L}$$

the maximal allowed time increment in Theorem 3.4. Let $s = t - \Delta_0$. From Theorem 3.4 we obtain

$$V(t) = V(s) + \mathcal{D}(s)\Delta_0 + \chi, \tag{3.60}$$

where

$$|\chi| \leq \frac{C\sqrt{\Delta_0} \log L}{L} = \frac{C\sqrt{\log L}}{L^{4/3}}.$$

Due to (3.58) we have

$$V_0(t) = V_0(s) + \chi_0 \tag{3.61}$$

with

$$|\chi_0| \leq \frac{E_0\varepsilon_0\Delta_0}{L} = \frac{E_0\varepsilon_0}{L^{5/3} \log L}.$$

For brevity, we set $U(s) = V(s) - V_0(s)$ for all s . Subtracting (3.61) from (3.60) gives

$$U(t) = U(s) + \mathcal{D}(s)\Delta_0 + \chi' \tag{3.62}$$

with $\chi' = \chi - \chi_0$, and hence

$$|\chi'| \leq \frac{2C\sqrt{\log L}}{L^{4/3}} \tag{3.63}$$

for large L . Without loss of generality we can assume that $U(t) > 0$. Since (3.45) fails at time t , we have

$$U(t) \geq \frac{C_0(\log L)^{3/2}}{L^{2/3}}. \tag{3.64}$$

Let us consider two cases. If $U(s) \leq 0$, then by (3.59)

$$U(t) \leq |\mathcal{D}(s)|\Delta_0 + |\chi'| \leq E_2|U(s)|\Delta_0 + |\chi'| \ll \frac{(\log L)^{3/2}}{L^{2/3}}$$

for large L , which contradicts (3.64). If $U(s) > 0$, then again by (3.62) and (3.59),

$$U(t) < U(s)[1 - E_1\Delta_0] + \chi',$$

and therefore

$$U(s) > \frac{U(t) - \chi'}{1 - E_1\Delta_0} > (U(t) - \chi')(1 + E_1\Delta_0) > U(t) + U(t)E_1\Delta_0 - 2\chi'. \tag{3.65}$$

If C_0 in (3.45) is large enough, say, $C_0 = 5C/E_1$, then $U(t)E_1\Delta_0 > 2\chi'$ by (3.64) and (3.63). This fact and (3.65) imply that $U(s) > U(t)$, and (3.45) fails at an earlier time $s < t$, a contradiction. Hence, (3.45) is proved for all $t < t_*$ and $C_0 = 5C/E_1$.

Combining (3.54) and (3.45) for large L and all $t < t_*$, we see that

$$|dX(t)/dt| < 2C_3\varepsilon_0 + C_4L^{-1}|X(t) - L/2|.$$

Use of the standard Gronwall inequality in differential equations (see, for example, Lemma 2.1 in [26]) gives

$$|X(t) - L/2| < 2\varepsilon_0 C_3 C_4^{-1} L (e^{C_4 L^{-1} t} - 1)$$

and

$$|V(t)| < 2\varepsilon_0 C_3 e^{C_4 L^{-1} t}$$

for all $t < t_*$. By (3.8),

$$|V(t)| < 2\varepsilon_0 C_3 e^{C_4 v_{\max}^{-1} t}. \tag{3.66}$$

We now choose $B = 3C_3 e^{C_4 v_{\max}^{-1}}$. Clearly, (3.66) then contradicts (3.38). This completes the proof of (3.44) for all $t < S_1$. Theorem 3.5 is now proved. \square

Corollary 3.6. *If $\varepsilon_0 > 0$ in (1.11) is small enough, then $S_1 = T_1$ for all large L and all $\omega \in \Omega_0^*$, that is, the previous theorems hold on the entire zero-recollision interval $(0, T_1)$.*

Proof. We recall that the slow interval $(0, S_1) \subset (0, T_1)$ is defined by two conditions, (a) and (b). If $S_1 < T_1$, then either (a) or (b) fails at S_1 . Clearly, (a) cannot fail ‘abruptly’, since (3.44) holds for all $t < S_1$.

Suppose that (b) fails at some $s < T_1$, while (a) still holds. The failure of (b) means that at time s the piston ‘collides’ with a point (x, v) such that $v_{\min} < |v| < v_{\max}$ and $(x, v) \notin G_0(s)$. Therefore, the backward trajectory $F^{-(s-t)}(x, v)$, $0 < t < s$, of the point (x, v) hits the piston at some time $t > 0$. During the time interval $(0, t)$ the piston covers the distance $|X(t) - L/2| \leq B\varepsilon_0 t$, and during the time interval (t, s) the trajectory of our point covers the distance $|v|(s-t) < v_{\max}(s-t)$. Hence,

$$\begin{aligned} L &\leq |X(t) - L/2| + |v|(s-t) + |X(s) - L/2| \\ &\leq |X(s) - L/2| + v_{\max}s - (v_{\max} - B\varepsilon_0)t. \end{aligned}$$

On the other hand, since $s < T_1$, we have

$$L > |X(s) - L/2| + v_{\max}s;$$

see the proof of Lemma 3.2. This contradiction shows that (b) cannot fail either. The proof of Corollary 3.6 is complete. \square

Remark. We have reset the initial velocity of the piston to $V(0) = W(0)$ here, while in §1 it was set to zero. If $V(0) = 0$, then Theorem 3.4 would imply that $V(t)$ converges to $V_0(t)$ exponentially fast in t until $V(t)$ becomes δ -close to $V_0(t)$ with $\delta = C_0(\log L)^{3/2}L^{-2/3}$. After that, all our results apply without change. The initial interval on which the convergence takes place is of order $\log L$, and in the hydrodynamic time it is $L^{-1} \log L$, which vanishes as $L \rightarrow \infty$. This is why we simply opted for the most convenient setting $V(0) = W(0)$ here.

The following theorem improves the results of Theorems 3.4 and 3.5.

Theorem 3.7. *If $\varepsilon_0 > 0$ in (1.11) is small enough, then there is a constant $C > 0$ such that*

$$|V(t) - V_0(t)| < \frac{C \log L}{L} \tag{3.67}$$

for all large L , all $\omega \in \Omega_0^*$, and all $t < T_1$, and

$$|V(t + \Delta t) - V(t)| < C \frac{\log L \sqrt{\Delta t}}{L} \tag{3.68}$$

for any time interval $(t, t + \Delta t) \subset (0, T_1)$ such that $L^{-2} < \Delta t \leq 1$.

Proof. Due to our choice of the initial velocity we have $V(0) = V_0(0)$, and hence (3.67) holds for at least small t . Assume that (3.67) fails at some $t_* < T_1$, and t_* is the earliest time of failure. Without loss of generality we can assume that

$$V(t_*) - V_0(t_*) \geq \frac{C \log L}{L}. \tag{3.69}$$

Let $0 < t_0 < t_*$ be the latest time when

$$V(t_0) - V_0(t_0) \leq \frac{C \log L}{2L}. \tag{3.70}$$

Then

$$V(t_0) - V_0(t_0) \leq V(t) - V_0(t) \leq V(t_*) - V_0(t_*) \tag{3.71}$$

for all $t \in (t_0, t_*)$. Let

$$\Delta t = \min\{1, t_* - t_0\}. \tag{3.72}$$

Let us analyze the dynamics of the piston during the time interval $(t_0, t_0 + \Delta t)$. By (3.58),

$$|V_0(t) - V_0(t_0)| < \delta V := E_0 \varepsilon_0 L^{-1} \Delta t \tag{3.73}$$

for all $t \in (t_0, t_0 + \Delta t)$, and hence (3.71) implies that

$$V(t) > V(t_0) - \delta V. \tag{3.74}$$

We note that Δt is not too small, namely, $\Delta t > (L^{2/3} \log L)^{-1}$. Indeed, otherwise we would have $t_* = t_0 + \Delta t$, and then (3.69), (3.70), and (3.73) would imply that

$$V(t_*) - V(t_0) \geq 2^{-1} C L^{-1} \log L - E_0 \varepsilon_0 L^{-5/3},$$

which would contradict Theorem 3.4, since $\mathcal{D}(t_0) < 0$ (because $V(t_0) > V_0(t_0)$).

Next, we develop a generalized version of the velocity decomposition (3.27) in the proof of Theorem 3.4. We partition the interval $(t_0, t_0 + \Delta t)$ into subintervals of length δ (to be chosen shortly) with endpoints $t_i = t_0 + i\delta$, $i = 0, 1, \dots, I$, where $I = \Delta t/\delta$. We select a δ for which

$$\frac{0.5}{L \log L} < \delta < \frac{1}{L \log L} \tag{3.75}$$

and $\Delta t/\delta$ is an integer (for convenience). The preliminary requirements in the proof of Theorem 3.4 enable us to adjust the time in such a way that $L^2 t_0$ and $L^2 \delta$ are integers, and hence $L^2 t_i$ is an integer for every i . The velocity decomposition in the proof of Theorem 3.4 now applies to each subinterval (t_i, t_{i+1}) of length δ . In particular, (3.27) implies that

$$V(t_{i+1}) - V(t_i) = -\varepsilon k_i V(t_i) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)}, \tag{3.76}$$

where k_i is the number of particles colliding with the piston during the time interval (t_i, t_{i+1}) and $v_j, 1 \leq j \leq k_i$, are their velocities. The fluctuation term $\chi_i^{(1)}$ can be bounded by the expression in (3.32):

$$|\chi_i^{(1)}| \leq 8a^2 v_{\max}^3 (\log L)^2 \delta^2. \tag{3.77}$$

In view of (3.74), the decomposition (3.76) can be rewritten as

$$V(t_{i+1}) - V(t_i) \leq -\varepsilon k_i V(t_i) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)} + \chi_i^{(2)} \tag{3.78}$$

with

$$|\chi_i^{(2)}| \leq \varepsilon k_i \delta V \leq 2av_{\max} E_0 \varepsilon_0 L^{-1} \log L \Delta t \delta,$$

where at the last step we used (3.31) and (3.73). Summing (3.78) over i yields

$$V(t_0 + \Delta t) - V(t_0) \leq -\varepsilon k V(t_0) + \varepsilon \sum_{j=1}^k v_j + \chi^{(3)}, \tag{3.79}$$

where k is the number of particles colliding with the piston during the time interval $(t_0, t_0 + \Delta t)$ and $v_j, 1 \leq j \leq k$, are their velocities, and we have

$$\begin{aligned} |\chi^{(3)}| &\leq 8a^2 v_{\max}^3 (\log L)^2 \delta \Delta t + 2av_{\max} E_0 \varepsilon_0 L^{-1} \log L (\Delta t)^2 \\ &\leq (8a^2 v_{\max}^3 + 2av_{\max} E_0 \varepsilon_0) L^{-1} \log L (\Delta t) \end{aligned} \tag{3.80}$$

(where we used (3.73) and the assumption that $\Delta t \leq 1$).

The decomposition (3.79) can be analyzed like (3.27) in the proof of Theorem 3.4. We define a region in the (x, v) -plane:

$$D_1 = \left\{ (x, v) : \frac{v - V(t_0) + \delta V}{x - X(t_0)} < -\frac{1}{\Delta t}, v_{\min} < |v| < v_{\max} \right\}. \tag{3.81}$$

It is the union of two trapezoids $D_1 = D_1^+ \cup D_1^-$, where D_1^- stands for the upper and D_1^+ for the lower one; see Fig. 4. The bound (3.74) implies that all the particles in the region D_1^+ necessarily collide with the piston during the time interval $(t_0, t_0 + \Delta t)$ and all the particles actually colliding with the piston on its left-hand side during this time interval are contained in D_1^- .

Since $v > V(t_0)$ for all particles $(x, v) \in D_1^-$ and $v < V(t_0)$ for all $(x, v) \in D_1^+$, we can remove from (3.79) the particles that do not belong in D_1^+ and simultaneously add to (3.79) the particles that belong in D_1^- but do not collide with the piston. This modification only makes the right-hand side of (3.79) larger, and therefore

$$V(t_0 + \Delta t) - V(t_0) \leq -\varepsilon k_1 V(t_0) + \varepsilon \sum_{j=1}^{k_1} v_j + \chi^{(3)}, \tag{3.82}$$

where k_1 is the number of particles in D_1 at time t_0 , and the summation runs over all those particles. Let $Z_1 = \sum_{(x,v) \in D_1} v$. Just as in the proof of Theorem 3.4, our main requirements stated there guarantee that

$$|k_1 - \mathbf{E}(k_1)| \leq c_3 L \log L \sqrt{\Delta t}$$

and

$$|Z_1 - \mathbf{E}(Z_1)| \leq c_4 L \log L \sqrt{\Delta t},$$

where the constants $c_3, c_4 > 0$ do not depend on the choice of C in (3.67), which we have not made yet. Computing the mean values of k_1 and Z_1 as in the proof of Theorem 3.4, we now arrive at the inequality

$$V(t_0 + \Delta t) - V(t_0) \leq \mathcal{D}(t_0)\Delta t + \chi^{(3)} + \chi^{(4)} \tag{3.83}$$

with

$$|\chi^{(4)}| \leq a(c_3 + c_4)L^{-1} \log L \sqrt{\Delta t} + c_5 \delta V \Delta t, \tag{3.84}$$

where $c_5 > 0$ is a constant independent of the choice of C in (3.67). The last term in (3.84) comes from the adjustment δV to the velocity $V(t_0)$ in (3.81). This term is bounded by $c_5 E_0 \varepsilon_0 L^{-1} (\Delta t)^2$ and can be incorporated into the first term in (3.84), since $\Delta t \leq 1$. We recall that $V(t_0) - V_0(t_0) \approx 2^{-1} C L^{-1} \log L > 0$ (here we have an approximation up to a quantity of order $1/L^2$, since the piston velocity changes by $O(1/L^2)$ at each collision). Then by (3.59),

$$\mathcal{D}(t_0) \leq -E_1(V(t_0) - V_0(t_0)) \approx 2^{-1} C E_1 L^{-1} \log L.$$

Therefore, combining the above estimates gives

$$V(t_0 + \Delta t) - V(t_0) \leq 2^{-1} C E_1 L^{-1} \log L \Delta t + \chi^{(5)} \tag{3.85}$$

with $\chi^{(5)} = \chi^{(3)} + \chi^{(4)}$ bounded in view of (3.80) and (3.84):

$$|\chi^{(5)}| \leq c_6 (L^{-1} \log L \Delta t + L^{-1} \log L \sqrt{\Delta t}),$$

where $c_6 > 0$ is a constant independent of the choice of C in (3.67). We now choose the constant C there as

$$C = \max\{c_6, 6E_1^{-1}c_6\}. \tag{3.86}$$

Then (3.85) implies that

$$V(t_0 + \Delta t) - V(t_0) \leq c_6 L^{-1} \log L (-2\Delta t + \sqrt{\Delta t}),$$

and therefore, due to (3.73),

$$V(t_0 + \Delta t) - V_0(t_0 + \Delta t) \leq V(t_0) - V_0(t_0) + c_6 L^{-1} \log L (-2\Delta t + \sqrt{\Delta t}) + E_0 \varepsilon_0 L^{-1} \Delta t. \tag{3.87}$$

We now have two cases. First, let $t_* - t_0 \leq 1$, and hence $\Delta t \leq 1$. The expression $-2\Delta t + \sqrt{\Delta t}$ has a maximum equal to $1/8$ at the point $\Delta t = 1/16$. Therefore, (3.87) implies that

$$V(t_*) - V_0(t_*) < V(t_0) - V_0(t_0) + 8^{-1} c_6 L^{-1} \log L + E_0 \varepsilon_0 L^{-1}.$$

This contradicts (3.69) and (3.70) when L is large enough (recall our choice of C in (3.86)). Let us consider the other case, $t_* - t_0 > 1$. Then $\Delta t = 1$, and (3.87) implies for large L that

$$V(t_0 + 1) - V_0(t_0 + 1) < V(t_0) - V_0(t_0) - 2^{-1} c_6 L^{-1} \log L,$$

which contradicts (3.71). This completes the proof of (3.67).

We now prove (3.68). If $\Delta t < (L^{2/3} \log L)^{-1}$, then we can use Theorem 3.4:

$$|V(t + \Delta t) - V(t)| \leq |\mathcal{D}(t)| \Delta t + CL^{-1} \log L \sqrt{\Delta t}.$$

The above estimates (3.59) and (3.67) imply that

$$|\mathcal{D}(t)| \leq E_2 |V(t) - V_0(t)| \leq CE_2 L^{-1} \log L, \tag{3.88}$$

so that (3.68) follows (with a larger value of C than that above).

Now let $(L^{2/3} \log L)^{-1} \leq \Delta t \leq 1$. Without loss of generality we can assume that $V(t + \Delta t) > V(t)$ and

$$V(s) > V(t) \quad \text{for } s \in (t, t + \Delta t). \tag{3.89}$$

Indeed, if this is not the case, then we can replace t by $t' = \max\{s < t + \Delta t : V(s) \leq V(t)\}$ and prove (3.68) for the smaller interval $(t', t + \Delta t)$.

Next, our plan is to apply some estimates from the proof of (3.67) and then argue along the lines of the proof of Theorem 3.4. We write $t_0 = t$ and partition the interval $(t_0, t_0 + \Delta t)$ into subintervals of length δ satisfying (3.75). Then we again have the decompositions (3.76)–(3.77). Due to (3.89) we have $V(t_i) > V(t_0)$ for all i , and hence (3.76) implies that

$$V(t_{i+1}) - V(t_i) \leq -\varepsilon k_i V(t_0) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)}.$$

Summing this over i gives (3.79) with the bound (3.80), in which the second term can simply be removed since we no longer have $\chi_i^{(2)}$. Next, the possible fluctuations

of the piston velocity $V(s)$ during the time interval $(t_0, t_0 + \Delta t)$ can be estimated with the help of (3.67) and (3.73):

$$|V(s) - V(t_0)| \leq \delta V := 2CL^{-1} \log L + E_0 \varepsilon_0 L^{-1} \Delta t$$

for all $s \in (t_0, t_0 + \Delta t)$. Then we estimate the random variables k and $Z = \sum_j v_j$ as in the proof of Theorem 3.4, starting with the construction of two domains D_1 and D_2 by using (3.29)–(3.30), and so on. Repeating the argument almost word by word, we arrive at an analogue of (3.41):

$$V(t + \Delta t) - V(t) < \mathcal{D}(t)\Delta t + \chi'$$

with

$$|\chi'| \leq \text{const} \cdot [L^{-1} \log L \sqrt{\Delta t} + \delta V \Delta t + L^{-1} \log L \Delta t],$$

where the last term comes from (3.80), which we now have instead of (3.32) (we note that (3.32) would no longer be nearly enough, since Δt is large; this is why we needed to partition the interval $(t, t + \Delta t)$ into smaller subintervals). We combine the above estimates with (3.88) and complete the proof of (3.68) and of Theorem 3.7. \square

Finally, we prove the convergence as $L \rightarrow \infty$ of the random trajectory of the piston to the solution $Y(\tau), W(\tau)$ of the hydrodynamic equations found in §2.

Theorem 3.8. *If $\varepsilon_0 > 0$ in (1.11) is small enough, then for all large L and all $\omega \in \Omega_0^*$ there is a constant $C > 0$ such that*

$$|Y_L(\tau, \omega) - Y(\tau)| \leq \frac{C \log L}{L}, \tag{3.90}$$

$$|W_L(\tau, \omega) - W(\tau)| \leq \frac{C \log L}{L} \tag{3.91}$$

for all $0 < \tau < \min\{\tau_1, T_1/L\}$, and

$$|\tau_1 - T_1/L| \leq \frac{C \log L}{L}. \tag{3.92}$$

Proof. In §2 we defined a function $F(Y, \tau)$ such that the hydrodynamic solution $Y(\tau)$ satisfies

$$dY(\tau)/d\tau = F(Y, \tau), \quad Y(0) = 1/2 \tag{3.93}$$

(see (2.29)). Theorem 3.7 implies that for all $\omega \in \Omega_0^*$ the random trajectory satisfies

$$\partial Y_L(\tau, \omega)/\partial \tau = F(Y, \tau) + \chi(\tau, \omega), \quad Y_L(0, \omega) = 1/2, \tag{3.94}$$

where

$$|\chi(\tau, \omega)| \leq \frac{C \log L}{L}.$$

We recall that $|\partial F(Y, \tau)/\partial Y| \leq \kappa$; see (2.30). Therefore, the difference $Z_L(\tau, \omega) := Y_L(\tau, \omega) - Y(\tau)$ satisfies

$$|Z'_L(\tau, \omega)| \leq \kappa |Z_L(\tau, \omega)| + \frac{C \log L}{L}$$

and $Z_L(0, \omega) = 0$. By the standard Gronwall inequality in differential equations (see, for example, Lemma 2.1 in [26]), we have

$$\begin{aligned} |Z_L(\tau, \omega)| &\leq \frac{C \log L}{\kappa L} (e^{\kappa\tau} - 1), \\ |Z'_L(\tau, \omega)| &\leq \frac{C \log L}{L} e^{\kappa\tau} \end{aligned}$$

for all $\tau < \min\{\tau_1, T_1/L\}$, which implies (3.90) and (3.91).

Lastly, let us verify (3.92). By (3.91), the random fluctuations of the piston velocities are bounded by $CL^{-1} \log L$. Hence, the random fluctuations of the velocities of particles that have had one collision with the piston are bounded by $2CL^{-1} \log L$. The random fluctuations of the positions of both the piston and particles at every moment of time $t < \min\{\tau_1 L, T_1\}$ are bounded by the same quantities (possibly with a different value of C) in the coordinate $y = x/L$. On the other hand, the relative velocity of the piston and the particles stays bounded away from zero (say, by $v_{\min} - 4B\varepsilon_0 > 0$). Hence, the time T_1/L of the first recollision can differ from τ_1 by at most $\text{const} \cdot L^{-1} \log L$. Theorem 3.8 is proved. \square

§ 4. Dynamics between the first and second recollisions

In this section we study the one-recollision interval (τ_1, τ_2) on which gas particles experience the second collision (that is, the first *recollision*) with the piston.

The particles that have collided with the piston no longer determine a Poisson process, and hence their distribution is much harder to control. This is our main trouble. On the other hand, we shall be satisfied with much weaker estimates than those in the previous section. Moreover, many arguments and constructions in this section are similar to those in § 3, and we omit some details and focus on new ideas.

Here our analysis is always restricted to the configurations $\omega \in \Omega_0^*$. Later we shall impose additional requirements on ω .

We recall that for $\omega \in \Omega_0^*$ the piston velocity is small, $|V(t)| < B\varepsilon_0$, on the zero recollision interval $(0, T_1)$; see (3.44). Hence, the velocities of gas particles that have experienced one collision with the piston on the interval $(0, T_1)$ are bounded,

$$v_{1,\min} < |v| < v_{1,\max}, \tag{4.1}$$

where

$$v_{1,\min} := v_{\min} - 2B\varepsilon_0 \quad \text{and} \quad v_{1,\max} := v_{\max} + 2B\varepsilon_0. \tag{4.2}$$

The earliest time $T_2 = T_2(\omega)$ of a second recollision is defined by $T_2 = \sup_{t>0} \{G_3^+ = \emptyset\}$; see (3.7). From (3.44) and (4.1) the following bound can easily be obtained as in the proof of Lemma 3.2:

$$T_2 \leq \frac{L}{v_{\max}} + \frac{L}{v_{\max} - 2B\varepsilon_0} \leq \frac{3L}{v_{\max}}. \tag{4.3}$$

Suppose that $(x, v) \in G^+$ and let $(x_t, v_t) = F^t(x, v)$ for $t > 0$. We denote by

$$s_1(x, v) = \min\{t : x_t = X(t)\}$$

the time of the first collision with the piston. The region

$$G_*^+(t) := \{(x_t, v_t) \in G_1^+(t) : s_1(x, v) < T_1\}$$

is occupied by points that by time t have experienced one collision with the piston before time T_1 . By removing the superscript $+$ in the above formula we define $G_*(t)$. Let $T_* \leq T_2$ be the earliest time the piston interacts with the particles

$$(x, v) \in [G_1^+(t) \setminus G_*^+(t)] \cup G_2^+(t).$$

The time T_* is a random analogue of the time τ_* introduced in §2. During the interval (T_1, T_*) the piston interacts with the particles from $G_0^+(t) \cup G_*^+(t)$ only, and hence their velocities must be bounded by (4.1). For each $t \in (T_1, T_*)$ we denote by

$$\begin{aligned} \mathcal{X}_1(t) = & \{(x, v) : x = X(t) + 0, -v_{1,\max} < v < -v_{1,\min}\} \\ & \cup \{(x, v) : x = X(t) - 0, v_{1,\min} < v < v_{1,\max}\} \end{aligned} \tag{4.4}$$

the union of the two immediate one-sided neighbourhoods of the piston which contain all incoming particles.

As in the previous section, we define a subinterval $(T_1, S_2) \subset (T_1, T_*)$ on which the motion of the piston is sufficiently slow.

Definition. Let $(T_1, S_2) \subset (T_1, T_*)$ be the maximal time interval during which

- (a) $|V(t)| < v_{1,\min}$,
- (b) $\mathcal{X}_1(t) \subset G_0(t) \cup G_*(t)$.

The condition (b) means that the particles with velocities $v_{1,\min} < |v| < v_{1,\max}$ which are about to collide with the piston at time t have interacted with it during the interval $(0, t)$ at most once, and, if they did, then the interaction occurred before time T_1 .

Next we estimate how large the interval (T_1, S_2) is. Suppose that

$$|V(t) - W(t/L)| \leq \Delta \tag{4.5}$$

for any $t < S_2$ and for a small Δ (we shall estimate Δ below and show that $\Delta \rightarrow 0$ as $L \rightarrow \infty$). This immediately implies that $|V(t)| \leq \Delta + \mathcal{B}\varepsilon_0$, according to (2.26). Integrating (4.5) with respect to t gives

$$|X(t) - LY(t/L)| \leq t\Delta,$$

and hence

$$|X(t) - L/2| \leq (\Delta + \mathcal{B}\varepsilon_0)t. \tag{4.6}$$

Proposition 4.1. *If (4.5) holds for $t < S_2$ with some small $\Delta > 0$, then $T_2 - S_2 \leq CL(\Delta + \varepsilon_0)$, where $C > 0$ is a constant.*

Proof. If $S_2 = T_2$, then the statement is trivial. If $S_2 < T_2$, then either $S_2 < T_*$, and hence at least one of the conditions (a) or (b) in the previous definition fails at time S_2 , or $S_2 = T_* < T_2$. We note that the condition (a) cannot fail abruptly, since we assume that $|V(t)| \leq \Delta + \mathcal{B}\varepsilon_0$ on $(0, S_2)$, that is, $V(t)$ remains small. If (b) fails, then at time S_2 the piston collides with a point (x, v) such that $v_{1,\min} \leq |v| \leq v_{1,\max}$ whose past trajectory $(x_t, v_t) := F^{t-S_2}(x, v)$ for $t \in (0, S_2)$ hits the piston at some time $t_1 \geq T_1$. If $S_2 = T_* < T_2$, then at time S_2 the piston recollides with a gas particle (x, v) whose past trajectory $(x_t, v_t) := F^{t-S_2}(x, v)$ experiences the first collision with the piston at some time $t_1 \geq T_1$. In the last case, by (4.5),

$$|v| \leq v_{\max} + 2|V(t_1)| \leq v_{\max} + 2(\Delta + \mathcal{B}\varepsilon_0).$$

In either of the above two cases, the trajectory (x_t, v_t) collides with the piston twice, namely, first at time $t_1 \geq T_1$ and then at $t_2 = S_2$. We denote by X_1, X_2 the positions of the piston and by V_1, V_2 its velocities at times t_1, t_2 , respectively. Without loss of generality we can assume that our trajectory (x_t, v_t) lies to the right of the piston. The speed $|v| = |v_t|$ for $t_1 < t < t_2$ satisfies

$$|v| \leq \max\{v_{1,\max}, v_{\max} + 2(\Delta + \mathcal{B}\varepsilon_0)\} \leq v_{\max} + 2(\Delta + B_1\varepsilon_0) \tag{4.7}$$

with $B_1 = \max\{B, \mathcal{B}\}$. Then by (4.6) and (4.7), the obvious identity

$$|v|(t_2 - t_1) = (L - X_1) + (L - X_2)$$

implies that

$$t_2 - t_1 \geq \frac{L - (\Delta + \mathcal{B}\varepsilon_0)(t_1 + t_2)}{v_{\max} + 2(\Delta + B_1\varepsilon_0)}.$$

On the other hand, let us consider a particle that experiences the *very first* recollision with the piston (this happens at time T_1). After the collision, that particle acquires a velocity satisfying $|v(T_1 + 0)| \geq v_{\max} - 2|V(T_1)|$. The next collision of this particle with the piston occurs *after* T_2 . Therefore,

$$\begin{aligned} T_2 - T_1 &\leq \frac{L/2 + |X(T_1) - L/2| + L/2 + |X(T_2) - L/2|}{v_{\max} - 2|V(T_1)|} \\ &\leq \frac{L + (\Delta + \mathcal{B}\varepsilon_0)(T_1 + T_2)}{v_{\max} - 2(\Delta + \mathcal{B}\varepsilon_0)}. \end{aligned}$$

Combining the above estimates gives

$$T_2 - S_2 \leq (T_2 - T_1) - (t_2 - t_1) \leq CL(\Delta + \varepsilon_0)$$

with some $C > 0$ determined by B and \mathcal{B} . \square

Next, we study the dynamics during the time interval (T_1, S_2) . We again define the density $q(v, t)$ of colliding particles on the piston by the equation (3.9), and the

functions $Q_i(t)$, $i = 0, 1, 2$, by (3.10)–(3.12). We emphasize that, unlike what we had in the previous section, the density $p(x, v, t)$ now depends *essentially* on ω (at least for $(x, v) \in G_*^+(t)$), and hence $q(v, t)$ and $Q_i(t)$ depend on ω not only through the piston position $X(t)$ but also through the surrounding density $p(x, v, t)$.

In the previous sections we also introduced the deterministic density $\tilde{p}(x, v, t)$. We can now define the corresponding deterministic density

$$\tilde{q}(v, t) = \begin{cases} \tilde{p}(X(t) + 0, v, t) & \text{for } v < 0, \\ \tilde{p}(X(t) - 0, v, t) & \text{for } v > 0 \end{cases} \tag{4.8}$$

on the piston (cf. (2.17)), and we can define the deterministic functions $\tilde{Q}_i(t)$, $i = 0, 1, 2$, by equations similar to (3.10)–(3.12) but with tildes over the corresponding functions. We use tildes to distinguish these deterministic functions from the random ones.

We now compare the random functions p, q, Q_i with their deterministic counterparts on the interval (T_1, S_2) . Since the transformations $F^t: G^+ \rightarrow G^+(t)$ and $\tilde{F}^t: \tilde{G}^+ \rightarrow \tilde{G}^+(t)$ (recall that $\tilde{G}^+(t) = \tilde{F}^t(G^+)$) are invertible area-preserving maps, so is the map

$$\Psi^t = \tilde{F}^t \circ F^{-t},$$

which takes $G^+(t)$ onto $\tilde{G}^+(t)$. The next lemma follows easily from Theorem 3.8.

Lemma 4.2. *Let $t < S_2$ and $\omega \in \Omega_0^*$. For every $(x, v) \in G_*^+(t)$ let $(\tilde{x}, \tilde{v}) := \Psi^t(x, v)$. Then*

$$|x - \tilde{x}| \leq C \log L \quad \text{and} \quad |v - \tilde{v}| \leq CL^{-1} \log L$$

with a constant $C > 0$. Moreover,

$$p(x, v, t) = p(\tilde{x}, \tilde{v}, t).$$

Next, the properties of the density $p(x, v, t)$ stated in Lemma 3.3 obviously hold for the deterministic density $\tilde{p}(x, v, t)$ on the entire region \tilde{G}^+ . That is, the density $\tilde{p}(x, v, t)$ is piecewise C^1 smooth with $|\partial \tilde{p}(x, v, t) / \partial x| \leq D_1 / L$, and the discontinuity lines of $\tilde{p}(x, v, t)$ have slopes of order $O(1/L)$ (thus, they are almost parallel to the x axis). Therefore, as in (3.13), we have

$$\left| \frac{\partial \tilde{Q}_i}{\partial X} \right| \leq \frac{\text{const}}{L}. \tag{4.9}$$

Lemma 4.2 allows us to compare Q_i and \tilde{Q}_i regarded as functions of X in the following way.

Lemma 4.3. *Let $t < S_2$ and $\omega \in \Omega_0^*$. Then for $i = 0, 1, 2$*

$$|Q_i(t) - \tilde{Q}_i(t)| \leq CL^{-1} \log L.$$

We note that $\tilde{Q}_i(t)$ is defined in terms of $\tilde{q}(v, t)$, which uses the (random) position of the piston $X(t)$ for the given ω ; see (4.8). The above lemma shows that the dependence of $Q_i(t)$ on ω through the density $p(x, v, t)$ (which itself depends on ω) is very weak, because $L^{-1} \log L$ is small. In other words, the density $p(x, v, t)$ of the gas surrounding the piston fluctuates with ω very little.

The following theorem is an analogue of Theorem 3.4.

Theorem 4.4. *For any sufficiently large L there exists a set $\Omega_1^* \subset \Omega_0^*$ of initial configurations of particles such that:*

(i) *there is a constant $c > 0$ for which*

$$P(\Omega_1^*) > 1 - L^{-c \log \log L}, \tag{4.10}$$

(ii) *for each configuration $\omega \in \Omega_1^*$ and for each time interval*

$$(t, t + \Delta t) \subset (T_1, S_2)$$

such that

$$\frac{(\log L)^2}{L^{1/3}} \leq \Delta t \leq \frac{1}{L^{1/7}} \tag{4.11}$$

the change of the velocity of the piston satisfies

$$V(t + \Delta t) - V(t) = \tilde{\mathcal{D}}(t)\Delta t + \chi, \tag{4.12}$$

where

$$\tilde{\mathcal{D}}(t) = a[\tilde{Q}_0(t)V^2(t) - 2\tilde{Q}_1(t)V(t) + \tilde{Q}_2(t)] \tag{4.13}$$

and

$$|\chi| \leq C \frac{\log L(\Delta t)^{1/4}}{L^{1/4}} \tag{4.14}$$

with some global constant $C > 0$.

Remark. We note that our bound (4.14) for the random fluctuations represented by χ is much weaker than (3.17) in Theorem 3.4. This is due to the lack of good control over large deviations for the distribution of gas particles, as will be clear from the proof.

Proof. Our argument goes basically along the lines of the proof of Theorem 3.4, but it involves a good deal of new construction, which we describe in detail. The first step is the velocity decomposition scheme (see (3.27))

$$V(t + \Delta t) = (1 - \varepsilon k)V(t) + \varepsilon \sum_{j=1}^k v_j + \chi^{(1)} + \chi^{(2)}. \tag{4.15}$$

The error terms $\chi^{(1)}$ and $\chi^{(2)}$ are defined in § 3 after (3.27), and they are bounded:

$$|\chi^u| \leq v_{\max} \varepsilon^2 k^2$$

for $u = 1, 2$; see (3.32). We shall see later that

$$k \leq \text{const} \cdot L^2 \Delta t, \tag{4.16}$$

and hence

$$|\chi^u| \leq \text{const} \cdot (\Delta t)^2 \tag{4.17}$$

for $u = 1, 2$.

Next, we need a crude upper bound δV for the possible fluctuations of the piston velocity $V(s)$ during the interval $(t, t + \Delta t)$, as defined by (3.28). One bound can easily be derived from (4.15):

$$\delta V \leq 2v_{\max}\varepsilon k \leq \text{const} \cdot \Delta t. \quad (4.18)$$

We now define two regions $D_1 = D_1^+ \cup D_1^-$ and $D_2 = D_2^+ \cup D_2^-$ in the (x, v) -plane by the equations (3.29)–(3.30), where v_{\min} and v_{\max} must be replaced by $v_{1,\min}$ and $v_{1,\max}$, respectively. As is explained in §3, all the particles in D_1 (at time t) necessarily collide with the piston during the interval $(t, t + \Delta t)$. All the particles actually colliding with the piston during that interval are contained in D_2 . Therefore,

$$k_1^\pm \leq k^\pm \leq k_2^\pm, \quad (4.19)$$

where k^\pm, k_1^\pm, k_2^\pm are defined above (3.34). This gives upper and lower bounds for the numbers of colliding particles.

Our next step is to estimate the numbers $k_i^\pm, i = 1, 2$. Since k_i^\pm no longer have a Poisson distribution, we estimate them by using a new approach. Let D be one of the four trapezoids $D_i^\pm, i = 1, 2$, and let $k_{D,\omega}$ be the (random) number of particles in D at time t . Obviously, $k_{D,\omega}$ is equal to the number of particles in $F^{-t}(D)$ at time zero.

We note that the trapezoid D has height $v_{1,\max} - v_{1,\min} = \text{const}$ and width $O(\Delta t)$, and hence its area is bounded:

$$d_1\Delta t \leq |D| \leq d_2\Delta t, \quad (4.20)$$

with some constants $0 < d_1 < d_2 < \infty$.

Let us examine the region $F^{-t}(D)$ more closely. Since $(t, t + \Delta t) \subset (T_1, S_2)$, it follows from the condition (b) in the definition of S_2 that $D \subset G_0(t) \cup G_*(t)$. We set $D_0 = D \cap G_0(t)$ and $D_1 = D \cap G_*(t)$; then $F^{-t}(D) = F^{-t}(D_0) \cup F^{-t}(D_1)$. To emphasize the dependence of the flow F^t on ω , we write F_ω^t for F^t . The part $F_\omega^{-t}(D_0)$ is obtained by a simple linear transformation of D_0 (because of the absence of collisions of particles in D_0 with the piston), and hence it is actually independent of ω . The part $F_\omega^{-t}(D_1)$ is also obtained by pulling the domain D_1 back in time, but one collision with the piston occurs along the way. Since the position and velocity of the piston are random, the domain $F_\omega^{-t}(D_1)$ depends on ω . Hence, the domain $D_\omega := F_\omega^{-t}(D)$ depends on ω . The initial number $k_{D,\omega}$ of particles in a randomly selected domain D_ω certainly need not be a Poisson random variable.

To estimate $k_{D,\omega}$ we fix some Δt satisfying (4.11) and construct finitely many domains $D_n \subset G, 1 \leq n \leq N_*$, which have the following property. For every $\omega \in \Omega_0^*$, every $t \in (T_1, S_2 - \Delta t)$, and every trapezoid D defined above there are two domains $D_{n'}$ and $D_{n''}$ such that

$$D_{n'} \subset F_\omega^{-t}(D) \subset D_{n''} \quad (4.21)$$

and the area of the difference is relatively small:

$$|D_{n''} \setminus D_{n'}| \leq \chi^{(3)}|D_\omega|, \quad (4.22)$$

where $\chi^{(3)} \rightarrow 0$ as $L \rightarrow \infty$. We say that $D_{n'}$ and $D_{n''}$ approximate D_ω ‘from inside’ and ‘from outside’, respectively. We denote the collection of domains D_n for a given Δt by

$$\mathcal{C} = \mathcal{C}_{\Delta t} = \{D_n\}_{n=1}^{N_*}.$$

We postpone the construction of the D_n for a moment and derive an immediate result from the above approximation. The inclusion (4.21) implies that

$$k_{n',\omega} \leq k_{D,\omega} \leq k_{n'',\omega}, \tag{4.23}$$

where $k_{n,\omega}$ is the (random) number of particles in the domain D_n at time zero. Since D_n is fixed (independent of ω) for each n , the random variable $k_{n,\omega}$ does have a Poisson distribution, with mean value

$$\lambda_n := L^2 \int_{D_n} p(x, v, 0) dx dv.$$

According to Lemma A.2,

$$P(\omega : |k_{n,\omega} - \lambda_n| > B\sqrt{\lambda_n}) \leq 2\varepsilon^{-cB^2} \tag{4.24}$$

for any $B < b\sqrt{\lambda_n}$, where $b > 0$ is a constant and $c = c(b) > 0$ is another constant. We shall specify the value of $B = B_{\Delta t}$ below (one value for all the D_n in $\mathcal{C}_{\Delta t}$). In view of (4.20) it is enough to require that

$$B_{\Delta t} < L\sqrt{\Delta t}. \tag{4.25}$$

Then we define $\Omega_1^*(\Delta t)$ as the set of configurations $\omega \in \Omega_0^*$ satisfying

$$|k_{n,\omega} - \lambda_n| \leq B_{\Delta t}\sqrt{\lambda_n} \quad \text{for } 1 \leq n \leq N_*. \tag{4.26}$$

Then by (4.24) we have

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \leq 2N_*e^{-cB_{\Delta t}^2}. \tag{4.27}$$

Next, for all $\omega \in \Omega_1^*(\Delta t)$ the bounds (4.23) and (4.26) imply that

$$\lambda_{n'} - B_{\Delta t}\sqrt{\lambda_{n'}} \leq k_{D,\omega} \leq \lambda_{n''} + B_{\Delta t}\sqrt{\lambda_{n''}}. \tag{4.28}$$

Furthermore, let us consider the quantity

$$\lambda_{D,\omega} = L^2 \int_{F_\omega^{-1}(D)} p(x, v, 0) dx dv = L^2 \int_D p(x, v, t) dx dv.$$

By (4.21), we have

$$\lambda_{n'} \leq \lambda_{D,\omega} \leq \lambda_{n''}.$$

It follows from (4.22) that

$$(1 - c\chi^{(3)})\lambda_{D,\omega} \leq \lambda_{n'} \leq \lambda_{n''} \leq (1 + c\chi^{(3)})\lambda_{D,\omega} \tag{4.29}$$

for some constant $c > 0$ (determined by π_{\max} , π_{\min} , v_1 , and v_2 in (P3)).

Let us consider the deterministic quantity

$$\tilde{\lambda}_D = L^2 \int_D \tilde{p}(x, v, t) dx dv.$$

It follows easily from Lemma 4.2 and the properties of $\tilde{p}(x, v, t)$ that

$$|\lambda_{D,\omega} - \tilde{\lambda}_D| \leq CL^{-1} \log L \tilde{\lambda}_D \tag{4.30}$$

with some constant $C > 0$. Combining (4.29) and (4.30), we arrive at the relations

$$(1 - \chi^{(4)})\tilde{\lambda}_D \leq \lambda_{n'} \leq \lambda_{n''} \leq (1 + \chi^{(4)})\tilde{\lambda}_D \tag{4.31}$$

with

$$\chi^{(4)} = \text{const} \cdot (\chi^{(3)} + L^{-1} \log L). \tag{4.32}$$

The bounds (4.28) and (4.31) give the desired estimate for the number $k_{D,\omega}$.

We now construct the domains D_n that approximate the domains $D_\omega = F_\omega^{-t}(D)$ for all $\omega \in \Omega_0^*$ and all trapezoids D defined above. We first fix an $\omega \in \Omega_0^*$ and a trapezoid D . Let us now construct two special domains D', D'' that approximate $D_\omega = F_\omega^{-t}(D)$ from inside and from outside, that is, $D' \subset D_\omega \subset D''$.

The domain D_ω is obtained by pulling D back in time. We consider its ‘trajectory’ $D_s^- = F_\omega^{s-t}(D)$ for $0 < s < t$. Thus, $D_0^- = D_\omega$ and $D_t^- = D$. Let us examine the shape of the domain D_s^- and how it changes as s runs from t down to 0. We recall that the trapezoid D is adjacent to the piston at time t (it is about to run into the piston at that time). As s goes from t downward, the domain D_s^- comes off the piston and travels to a wall (as in a movie running backward). During that period, the map $F_\omega^{-(t-s)}$ restricted to D is linear, and hence D_s^- is still a trapezoid. Since the velocities of points $(x, v) \in D$ vary (from $v_{1,\min}$ to $v_{1,\max}$), the trapezoid D_s^- becomes ‘skewed’, namely, its ‘outer’ edge $|v| = v_{1,\max}$ moves toward the wall faster than the other edge $|v| = v_{1,\min}$. By the time it reaches the wall, D_s^- becomes a long slanted trapezoid stretched by a distance $O(L)$ along the x axis. Every vertical line (parallel to the v axis) intersects D_s^- along a segment of length $O(\Delta t/L)$. As s goes farther down, a collision with the wall occurs, and a new part of D_s^- appears directly across the x axis, moving now toward the piston. The shape of this part is also that of a long narrow trapezoid whose vertical ‘thickness’ is $O(\Delta t/L)$. Eventually it moves all the way (along a distance $O(L)$) from the wall to the piston and contacts the piston at some time $s_* = s_*(D)$; see Fig. 5. We note that, so far, D_s^- is completely independent of ω .

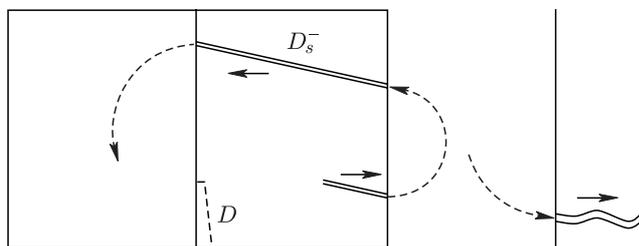


Figure 5. The domain D_s^- . The arrows show its motion as s decreases. The ‘outgoing’ part of D_s^- , as it comes off the piston, is shown on the right

After the collision with the piston, a new part of D_s^- appears across the x axis, coming off the piston and moving back to the wall. This is the most interesting part; it is determined by the piston position and velocity, and hence it actually depends on ω . We recall that the part of D_s^- running into the piston before the collision (the ‘incoming’ part) is a narrow trapezoid bounded by two almost horizontal lines with slope $O(1/L)$ lying a distance $O(\Delta t/L)$ apart. We call them the upper and lower boundaries of D_s^- . The part of D_s^- coming off the piston after the collision (the ‘outgoing’ part) also has upper and lower boundaries (due to the reversal of velocities at collision, although the upper boundary after the collision is the image of the lower boundary before the collision, and vice versa); see Fig. 5. The boundaries of the outgoing part of D_s^- are quite irregular and depend on ω . It is they that need careful approximation.

The shape of the outgoing boundary of D_s^- is determined by the piston velocity $V(u)$ during the interval $s < u < s_*$. Since the collision between D_s^- and the piston occurs during the zero-recollision interval ($s_* < T_1$), we can apply the results of § 3. By Theorem 3.7 the piston velocity behaves like a Hölder-continuous³ function:

$$|V(u+h) - V(u)| \leq CL^{-1} \log L \sqrt{h} =: d_h \tag{4.33}$$

for all $h \in (L^{-2}, 1)$. We consider only h satisfying $L^{-a_1} < h < L^{-a_2}$ with some $0 < a_2 < a_1 < 2$. Note that in this case

$$Ld_h/h \rightarrow \infty \quad \text{and} \quad d_h/h \rightarrow 0 \tag{4.34}$$

as $L \rightarrow \infty$.

Let us consider the upper boundary of the outgoing domain D_s^- after the collision. It is a continuous curve that can be parameterized by the collision time (or, in other words, the contact time between D_s^- and the piston), denoted by u . This gives a parametric curve $(x_s(u), v_s(u))$ with parameter $u \in (s, s_*)$. In particular,

$$F^w(x_s(u), v_s(u)) = (x_{s+w}(u), v_{s+w}(u))$$

for $s < u < s_*$ and $w \leq u - s$, and the point

$$F^{u-s}(x_s(u), v_s(u)) = (x_u(u), v_u(u))$$

is the endpoint of the upper boundary of D_u^- , that is, $x_u(u) = X(u)$ is the piston coordinate at time u .

Claim 1. For each $u \in (0, s_* - h)$ and all sufficiently large L

$$|x_{u+h}(u+h) - x_u(u)| \leq B\varepsilon_0 h \tag{4.35}$$

and

$$|v_{u+h}(u+h) - v_u(u)| \leq 3d_h. \tag{4.36}$$

³For any $d > 0$, d -dense sets in the space of Hölder-continuous functions were constructed by Kolmogorov and Tikhomirov [16], and our constructions here are in the same spirit.

Proof. The first bound means that $|X(u + h) - X(u)| \leq B\varepsilon_0 h$ and follows from (3.44). To prove the second inequality we use the collision rule

$$v_u(u) = -v_u^-(u) + 2V(u),$$

where $v_u^-(u)$ is the v coordinate of the lower boundary of the incoming part of D_u where it contacts the piston (at time u). Similarly,

$$v_{u+h}(u + h) = -v_{u+h}^-(u + h) + 2V(u + h).$$

Since the lower boundary of the incoming part of D_s^- is a straight line with slope $O(1/L)$, and the piston velocity is bounded, we have

$$|v_{u+h}^-(u + h) - v_u^-(u)| \leq \text{const} \cdot h/L \leq d_h,$$

where the last bound follows from (4.34). This and (4.33) imply (4.36). \square

We now fix some h satisfying

$$(\Delta t/L)^{1/2} \leq h \leq 2(\Delta t/L)^{1/2}. \tag{4.37}$$

With our restrictions (4.11) on Δt , this implies that $L^{-2/3} \leq h \leq L^{-4/7}$, and hence (4.34) holds. We also note that

$$d_h/(\Delta t/L) \rightarrow 0 \quad \text{as } L \rightarrow \infty, \tag{4.38}$$

and therefore d_h is much less than the ‘thickness’ of the domain D_s^- .

Next we take a lattice in the (x, v) -plane with x -spacing rh and v -spacing rd_h , where $r > 0$ is a sufficiently small constant (for example, $r = B\varepsilon_0/10$). The lattice sites are

$$(x_i, v_j) = (rhi, rd_h j), \quad i, j \in \mathbb{Z}. \tag{4.39}$$

We are now ready to define the boundary of the domain $D' \subset D_0^- = F_\omega^{-t}(D)$ approximating D_0^- from inside. We start with the upper boundary of the outgoing part of D_0^- . It is parameterized as $(x_0(u), v_0(u))$ (see above), with $0 < u < s_*$. We consider the discrete parameter values $u_k = kh, k = 0, 1, \dots, [s_*/h]$. For each such k we consider the point

$$F^{kh}(x_0(kh), v_0(kh)) = (x_{kh}(kh), v_{kh}(kh)).$$

This is the endpoint of the upper boundary of D_{kh}^- , and therefore $x_{kh}(kh) = X(kh)$ is the piston coordinate at time kh . Now we pick a lattice site (4.39) closest to the point $(x_{kh}(kh), v_{kh}(kh))$ and lying on the same side of the piston as the domain D_{kh}^- ; denote this site by (x_k^s, y_k^s) . Next, we adjust this site by moving it down (along the v axis) the fixed distance $10d_h$ and we obtain the adjusted point (x_k^a, v_k^a) ,

$$x_k^a = x_k^s, \quad v_k^a = v_k^s - 10d_h. \tag{4.40}$$

We note that Claim 1 (together with the smallness of r) implies that

$$|x_k^a - x_{k+1}^a| = |x_k^s - x_{k+1}^s| \leq 2B\varepsilon_0 h \tag{4.41}$$

and

$$|v_k^a - v_{k+1}^a| = |v_k^s - v_{k+1}^s| \leq 4d_h. \tag{4.42}$$

Next, let

$$(x_k^0, v_k^0) := F^{-kh}(x_k^a, v_k^a).$$

We join the points (x_k^0, v_k^0) and (x_{k+1}^0, v_{k+1}^0) by a straight line segment, say, L_k . Then the upper boundary of the domain D' is made up of the segments L_k ; in other words, it is the union

$$\bigcup_{k=0}^{\lfloor s_*/h \rfloor - 1} L_k.$$

Claim 2. *The upper boundary of D' lies completely inside the domain $D_0^- = F_\omega^{-t}(D)$.*

Proof. It is enough to show that each link L_k lies inside D_0^- . It is easy to see that the map F^{kh} restricted to the link L_k is a linear map, and the image $L_k^a := F^{kh}(L_k)$ is a straight line segment joining the points

$$(x_k^a, v_k^a) \quad \text{and} \quad F^{-h}(x_{k+1}^a, v_{k+1}^a) = (x_{k+1}^a - v_{k+1}^a h, v_{k+1}^a).$$

We denote by

$$|L_k^a|_x = |x_{k+1}^a - x_k^a - v_{k+1}^a h|, \quad |L_k^a|_v = |v_{k+1}^a - v_k^a|$$

the lengths of the projections of L_k^a on the x and v axes, respectively. We note that $v_{1,\min} - B\varepsilon_0 \leq |v_{k+1}^a| \leq v_{1,\max} + B\varepsilon_0$. Hence, (4.35) implies that

$$(v_{1,\min} - 2B\varepsilon_0)h \leq |L_k^a|_x \leq (v_{1,\max} + 2B\varepsilon_0)h,$$

that is, $|L_k^a|_x$ is of order h . By (4.36), $|L_k^a|_v < 3d_h$. Thus, for the slope of L_k^a we have $O(d_h/h) \rightarrow 0$ by (4.34). Therefore, L_k^a is almost a horizontal segment. The inequality (4.36) shows that the upper boundary $(x_{kh}(u), v_{kh}(u))$ of D_{kh}^- for $kh \leq u \leq kh + h$, which lies directly above L_k^a , oscillates in the v direction by less than $3d_h$. The adjustment (4.40) then ensures that the segment L_k^a lies entirely below the upper boundary of the domain D_{kh}^- . Moreover, (4.38) implies that D_{kh}^- is much ‘thicker’ than the distance between its upper boundary and L_k^a , and hence L_k^a lies entirely inside that domain. Thus, $L_k = F^{-kh}(L_k^a)$ lies entirely inside $D_0^- = F^{-kh}(D_{kh}^-)$, proving our claim. \square

It is also clear from the above argument that the segment L_k^a is at a distance $< 20d_h$ below the actual upper boundary of D_{kh}^- , and therefore the area between them is bounded by $\text{const} \cdot hd_h$. Hence, the total area between the upper boundaries of D_0^- and D' is bounded by

$$\text{const} \cdot hd_h \cdot (s_*/h) \leq \text{const} \cdot Ld_h.$$

In the same way we construct the lower boundary of the new domain D' . The only difference is that we adjust the selected sites of the lattice (4.39) by moving them up, so that the segments joining them are inside D_0^- again. The upper and lower parts of the boundary give us a new domain approximating the outgoing part of D_0^- after the collision with the piston, that is, the part $F^{-t}(D \cap G_*(t))$. We note that this part may itself experience one collision with the wall (during the interval $(0, s_*)$), and then it consists of two connected components lying across the x axis.

It remains to approximate the ‘good’ part of $F_\omega^{-t}(D) = D_0^-$ which has not interacted with the piston, that is, the part $F_\omega^{-t}(D \cap G_0(t))$. This part, as described above, consists of one or two trapezoids bounded by two almost horizontal straight lines a distance $O(\Delta t/L)$ apart. This is an easy task. We simply replace the upper (lower) boundary of the domain $F^{-t}(D \cap G_0(t))$ by a polygonal line joining some sites of the lattice (4.39), picked one on each vertical line $x = x_i$ crossing the domain D_s^- and lying inside (outside) this domain and no farther than d_h from the original boundary (recall that r in (4.39) is very small, so the above choice is possible). This gives the boundary of D' . We note that, if we enumerate the selected sites consecutively (say, from left to right) and call them (x_k^s, y_k^s) , then the neighbouring sites again satisfy (4.41)–(4.42).

The complicated construction described above produces a domain $D' \subset D_\omega = F_\omega^{-t}(D)$ such that

$$|D_\omega \setminus D'| \leq \text{const} \cdot d_h L = \text{const} \cdot \log L \sqrt{h}. \tag{4.43}$$

In a completely similar way we construct another domain D'' that approximates D_ω from outside and satisfies

$$|D'' \setminus D_\omega| \leq \text{const} \cdot d_h L = \text{const} \cdot \log L \sqrt{h} \tag{4.44}$$

(we just need to adjust the sites along the upper boundary by moving them up and the sites along the lower boundary by moving them down in such a way that the new boundaries are completely outside D_ω). Since $|D| = O(\Delta t)$, the inequalities (4.43)–(4.44) imply that

$$|D'' \setminus D'| \leq \text{const} \cdot \log L \sqrt{h} (\Delta t)^{-1} |D|,$$

and hence we obtain (4.22) with

$$\chi^{(3)} = \text{const} \cdot \log L \sqrt{h} (\Delta t)^{-1}. \tag{4.45}$$

We note that $\chi^{(3)} \rightarrow 0$ as $L \rightarrow \infty$ due to our choice of h in (4.37) and to the restrictions on Δt in (4.11).

Our domains D' and D'' are constructed around D_ω , and hence they depend on ω . However, the boundaries of D' and D'' are polygonal lines whose vertices are selected from the sites of the fixed lattice (4.39). Therefore, the total number N_* of *distinct* domains D' and D'' is finite. Let us estimate N_* .

First, the total number of sites of the lattice (4.39) in the relevant area $0 < x < L$, $|v| < v_{1,\max}$ is

$$K_1 = \text{const} \cdot (L/h) \cdot (2v_{\max}/d_h) \leq \text{const} \cdot L(hd_h)^{-1}. \tag{4.46}$$

We recall that the domain $D_0 = F^{-t}(D)$ consists of at most four connected components. The part $F^{-t}(D \cap G_*(t))$, or the outgoing part of D_0 created after the collision with the piston, consists of at most two components. And the part $F^{-t}(D \cap G_0(t))$ consists of at most two components, which are regular trapezoidal regions. Then the upper (lower) part of the boundary of D' (D'') consists of at most four disjoint polygonal lines constructed on lattice sites (4.39). In each polygonal line, the first point can be constructed from any of the K_1 sites of the lattice; see (4.46). But the other points are constructed consecutively, and each point must be constructed from a site selected from

$$K_0 := \frac{4B\varepsilon_0}{r} \cdot \frac{8}{r} = \text{const} \tag{4.47}$$

nearest neighbours of the previously selected site, according to (4.41), (4.42). It is also clear that each polygonal line has at most

$$M := \text{const} \cdot L/h$$

links (vertices). Therefore, the total number of ways to construct one polygonal line does not exceed $K_1 K_0^M$. The total number of ways to construct the entire boundary of D' (D'') is then less than $K_1^8 K_0^{8M}$. This gives an upper bound for the number of distinct domains D' (D''),

$$N_* \leq K_1^8 K_0^{8M} \leq \text{const} \cdot L^8 (hd_h)^{-8} e^{\text{const} \cdot L/h}. \tag{4.48}$$

Of course, the exponential factor is dominant here and can absorb all the others.

We now set the value of $B_{\Delta t}$ in (4.25)–(4.28),

$$B_{\Delta t} = c_7 \sqrt{L/h}, \tag{4.49}$$

where the constant $c_7 > 0$ is sufficiently small. Then (4.27) and (4.48) imply that

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \leq \text{const} \cdot e^{-c_8 L/h} \tag{4.50}$$

with some constant $c_8 > 0$. We recall that h satisfies (4.37), and hence

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \leq \text{const} \cdot e^{-\text{const} \cdot L^{3/2} (\Delta t)^{-1/2}}.$$

Next, since Δt satisfies (4.11), we have

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \leq \text{const} \cdot e^{-\text{const} \cdot L^{11/7}}. \tag{4.51}$$

Moreover, by (4.49) and (4.37)

$$B_{\Delta t} \leq c_7 L^{3/4} (\Delta t)^{-1/4},$$

which implies (4.25) since $\Delta t \gg L^{-1/3}$.

Next, for each $\omega \in \Omega_1^*(\Delta t)$, each $t \in (T_1, S_2 - \Delta t)$, and any trapezoid D defined above, the number $k_{D,\omega}$ of particles in D satisfies (4.28)–(4.32) with $\chi^{(3)}$ given by (4.45) and h given by (4.37), and hence

$$k_{D,\omega} = \tilde{\lambda}_D + \chi_{D,\omega},$$

with

$$|\chi_{D,\omega}| \leq \text{const} \cdot (B_{\Delta t} L \sqrt{\Delta t} + L^2 \log L \sqrt{h} + L \log L) \leq \text{const} \cdot L^{7/4} \log L (\Delta t)^{1/4}.$$

We note that $\tilde{\lambda}_D = O(L^2 \Delta t)$, and therefore $|\chi_{D,\omega}| \ll \tilde{\lambda}_D$ for all Δt satisfying (4.11). Incidentally, this easily implies a rough bound $k_{D,\omega} \leq 2\tilde{\lambda}_D \leq \text{const} \cdot L^2 \Delta t$, which justifies (4.16), and hence (4.18).

Further, using (4.19) and applying the above estimates to each k_i^\pm , $i = 1, 2$, we have

$$k = L^2 \int_{D_1} \tilde{p}(x, v, t) dx dv + \chi'$$

with

$$|\chi'| \leq \text{const} \cdot [L^{7/4} \log L (\Delta t)^{1/4} + L^2 \delta V \Delta t].$$

In a similar way we can estimate the other random factor in the main decomposition formula (4.15), which is

$$Z = \sum_{j=1}^k v_j,$$

and obtain

$$Z = L^2 \int_{D_1} v \tilde{p}(x, v, t) dx dv + \chi''$$

with the same upper bound for χ'' as for χ' .

Use of the smoothness properties of the function $\tilde{p}(x, v, t)$ described above (4.9) gives (cf. (3.38)–(3.40))

$$\begin{aligned} \int_{D_1} \tilde{p}(x, v, t) dx dv &= (\tilde{Q}_1(t) - \tilde{Q}_0(t))\Delta t + \chi^{(4)}, \\ \int_{D_1} v \tilde{p}(x, v, t) dx dv &= (\tilde{Q}_2(t) - \tilde{Q}_1(t))\Delta t + \chi^{(5)}, \end{aligned}$$

where

$$|\chi^{(u)}| \leq \text{const} \cdot [(\Delta t)^2/L + \delta V \Delta t]$$

for $u = 4, 5$.

Therefore, we obtain

$$V(t + \Delta t) - V(t) = \tilde{D}(t)\Delta t + \chi \tag{4.52}$$

with

$$|\chi| \leq \text{const} \cdot [L^{-1/4} \log L (\Delta t)^{1/4} + \delta V \Delta t + (\Delta t)^2]. \tag{4.53}$$

We recall that $\delta V = O(\Delta t)$ by (4.18). It is now easy to check that the second and third terms are much less than the first. This proves (4.12)–(4.14) but does not yet prove Theorem 4.4, because we have (arbitrarily) fixed a value of Δt , and our set Ω_1^* depends on Δt .

Actually, for our main purpose it is enough to prove Theorem 4.4 for just one value of Δt , namely for $\Delta t = 1/L^{1/7}$, as we will see later. But with a little extra effort we can prove our theorem for all Δt satisfying (4.11). We do that next. Let us divide the interval (4.11) into subintervals of length e^{-L} . That is, we fix a finite collection of points $(\Delta t)_n = ne^{-L}$ for $n = n_1, \dots, n_2$ with $n_1 = L^{-1/3}(\log L)^2 e^L$ and $n_2 = L^{-1/7} e^L$. Then we define

$$\Omega_1^* = \bigcap_{n=n_1}^{n_2} \Omega_1^*((\Delta t)_n).$$

The bound (4.51) implies that

$$P(\Omega_0^* \setminus \Omega_1^*) \leq \text{const} \cdot e^{L - \text{const} \cdot L^{11/7}},$$

which is obviously sufficient to maintain the bound (4.10).

Now for any Δt satisfying (4.11) we find a $(\Delta t)_n$ such that $|\Delta t - (\Delta t)_n| \leq e^{-L}$. For any $\omega \in \Omega_1^* \subset \Omega_1^*((\Delta t)_n)$ we have all the above estimates with Δt replaced by $(\Delta t)_n$. This replacement causes only an exponentially small error e^{-L} in our estimates, and does not spoil our bounds, which are all polynomial in L . This completes the proof of Theorem 4.4. \square

The next theorem is an analogue of Theorem 3.5.

Theorem 4.5. *If $\varepsilon_0 > 0$ is small enough, then the following assertions hold for all sufficiently large L , for each configuration $\omega \in \Omega_1^*$, and for all $t \in (T_1, S_2)$.*

- (i) *There is a constant $B > 0$ such that*

$$|V(t)| < B\varepsilon_0; \tag{4.54}$$

- (ii) *There is a constant $C_0 > 0$ such that*

$$|V(t) - V_0(t)| < C_0 L^{-1/7} \log L, \tag{4.55}$$

where $V_0(t)$ is defined by

$$V_0(t) = \frac{\tilde{Q}_1(t) - \sqrt{\tilde{Q}_1^2(t) - \tilde{Q}_0(t)\tilde{Q}_2(t)}}{\tilde{Q}_0(t)} \tag{4.56}$$

whenever $\tilde{Q}_0(t) \neq 0$ and by

$$V_0(t) = \frac{\tilde{Q}_2(t)}{2\tilde{Q}_1(t)} \tag{4.57}$$

otherwise.

Proof. This proof repeats in large part that of Theorem 3.5. The first half, up to the formula (3.59), can be copied almost verbatim, except with the replacement of S_1 by S_2 , Q_i by \tilde{Q}_i , and \mathcal{D} by $\tilde{\mathcal{D}}$. We omit this part. The rest of the proof requires more substantial modifications, and we give it in detail.

First, we recall that Theorem 3.5 deals with $t \in (0, S_1)$. On the interval $(0, S_1)$ the functions $Q_i(t)$, $i = 0, 1, 2$, are independent of ω ; they are defined by equations (3.9)–(3.12), where $p(x, v, t)$ is in fact the deterministic density now denoted by $\tilde{p}(x, v, t)$. Therefore, for $t > T_1$ our functions $\tilde{Q}_i(t)$ are natural extensions of $Q_i(t)$ beyond the interval $(0, S_1) = (0, T_1)$, and they have the same properties; cf. (3.13) and (4.9). Hence, the function $V_0(t)$ defined by (4.56)–(4.57) for $t > T_1$ is an extension of the function $V_0(t)$ defined by (3.46)–(3.47) on the interval $(0, T_1)$.

Next, in the first half of the proof (which we omitted since it is almost the same as the first half of the proof of Theorem 3.5) we must introduce $t_* < S_2$ as the first time (4.54) fails. We now prove (4.55) for all $t < t_*$ with some constant $C_0 > 0$ (independent of the choice of B in (4.54), which is yet to be made).

The bound (3.45) proved for all $t \leq T_1$ implies that (4.55) holds for at least some $t > T_1$. Next, if (4.55) fails for any $t < t_*$, then let $t \in (T_1, t_*)$ be the first time it fails. We denote by

$$\Delta_0 = L^{-1/7}$$

the maximal allowed time increment in Theorem 4.4. Let $s = t - \Delta_0$. Due to Theorem 4.4,

$$V(t) = V(s) + \tilde{\mathcal{D}}(s)\Delta_0 + \chi \tag{4.58}$$

with

$$|\chi| \leq CL^{-1/4} \log L(\Delta_0)^{1/4} = CL^{-2/7} \log L.$$

Due to the analogue of (3.58) obtained in the first half of the proof of the theorem,

$$V_0(t) = V_0(s) + \chi_0 \tag{4.59}$$

with

$$|\chi_0| \leq \frac{E_0 \varepsilon_0 \Delta_0}{L} = \frac{E_0 \varepsilon_0}{L^{8/7}}.$$

For brevity, let $U(s) = V(s) - V_0(s)$ for all s . Subtracting (4.59) from (4.58) gives

$$U(t) = U(s) + \tilde{\mathcal{D}}(s)\Delta_0 + \chi' \tag{4.60}$$

with $\chi' = \chi - \chi_0$, and hence for large L

$$|\chi'| \leq 2CL^{-2/7} \log L. \tag{4.61}$$

Without loss of generality we can assume that $U(t) > 0$. Since (4.55) fails at time t , we have

$$U(t) \geq C_0 L^{-1/7} \log L. \tag{4.62}$$

Let us consider two cases. If $U(s) \leq 0$, then by the analogue of (3.59)

$$U(t) \leq |\tilde{\mathcal{D}}(s)|\Delta_0 + |\chi'| \leq E_2|U(s)|\Delta_0 + |\chi'| \ll L^{-1/7} \log L$$

for large L , which contradicts (4.62). If $U(s) > 0$, then, again due to (4.60) and the analogue of (3.59),

$$U(t) < U(s)[1 - E_1\Delta_0] + \chi',$$

and hence

$$U(s) > \frac{U(t) - \chi'}{1 - E_1\Delta_0} > (U(t) - \chi')(1 + E_1\Delta_0) > U(t) + U(t)E_1\Delta_0 - 2\chi'. \quad (4.63)$$

Now, if C_0 in (4.55) is large enough, say $C_0 = 3C/E_1$, then $U(t)E_1\Delta_0 > 2\chi'$ by (4.62) and (4.61). This fact and (4.63) imply that $U(s) > U(t)$, and therefore (4.55) fails at an earlier time $s < t$, a contradiction. Hence, (4.55) is proved for all $t < t_*$ and $C_0 = 3C/E_1$.

Lastly, the remaining part of the proof of Theorem 3.5 can be repeated verbatim, concluding the proof of Theorem 4.5. \square

Let us finally prove the convergence as $L \rightarrow \infty$ of the random trajectory of the piston to the solution $Y(t/L), W(t/L)$ of the hydrodynamic equations described in §2 for all $t \in (T_1, S_2)$.

Theorem 4.6. *If $\varepsilon_0 > 0$ in (P5) is small enough, then for all large L and all $\omega \in \Omega_1^*$ there is a constant $C > 0$ for which*

$$|Y_L(\tau, \omega) - Y(\tau)| \leq \frac{C \log L}{L^{1/\tau}} \quad (4.64)$$

and

$$|W_L(\tau, \omega) - W(\tau)| \leq \frac{C \log L}{L^{1/\tau}} \quad (4.65)$$

for all τ such that $\min\{\tau_1, T_1/L\} < \tau < \min\{\tau_2, S_2/L\}$. Moreover,

$$T_2 - S_2 \leq C\varepsilon_0 L \quad (4.66)$$

and

$$\tau_2 - S_2/L \leq C\varepsilon_0. \quad (4.67)$$

The fluctuations of the function $S_2 = S_2(\omega)$ for $\omega \in \Omega_1^*$ are bounded by

$$\sup_{\omega, \omega' \in \Omega_1^*} |S_2(\omega) - S_2(\omega')| \leq \frac{C \log L}{L^{1/\tau}}. \quad (4.68)$$

Proof. According to (2.29), the deterministic function $Y(\tau)$ satisfies

$$dY(\tau)/d\tau = F(Y, \tau), \quad Y(0) = 1/2. \quad (4.69)$$

Theorems 3.7 and 4.5 imply that for all $\omega \in \Omega_1^*$ the random trajectory satisfies

$$\partial Y_L(\tau, \omega)/\partial\tau = F(Y, \tau) + \chi(\tau, \omega), \quad Y_L(0, \omega) = 1/2, \quad (4.70)$$

with some

$$|\chi(\tau, \omega)| < \frac{C \log L}{L^{1/\tau}}.$$

We recall that $|\partial F(Y, \tau)/\partial Y| \leq \kappa$; see (2.30). Therefore, the difference

$$Z_L(\tau, \omega) := Y_L(\tau, \omega) - Y(\tau)$$

satisfies

$$|Z'_L(\tau, \omega)| \leq \kappa |Z_L(\tau, \omega)| + \frac{C \log L}{L^{1/\tau}}$$

and $Z_L(0, \omega) = 0$. Use of the standard Gronwall inequality in differential equations (see, for example, Lemma 2.1 in [26]) gives

$$|Z_L(\tau, \omega)| \leq \frac{C \log L}{\kappa L^{1/\tau}} (e^{\kappa\tau} - 1)$$

and

$$|Z'_L(\tau, \omega)| \leq \frac{C \log L}{L^{1/\tau}} e^{\kappa\tau}$$

for all $\tau < S_2/L$, which imply (4.64) and (4.65).

Next, (4.65) enables us to apply Proposition 4.1 with $\Delta = CL^{-1/\tau} \log L$ and thus prove (4.66). We now employ the same argument as in the proof of Theorem 3.8. By (4.65), the random fluctuations of the piston velocity are bounded by $CL^{-1/\tau} \log L$. Hence, the random fluctuations of the velocities of particles that have had one or two collisions with the piston are bounded by $4CL^{-1/\tau} \log L$. The random fluctuations of the positions of both piston and particles at every moment of time $t < \min\{\tau_2 L, S_2\}$ are bounded by the same quantities (possibly with a different value of C) in the coordinate $y = x/L$. This implies (4.68) in the same way as for (3.92) in Theorem 3.8. We can now apply Proposition 4.1 to the deterministic dynamics constructed in §2 with the same result, and thus prove (4.67). \square

The estimate (4.68) shows that the limit

$$\tau_{**} := \lim_{L \rightarrow \infty} S_2(\omega)/L$$

does not depend on $\omega \in \Omega_1^*$. We have $|\tau_{**} - 2/v_{\max}| \leq \text{const} \cdot \varepsilon_0$, due to (4.67) and the estimates in Lemma 2.9.

The convergence claimed in Theorem 1.2 is now proved on the interval $(0, \tau_{**})$, but generally $\tau_{**} < \tau_*$ with $\tau_* - \tau_{**} = O(\varepsilon_0)$, so we may still be $O(\varepsilon_0)$ short of our target value τ_* . To extend our results all the way to τ_* , we need to redefine more accurately S_2 and the neighbourhood \mathcal{X}_1 introduced by (4.4). We need to set

$$\begin{aligned} \mathcal{X}_1(t) = & \{(x, v) \in G^+(t) : x = X(t) + 0, v < 0\} \\ & \cup \{(x, v) \in G^+(t) : x = X(t) - 0, v > 0\}. \end{aligned} \tag{4.71}$$

The domain $G^+(t) = F^t(G^+)$ is random (it depends of ω), that is, we must write $G^+(t) = G^+(t, \omega)$, and this is why we could not adopt the above definition of $\mathcal{X}_1(t)$ earlier and opted for the cruder definition (4.4). But now, as we have just shown in the

proof of Theorem 4.6, the random fluctuations of the velocities and positions (in the y coordinate) of the particles and the piston are bounded by $\text{const} \cdot L^{-1/7} \log L$. Hence, at every time $t < \min\{\tau_2 L, S_2\}$ all the domains $G^+(t, \omega)$ are close to each other, namely, the distance between $G^+(t, \omega)$ and $G^+(t, \omega')$ in the Hausdorff metric on the coordinate plane y, v is bounded for $\omega, \omega' \in \Omega_1^*$ by $\text{const} \cdot L^{-1/7} \log L$. For the same reason, every domain $G^+(t, \omega)$ is $O(L^{-1/7} \log L)$ -close in the Hausdorff metric to the deterministic domain $\mathcal{G}^+(t/L)$ defined in §2. Hence, we can easily make $G^+(t)$ in (4.71) independent of ω , say, by taking the union

$$G^+(t) = \bigcup_{\omega \in \Omega_1^*} G^+(t, \omega). \quad (4.72)$$

It is important to note that all the gas particles colliding with the piston at time t for every $\omega \in \Omega_1^*$ are in the \mathcal{X}_1 defined by (4.71), (4.72). With this new definition of \mathcal{X}_1 replacing (4.4) and with S_2 changed accordingly, we have

$$\lim_{L \rightarrow \infty} |S_2/L - \tau_*| = 0,$$

which follows from the $O(L^{-1/7} \log L)$ -closeness of the random dynamics to the deterministic dynamics in the (y, v) -plane. Thus, we have extended all our results to the interval $(0, \tau_*)$.

§ 5. Beyond the second recollision

The main goal of our analysis in § 3 and § 4 is to prove that under suitable initial conditions the random fluctuations in the motion of a massive piston in a closed container filled with an ideal gas are small and vanish in the thermodynamic limit. However, we are able to control those fluctuations effectively only as long as the surrounding gas of particles can be described by a Poisson process, that is, during the zero-recollision interval $0 < \tau < \tau_1$. In that case the random fluctuations are bounded by $\text{const} \cdot L^{-1} \log L$; see Theorem 3.8. Up to the logarithmic factor, this bound is optimal, according to Theorem 1.1 due to Holley.

During the one-recollision interval $\tau_1 < \tau < \tau_*$, the situation is different. The probability distribution of gas particles that have experienced one collision with the piston is no longer a Poisson process, it has intricate correlations. We can only show that the random fluctuations remain bounded by $L^{-1/7}$; see Theorem 4.6. Our bound is possibly far from optimal, but numerical experiments discussed below demonstrate that the random fluctuations actually grow during the one-recollision interval and beyond.

At present we do not know whether our methods or results can be extended beyond the critical time τ_* , and this remains an open question. However, we emphasize that our Theorem 4.6 is the first rigorous treatment of the evolution of a piston in an ideal gas in which most or all of the particles experience more than one collision with the piston.

In order to understand what happens beyond the critical time τ_* , and in particular whether the random fluctuations grow or remain small, we undertook experimental and heuristic studies of the piston dynamics on a large time scale. Below

we describe our findings and discuss further research in this direction. A detailed account of our work can be found in [4].

We set the initial density of the gas to

$$\pi_0(y, v) = \pi_0(|v|) = \begin{cases} 1 & \text{if } 0.5 \leq |v| \leq 1, \\ 0 & \text{elsewhere.} \end{cases} \quad (5.1)$$

It satisfies our requirements (P1)–(P5); in particular, $v_{\min} = 0.5$ and $v_{\max} = 1$, and most importantly $\varepsilon_0 = 0$. Therefore, by Corollary 2.13, the solution of the hydrodynamic equations is trivial: $Y(\tau) \equiv 0.5$, $W(\tau) \equiv 0$, and $\pi(y, v, \tau) \equiv \pi_0(y, v)$ for all $\tau > 0$.

To generate an initial configuration of particles, we used a random number generator described in [23]. For our density (5.1), the x and v coordinates of all the particles are independent random variables uniformly distributed in their ranges $0 < x < L$ and $v_{\min} \leq |v| \leq v_{\max}$. Our computer program first selects the number N of particles according to the Poisson law with mean L^3 and then generates all (x_i, v_i) , $1 \leq i \leq N$, independently according to their uniform distributions. The parameter L varied in our simulations from $L = 30$ to $L = 300$. For $L = 300$ the system contains $\approx L^3 = 27,000,000$ particles.

Once the initial data is generated randomly, the program computes the dynamics by using the elastic collision rules (1.2)–(1.3). All calculations were performed in double precision, with coordinates and velocities of all particles stored and computed individually.

Fig. 6 presents a typical trajectory of the piston. Here $L = 100$. The position and time are measured in the hydrodynamic variables $Y = X/L$, $0 < Y < 1$, and $\tau = t/L$.

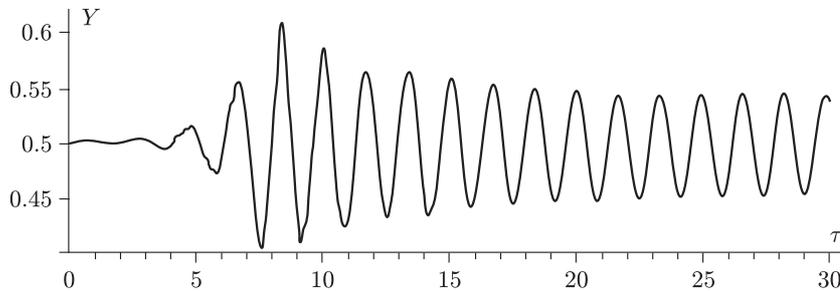


Figure 6. The piston coordinate Y as a function of the time τ . Here $L = 100$, $N = 1,000,229$

Initially, the piston barely moves about its stationary point $y = 0.5$. Then, at times τ between 3 and 5, the random vibrations of the piston grow and become quite visible on the y scale, but for a short while they look random, like a trajectory of a Brownian motion. After that the piston starts travelling back and forth along the y axis in a more regular manner, making excursions farther and farther away from the stationary point $y = 0.5$. Very soon, at $\tau = \tau_{\max} \approx 8$, the swinging motion of the piston reaches its maximum, $(\Delta Y)_{\max} = \max |Y_L(\tau) - 0.5| \approx 0.1$.

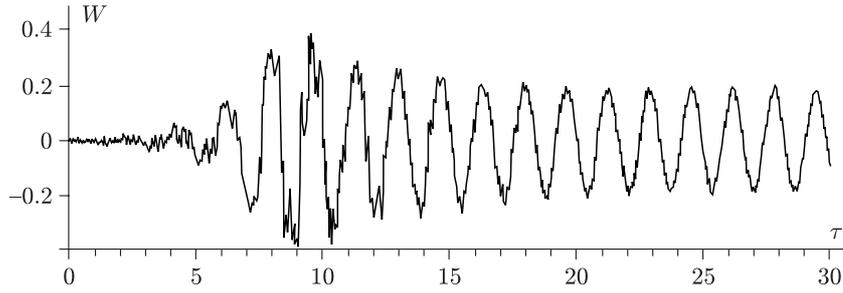


Figure 7. The piston velocity W as a function of the time τ . The same run as in Fig. 6

Then the oscillations of the piston dampen in size and seem to stabilize at an amplitude $A \approx 0.04$. At the same time the trajectory of the piston smoothes out and enters an oscillatory mode with period $\tau_{\text{per}} \approx 1.63$. The velocity $W(\tau)$ of the piston follows similar patterns; see Fig. 7.

Both the coordinate and velocity of the piston continue almost perfect harmonic oscillations for a long time with the same period $\tau_{\text{per}} \simeq 1.63$ (this is independent of L), but the amplitudes of both $Y(\tau)$ and $W(\tau)$ are slowly decreasing; see Fig. 8.

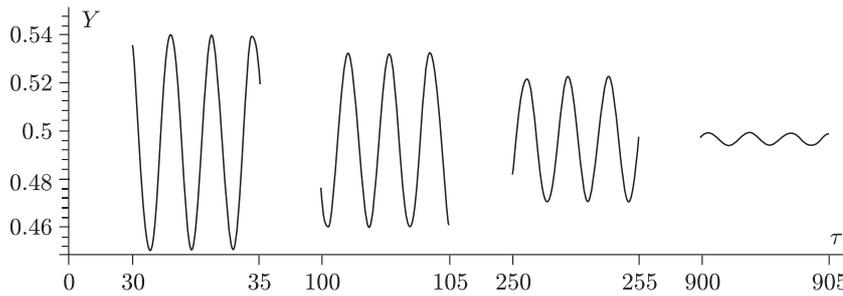


Figure 8. The piston coordinate Y in the intervals $(30, 35)$, $(100, 105)$, $(250, 255)$, and $(900, 905)$. The same run as in Figs. 6 and 7

The oscillations of the piston with decaying amplitude can be described approximately in the interval $20 < \tau < 1000$ by

$$Y(\tau) \approx Ae^{-\lambda(\tau-20)} \sin \omega(\tau - \alpha) \tag{5.2}$$

with $A = 0.046$ and some constant $\lambda > 0$. Correspondingly, $W = dY/d\tau$ in the same interval $20 < \tau < 1000$ has the form

$$\begin{aligned} W(\tau) &\approx -\lambda Y(\tau) + Ae^{-\lambda(\tau-20)} \omega \cos \omega(\tau - \alpha) \\ &= Ae^{-\lambda(\tau-20)} [-\lambda \sin \omega(\tau - \alpha) + \omega \cos \omega(\tau - \alpha)] \\ &= A_1 e^{-\lambda(\tau-20)} \sin \omega(\tau - \beta) \end{aligned} \tag{5.3}$$

with $A_1 = A\sqrt{\omega^2 + \lambda^2}$ and some β related to α .

To check how well our prediction (5.2) agrees with the experimental data, we computed the amplitude $A(\tau)$ as a function of time τ by fitting a sine function $Y(\tau) = A \sin \omega(\tau - \alpha)$ ‘locally’ on the interval $(\tau - 5, \tau + 5)$ for each τ . Fig. 9 shows $A(\tau)$ on the logarithmic scale, which looks almost linear on the interval $30 < \tau < 800$.

We used a least squares fit to estimate $\lambda = 0.00264$ for the run shown in Figs. 6–9. Since λ is small, the oscillations really die out very slowly. The ‘half-life’ time (the time it takes to reduce the amplitude by a factor of two) is $\tau_{1/2} = \lambda^{-1} \log 2 \approx 263$. The parameter λ , and hence $\tau_{1/2}$, depend on the system size L . We estimated numerically that $\tau_{1/2} \sim L^{1.3}$, and hence $\lambda \sim L^{-1.3}$.

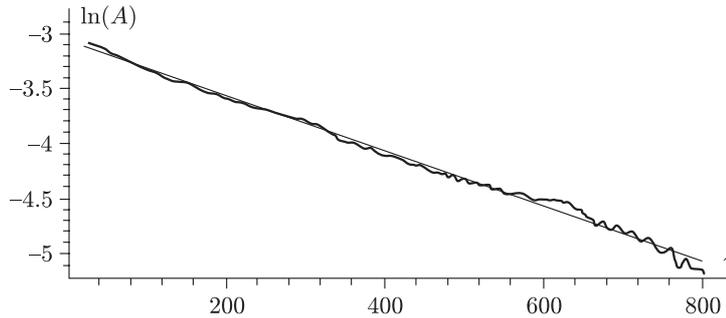


Figure 9. The amplitude $A(\tau)$ on the logarithmic scale: the experimental curve (bold) and a linear fit (thin). The same run as the one shown in Figs. 6, 7, and 8

The key characteristics of the piston trajectory described above, in particular, $(\Delta Y)_{\max}$, W_{\max} , A , and τ_{per} , appear to be independent of L . Even for $L = 300$ (the largest system tested experimentally) the piston experiences large oscillations very similar to the ones shown in Figs. 6 and 7. Some other quantities, such as $\tau_{1/2}$ and the related λ , depend in a systematic way on L .

But most importantly, the time τ_{\max} of the largest oscillations and the related time τ_c of the onset of instability (see below) seem to slowly grow with L , very likely as $\log L$. To understand this, we looked into the mechanism of the build-up of random fluctuations of the piston position and velocity displayed in Figs. 6 and 7. To this end, we plotted the histogram of the (empirical) density of gas particles in the (y, v) -plane at various times $0 < \tau < 30$; see samples in Fig. 10. The initial density (at time zero) is almost uniform over the domain $0 < x < L$ and $v_{\min} \leq |v| \leq v_{\max}$ (variations in the initial configuration always exist, because it is generated randomly). Then for $0 < \tau < 1$ the piston experiences random collisions with particles and acquires a speed of order $M^{-1/2} = O(1/L)$; see Theorem 1.1. These small fluctuations of the piston velocity result in bigger changes of the velocities of the particles which leave the piston after collisions according to the rule (1.3). In particular, the outgoing particles on the right-hand side of the piston have velocities in the interval $(v_{\min} + 2W(\tau), v_{\max} + 2W(\tau))$ while those on the left-hand side of the piston have velocities in the interval $(-v_{\min} + 2W(\tau), -v_{\max} + 2W(\tau))$. Hence, the region in the (y, v) -plane where the density of the particles is positive is no longer a rectangle with straight sides, and now its boundaries are curves whose

shape nearly repeats the graph of the randomly evolving piston velocity $W(\tau)$. While the variations of order $O(1/L)$ of these boundary curves may seem small, it is crucial that on opposite sides of the piston they go in opposite directions. Indeed, when $W(\tau) > 0$, the outgoing particles on the right-hand side accelerate and those on the left-hand side slow down. The opposite happens for $W(\tau) < 0$.

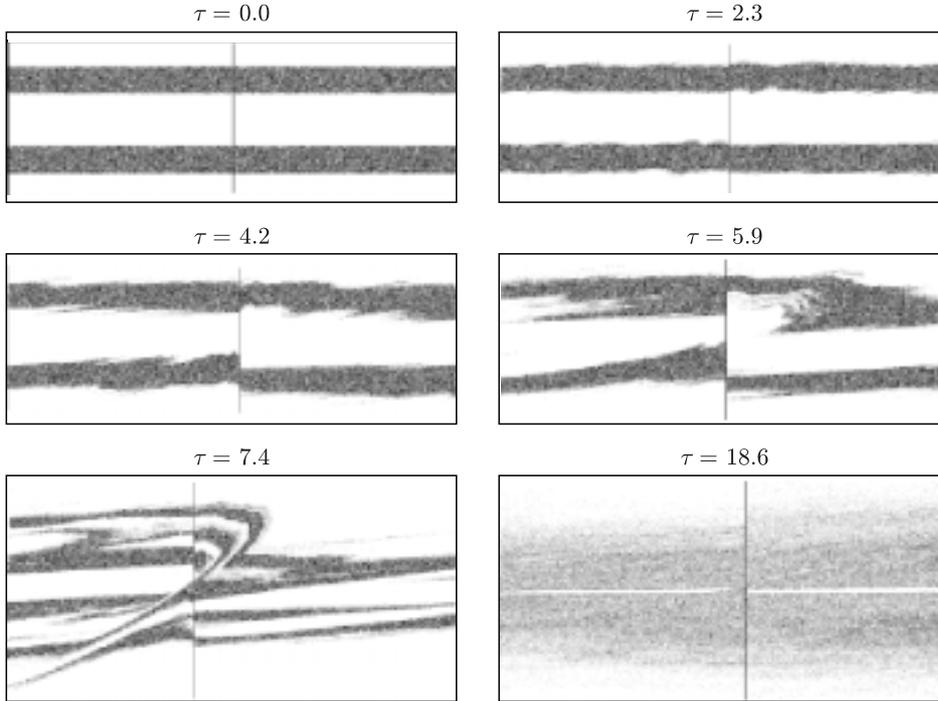


Figure 10. Six snapshots of the empirical gas density (in the (x, v) -plane) at times $\tau = 0, 2.3, 4.2, 5.9, 7.4,$ and 18.6

Next, the particles that have collided with the piston travel to the wall and come back to the piston. Their densities are now less regular than they were initially, namely, the regions in the (x, v) -plane in which the density is positive are curvilinear domains. When the particles hit the piston, they shake it back and forth more forcefully than before, because the velocities of the incoming particles on the opposite sides of the piston are now negatively correlated. When particles on the right-hand side are fast, those on the left-hand side are slow, and vice versa. The fluctuations of the gas densities on the different sides of the piston thus ‘cooperate’ to push the piston harder, with ‘double’ force. This produces a resonance-type effect destabilizing the piston dramatically, and the velocity of the piston $W(\tau)$ experiences larger fluctuations than before. The velocities of the newly outgoing particles go up and down in opposite direction again, on a greater scale than before.

As time goes on, the above phenomenon repeats over and over, with larger and larger fluctuations of the gas and piston velocities, until the distribution of gas particles completely breaks down. For $L = 100$ two large clusters of particles are

formed at times $\tau \sim 10$, one on each side of the piston. When one cluster bombards the piston, the other moves away from it and hits the wall, and then they exchange roles. The clusters have sizes of about 0.3–0.5 in the y direction, and the particle velocities range from about 0.2 to just over 1. The average velocity is about 0.5–0.6, and so the clusters hammer the piston periodically with period 1.6–2.0, which is close to the experimentally determined period of piston oscillations; see above.

Fig. 10 shows six snapshots of the empirical density of gas particles taken at different times. At $\tau = 0$ the gas fills (almost uniformly) two rectangles $\{(y, v) : 0.5 < |v| < 1, 0 < y < 1\}$. At $\tau = 2.3$ one can see some ripples on the boundaries of these rectangles. At time $\tau = 4.2$ the irregularities grow, and by $\tau = 5.9$ the rectangular shape has broken down. Two large clusters of particles are formed, both in the upper half-plane $v > 0$, that is, at that time both clusters move to the right (one toward the piston, the other away from it). The density later undergoes strange formations ($\tau = 7.4$), but eventually smoothes out and enters a slow process of convergence to the Maxwellian distribution ($\tau = 18.6$) described below.

The above analysis suggests that the fluctuations of the piston velocity roughly increase by a constant factor during each time interval of length one. Indeed, the initial random fluctuations $W_a \sim O(1/L)$ result in additional changes of the velocities of the outgoing particles by $2W_a$. When those particles come back to the piston (in a time $\Delta\tau \approx 1$), they kick its velocity to the level of $2W_a$. Then the newly outgoing particles acquire an additional velocity $4W_a$, and so on. Over each time interval of length one the fluctuations double in size. This is an obvious oversimplification of the real dynamics, but it leads to a reasonable conjecture:

$$W_a(\tau) \approx \frac{CR^\tau}{L}, \quad (5.4)$$

where $W_a(\tau)$ are typical fluctuations of the piston velocity at time τ and $C, R > 0$ are constants.

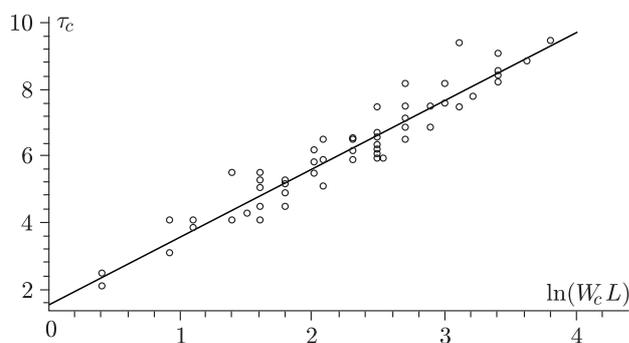


Figure 11. The value τ_c as a function of $\log(W_c L)$: experimental points and a linear fit

We tested the above formula numerically as follows. Let $W_c > 0$ be some preset critical value of the piston speed and let $\tau_c = \inf\{\tau > 0 : |W(\tau)| > W_c\}$ be the

(random) time when W_c is first exceeded. This time plays the role of the ‘onset’ of large fluctuations of the piston velocity. One would expect (by (5.4)) that

$$\tau_c \approx \frac{\log(W_c L/C)}{\log R}, \quad (5.5)$$

that is, τ_c grows as $\log L$ if L increases.

We found τ_c experimentally for $W_c = 0.1$ and $W_c = 0.15$ and checked that (5.5) agreed well with the data; see Fig. 11. By a least squares fit we estimated $C = 0.45$ and $R = 1.6$.

To summarize our experimental observations, we conclude that the random fluctuations of the density function $p_L(y, v, \tau)$ and the piston coordinate $Y_L(\tau)$ grow exponentially in time τ , and at times $\tau \sim \log L$ they become large even on a macroscopic scale. At that point the evolution of the system deviates far from the solution of the hydrodynamic equations (H1)–(H4), and they become completely separated afterwards.

Interestingly, our observations do not indicate that the convergence (1.12)–(1.13) claimed in Theorem 1.2 fails on any interval of time. In fact, if the random fluctuations behave like CR^τ/L as predicted by (5.4), then (1.12)–(1.13) must hold as $L \rightarrow \infty$ on *every* finite interval $(0, \tau_*)$. However, a rigorous proof of this would be a very challenging task.

Next, we also examined numerically and heuristically how the system behaves asymptotically as $\tau \rightarrow \infty$. On physical grounds [1], we expect the system to approach thermal equilibrium (see § 1), that is, the velocity distribution of gas particles must converge to a Maxwellian distribution.

We used the Kolmogorov–Smirnov statistical test to verify the convergence of the velocity distribution to a normal law. At any given time $\tau > 0$ let

$$F_\tau(u) = \#\{i : v_i < u\}/N$$

be the empirical (cumulative) distribution function of particle velocities. For the corresponding normal distribution function $\Phi(x)$ we compute

$$D_\tau = \sup_{-\infty < u < \infty} |F_\tau(u) - \Phi(u)|.$$

We initially have $D_0 \approx 0.245$ for our choice of $\pi_0(v)$ in (5.1). If the velocities v_i were independent normal random variables, then D_τ would be of order $O(1/\sqrt{N})$, and the product $D_\tau\sqrt{N}$ would have a certain limit distribution; see, for example, [22]. In particular, it is known that $P(D_\tau\sqrt{N} > 1) \approx 0.2$. Using this fact, we opted to define the time of convergence to equilibrium by

$$\tau_{\text{eq}} = \inf\{\tau > 0 : D_\tau\sqrt{N} < 1\}. \quad (5.6)$$

We estimated τ_{eq} for different values of L and found that $\tau_{\text{eq}} \approx aL^b$ with some constants $a, b > 0$. By a least squares fit to experimental points we found $a = 0.18$ and $b = 2.47$.

The plot of the product $S = D_\tau\sqrt{N}$ versus τ is given in Fig. 12 (for a particular run with $L = 40$). It shows that, after an initial sharp drop over the period

$0 < \tau < 20$, the statistic S decreases exponentially in τ . Another commonly used (and popular among experimentalists) statistic to measure closeness to a normal distribution is

$$S' = 3 - \frac{M_4}{M_2^2},$$

where M_2 and M_4 are the second and the fourth sample moments of the empirical velocity distribution, respectively. Fig. 12 shows that S' converges to zero in a similar way (for the same run with $L = 40$).

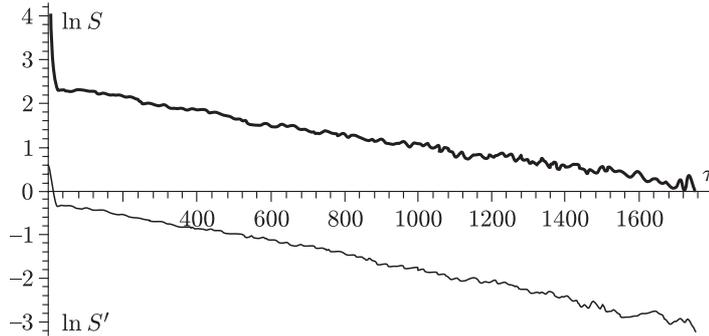


Figure 12. $\log S$ (thick line) and $\log S'$ (thin line) as functions of τ

The convergence to a thermal equilibrium can also be justified mathematically. If one fixes three integrals of motion, namely, the total energy E and the numbers N_L and N_R of particles in the left and right compartments, then the dynamics can be reduced to a billiard system in a high-dimensional polyhedron by standard techniques, as we show next.

Let $\{x_i\}$, $i = 1, \dots, N_R$, be the x -coordinates of the particles to the right of the piston, and $\{x_i\}$, $i = -1, \dots, -N_L$, those to the left of it (ordered arbitrarily). We set $x_0 = X\sqrt{M}$, where X is the coordinate of the piston and M is its mass. Then the configuration space of the system (in the coordinates x_i , $-N_L \leq i \leq N_R$) is a polyhedron $Q \subset \mathbb{R}^{N+1}$ (we recall that $N = N_L + N_R$) defined by inequalities

$$0 \leq x_{-N_L}, \dots, x_{-1} \leq x_0/\sqrt{M} \leq x_1, \dots, x_{N_R} \leq L.$$

It is known that the dynamics of our mechanical system corresponds to the billiard dynamics in Q ; see [6]. That is, the configuration point $\mathbf{q} \in Q$ moves freely and experiences perfectly elastic reflections at the boundary ∂Q . The velocity vector

$$\mathbf{p} = \dot{\mathbf{q}} = \{v_{-N_L}, \dots, v_{-1}, V\sqrt{M}, v_1, \dots, v_{N_R}\}$$

has constant length, since $\|\mathbf{p}\|^2 = 2E = \text{const}$. Therefore, the phase space of the billiard system is $\mathcal{M} = Q \times S_\rho^N$, where S_ρ^N is the N -dimensional sphere of radius $\rho = \sqrt{2E}$.

The billiard system has the Liouville invariant measure μ on \mathcal{M} , which is the product of a uniform measure on the polyhedron Q and a uniform (Lebesgue)

measure on the sphere S_ρ^N , that is, $d\mu = dq dp$. The properties of billiard dynamics depend heavily on the curvature of the boundary ∂Q . In our case Q is a polyhedron, and hence its boundary consists of flat sides with zero curvature. A prototype of such systems is a billiard ball in a polygon. The following assertion is well known (see, for example, [5]).

Fact. For billiards in polygons and polyhedra (and hence for our mechanical model of a piston in the ideal gas) all Lyapunov exponents vanish, and so does the measure-theoretic (Kolmogorov–Sinai) entropy.

Systems with zero Lyapunov exponents and zero entropy are not regarded as truly chaotic, but they may still be ergodic. In fact, billiards in generic polygons *are* ergodic [14]. Moreover, for many non-ergodic polygons the phase space is foliated by invariant subsurfaces on which the dynamics is ergodic.

Even though there are no similar results, to our knowledge, for billiards in high-dimensional polyhedra, one can expect that they have similar properties. That is, they are generically ergodic or become ergodic after trivial reductions. In our case, the billiard ball in Q is perhaps ergodic for typical values of M , or else the phase space is foliated by invariant submanifolds on which the dynamics is ergodic, and those submanifolds fill \mathcal{M} sufficiently densely. In the latter case, one could hardly distinguish experimentally between such a non-ergodic system and a truly ergodic one.

Hence, we can assume that our system is ergodic or very close to ergodic in the above sense. Then almost every trajectory eventually behaves according to the invariant measure μ , independently of the initial state. In particular, for any initial gas density and velocity distribution (given by the function $\pi_0(y, v)$; see § 1) the hydrodynamic regime for a finite L is valid only on a finite interval of time, and the system eventually relaxes to a thermal equilibrium. We expect in fact that in terms of the ‘macroscopic’ variables, say, the one-particle distribution function, the system relaxes to an effective equilibrium in the time τ_{eq} defined by (5.6), which is much less than the exponentially long time (with respect to L) required for the ergodic theorem. Thus, the real question is how this time depends on L . According to our earlier discussion, $\tau_c \sim \log L$ and $\tau_{\text{eq}} \sim L^{5/2}$.

At equilibrium, the distributions of coordinates x_i and velocities v_i are determined by the Liouville measure μ , which is uniform in the phase space. Its marginal measures, that is, the projections on lower-dimensional subspaces, are physically interesting (and only they are observable). The marginal measures of the velocities are approximately normal for large N .

In particular, each individual velocity v_i converges in distribution to a Maxwellian (that is, normal) random variable with zero mean and variance $2E/N = \text{const}$. The same holds for the ‘piston’ component of the velocity, $\dot{x}_0 = V\sqrt{M}$, and hence the piston velocity V is normally distributed with zero mean and standard deviation $\text{const}/\sqrt{M} = \text{const}/L$ as $L \rightarrow \infty$. In our case V has the standard deviation $\sqrt{7/24}/L \approx 0.5/L$. This conclusion agrees well with Holley’s theorem (Theorem 1.1) and our numerical data.

The equilibrium distribution of the piston coordinate X is also determined by the projection of the uniform measure dq on Q on the x_0 axis. Before we do that, let us get rid of M in the definition of both Q and x_0 . A simple change of variable

$X = x_0/\sqrt{M}$ allows us to redefine Q by

$$0 \leq x_{-N_L}, \dots, x_{-1} \leq X \leq x_1, \dots, x_{N_R} \leq L.$$

Furthermore, rescaling $Y = X/L$ and $y_i = x_i/L$ gives a simpler definition of Q :

$$0 \leq y_{-N_L}, \dots, y_{-1} \leq Y \leq y_1, \dots, y_{N_R} \leq 1.$$

This is a variation of the so-called Brownian bridge. ‘Integrating away’ the variables y_i gives us for Y the equilibrium density

$$f(Y) = cY^{N_L}(1 - Y)^{N_R} \tag{5.7}$$

for $0 < Y < 1$, where c is a normalizing factor that can be computed explicitly:

$$c^{-1} = \int_0^1 Y^{N_L}(1 - Y)^{N_R} dY = \frac{N_L! N_R!}{(N_L + N_R + 1)!}. \tag{5.8}$$

Asymptotically as $L \rightarrow \infty$ we have $N_L \sim L^3/2$ and $N_R \sim L^3/2$. We assume for simplicity that $N_L = N_R = N/2$ and write $K = N/2$; then

$$c = \frac{(2K + 1)!}{(K!)^2} \simeq \frac{2 \cdot 4^K \sqrt{K}}{\sqrt{\pi}}.$$

Let $z = (Y - 0.5)\sqrt{8K}$. Then the density of z is given asymptotically by

$$\begin{aligned} f(z) &= \frac{c}{\sqrt{8K}} \left(0.5 + \frac{z}{\sqrt{8K}}\right)^K \left(0.5 - \frac{z}{\sqrt{8K}}\right)^K \\ &= \frac{c}{4^K \sqrt{8K}} \left(1 - \frac{z^2}{2K}\right)^K \approx \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \end{aligned}$$

Thus, Y is asymptotically Gaussian with mean 0.5 and variance $(4N)^{-1} = (4L^3)^{-1}$. Therefore, in equilibrium

$$|Y - 0.5| \sim \frac{1}{2L\sqrt{L}} \sim \frac{1}{2\sqrt{N}}.$$

We note that this estimate is independent of the piston mass.

Next, by using (5.7) one can easily compute the probability that the piston coordinate Y deviates from its mean value 0.5 by a fixed amount $d > 0$, say $d = 0.1$: this probability is $< \text{const} \cdot e^{-aN}$ with some $a = a(d) > 0$. Therefore, we have observed experimentally a very rare event whose probability is exponentially small in N . If we started with an equilibrium state, then such an observation would be practically impossible. But we started with a state described by the density (5.1), which itself has a probability less than $\text{const} \cdot e^{-bN}$, $b > 0$, with respect to the equilibrium measure μ . So we have only observed how one highly improbable initial state evolved to another highly improbable state along its (very slow) transformation to an equilibrium state. It would be interesting to understand

why the system ‘chooses’ such a strange evolution to equilibrium, that is, why, starting with a state described by a ‘double-peaked’ distribution (5.1), the system behaves as a damped harmonic oscillator over an extended time interval, with initial oscillations as large as $1/10$ of the system size.

We are currently working on this problem and shall report results in a separate paper [2]. Below we outline our program and mention some preliminary findings. For simplicity we assume that $\pi_0(y, v) = \pi_0(|v|)$ and $X(0) = L/2$, $V(0) = 0$. Corollary 2.13 ensures that the hydrodynamic equations have the trivial solution $Y(\tau) \equiv 0.5$, $W(\tau) \equiv 0$, and $\pi(y, v, t) \equiv \pi_0(y, v)$ for all $\tau > 0$. Since the initial configuration of particles is generated randomly from a Poisson process with density $\pi_0(y, v)$, the ‘actual’ (empirical) density of the particles, such as the one shown on Fig. 10 at $\tau = 0$, does not coincide exactly with $\pi_0(y, v)$. The random fluctuations of the empirical density are typically of order $O(1/L)$. Hence, the ‘actual’ initial distribution of particles can be thought of as a small perturbation of the function $\pi_0(y, v) = \pi_0(|v|)$ and can be written as $\pi_0(|v|) + \varepsilon\pi_1(y, v)$ with $\varepsilon = 1/L$ and some (random) function $\pi_1(y, v)$ of order one.

We conjecture now that the evolution of the mechanical system closely follows the solutions of the hydrodynamic equations (H1)–(H4) with a perturbed initial density $\pi_0(|v|) + \varepsilon\pi_1(y, v)$ rather than the stationary solution corresponding to the unperturbed density $\pi_0(|v|)$. This accounts for significant differences between the behaviour of the mechanical system and the stationary solution if the latter is unstable. In particular, two trajectories which are initially ε -close (in our case $\varepsilon = 1/L$) can deviate from each other exponentially fast in time τ , and look completely different at times $\tau \sim -\log \varepsilon = \log L$. This would be in agreement with our experimental observations and the estimate (5.5) of the time $\tau_c \sim \log L$ of the onset of ‘instability’.

To test our conjecture, we solved the hydrodynamic equations (H1)–(H4) numerically starting with a perturbed initial density obtained by adding to (5.1) a function ε -small in the L^1 metric (with $\varepsilon \simeq 0.01$). We found that the corresponding solution resembled strikingly well the evolution of the mechanical system described above. In particular, the coordinate and velocity of the piston followed large nearly harmonic oscillations during the interval $10 < \tau < 30$. The corresponding plots of the piston position and velocity along the perturbed solutions of (H1)–(H4) were almost indistinguishable from our Figs. 6 and 7. Hence, the behaviour of the mechanical system can be traced to that of the perturbed solutions of the hydrodynamic equations, and the instability of the latter becomes an important issue.

When we were finishing the present paper, we received a message from E. Caglioti and E. Presutti, who (a) proved that the solutions of the hydrodynamic equations (H1)–(H4) are stable when the function $\pi_0(|v|)$ is monotonically non-increasing in $|v|$, that is, $\pi_0'(|v|) \leq 0$, and (b) suggested that they might be unstable for our class of non-monotone functions $\pi_0(|v|)$. We checked the suggestion (b) for our particular density (5.1) and found that it was indeed correct; we proved that small perturbations grow exponentially in τ .

Conversely, when we simulated the particle dynamics with a non-increasing initial density $\pi_0(|v|)$, the oscillations essentially disappeared (to this end, we tried a uniform ‘flat’ function given by $\pi_0(|v|) = 1$ for $|v| \leq v_{\max}$ and a triangular one

$\pi_0(|v|) = 1 - |v|/v_{\max}$). On the other hand, the particle velocity distribution still approached a Maxwellian distribution, albeit at a somewhat slower pace.

Finally, we describe some other open problems related to the piston dynamics.

1. It is clear that recollisions of gas particles with the piston have a very ‘destructive’ effect on the dynamics in the system. However, we need to distinguish between two types of recollisions.

We say that a recollision of a gas particle with the piston is *long* if the particle hits a wall $x = 0$ or $x = L$ between the two consecutive collisions with the piston. Otherwise a recollision is said to be *short*. Long recollisions require some time, as the particle has to travel all the way to a wall, bounce off it, and then travel back to the piston before it hits it again. Short recollisions can occur in rapid succession.

We have imposed the velocity cutoff (P4) in order to avoid any recollisions for at least some initial period of time (which we called the zero-recollision interval). More precisely, the upper bound v_{\max} guarantees the absence of long recollisions. Without it, we would have to deal with arbitrarily fast particles that dash between the piston and the walls many times in any interval $(0, \tau)$. On the other hand, the lower bound v_{\min} was assumed to exclude short recollisions.

However, there are good reasons to believe that short recollisions may not be so destructive for the piston dynamics. Indeed, let a particle experience two or more collisions with the piston in rapid succession (that is, without hitting a wall in between). This can occur in two cases: (i) the particle’s velocity is very close to that of the piston, or (ii) the piston’s velocity changes very rapidly. The latter should be very unlikely, since the deterministic acceleration of the piston is very small; cf. Theorem 2.12c. In case (i), the recollisions should have very little effect on the velocity of the piston according to the rule (1.2), so that they may be safely ignored, as it was done already in some earlier studies [13], [8].

We therefore expect that our results can be extended to velocity distributions without a cutoff from zero, that is, allowing $v_{\min} = 0$.

2. In our paper, L plays a dual role: it parameterizes the mass of the piston ($M \sim L^2$), and it represents the length of the container ($0 \leq x \leq L$). This duality comes from our assumption that the container is a cube.

However, our model is essentially one-dimensional, and the mass M of the piston and the length of the interval $0 \leq x \leq L$ can be treated as two independent parameters. In particular, we can assume that the container is infinitely long in the x direction (in other words, *this* L is infinite), but the mass of the piston is still finite and given by $M \sim L^2$ (*this* L is the size of the container in the y and z directions). In this case there are no recollisions with the piston, as long as its velocity remains small. Hence, our zero-recollision interval is effectively infinite. As a result, Theorem 1.2 can be extended to arbitrarily large times. More precisely, for any $T > 0$ we can prove the convergence in probability:

$$\mathbb{P}\left(\sup_{0 \leq \tau \leq T} |Y_L(\tau, \omega) - Y(\tau)| \leq C_T \frac{\log L}{L}\right) \rightarrow 1$$

and

$$\mathbb{P}\left(\sup_{0 \leq \tau \leq T} |W_L(\tau, \omega) - W(\tau)| \leq C_T \frac{\log L}{L}\right) \rightarrow 1$$

as $L \rightarrow \infty$, where $C_T > 0$ is a constant, and $Y(\tau)$ and $W(\tau) = \dot{Y}(\tau)$ are the solutions of the hydrodynamic equations described in § 2.

3. Along the same lines as above, we can assume that the container is d -dimensional with $d \geq 4$. Then the mass of the piston and the density of the particles are proportional to L^{d-1} rather than L^2 .

When d is large, the gas particles are very dense in the (x, v) -plane. This leads to a much better control over fluctuations of the particle distribution and the piston trajectory. During the zero-recollision interval, for example, the piston trajectory is $L^{-(d-1)/2}$ -close to its deterministic trajectory. This is an easy modification of the results of our § 3. During the one-recollision interval, the piston trajectory is $L^{-(2d-5)/7}$ -close to its deterministic trajectory. This can be shown with the methods developed in § 4 but requires some extra work. Moreover, the methods and results of that section can be extended to the k -recollision interval (τ_k, τ_{k+1}) for any $k \geq 1$. It can be shown that there is a $d_k \geq 3$ such that for all $d \geq d_k$ we have

$$\mathbf{P}\left(\sup_{\tau_k < \tau < \tau_{k+1}} |W_L(\tau, \omega) - W(\tau)| \leq L^{-b}\right) \rightarrow 1$$

as $L \rightarrow \infty$, where $b > 0$ depends on k and d . However, this extension requires quite different methods, which are beyond the scope of this article. The upshot is that a high-dimensional piston is more stable than a lower-dimensional one.

It would be interesting to investigate other modifications of our model that lead to more stable regimes. For example, let the initial density $\pi_0(y, v)$ of the gas depend on the factor $a = \varepsilon L^2$ in such a way that $\pi_0(y, v) = a^{-1} \rho(y, v)$, where $\rho(y, v)$ is a fixed function. Then the particle density grows as $a \rightarrow 0$. This is another way to increase the density of the particles, but without changing the dimension. One may expect better control over random fluctuations in this case, too.

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Appendix

In this section we derive various probabilistic estimates of the distribution of gas particles and their velocities. The number $K = N_D$ of particles in any domain D in the (x, v) -plane at time $t = 0$ is a Poisson random variable, and we need bounds for its large deviations. Perhaps some of our estimates are known in probability theory, but we include proofs for the sake of completeness.

Lemma A.1. *Let K be a Poisson random variable with parameter $\lambda > 0$. Then for any $A > \lambda$*

$$\mathbf{P}(K > A) \leq e^{A-\lambda-A \log(A/\lambda)}$$

and for any $A < \lambda$

$$\mathbf{P}(K < A) \leq e^{A-\lambda-A \log(A/\lambda)}.$$

Proof. The moment generating function of K is

$$\varphi_K(t) = \mathbf{E}(e^{tK}) = e^{\lambda(e^t-1)}.$$

First, let $A > \lambda$. Then, obviously, for all $t > 0$

$$\varphi_K(t) \geq e^{At} \mathbf{P}(K > A).$$

Hence, for all $t > 0$

$$\mathbf{P}(K > A) \leq e^{\lambda(e^t-1)-At}.$$

The expression on the right-hand side takes its minimum at

$$t = \log(A/\lambda) > 0.$$

This proves the first part of the lemma.

Now let $A < \lambda$. Then for all $t < 0$

$$\varphi_K(t) \geq e^{At} \mathbf{P}(K < A).$$

Hence, for all $t < 0$

$$\mathbf{P}(K < A) \leq e^{\lambda(e^t-1)-At}.$$

The expression on the right-hand side takes its minimum at

$$t = \log(A/\lambda) < 0.$$

This proves the second part. \square

Lemma A.2. *Let K be a Poisson random variable with parameter $\lambda > 0$. For any $b > 0$ there is a $c > 0$ such that for all $0 < B < b\sqrt{\lambda}$*

$$\mathbf{P}(|K - \lambda| > B\sqrt{\lambda}) \leq 2e^{-cB^2}.$$

Proof. A direct application of the previous lemma gives

$$\mathbf{P}(|K - \lambda| > B\sqrt{\lambda}) \leq 2e^{-B^2g(q)}, \tag{A.1}$$

where

$$g(q) = \frac{(1+q)\log(1+1)-q}{q^2} = \frac{\int_0^q \log(1+s) ds}{2 \int_0^q s ds} \tag{A.2}$$

and $q = B/\sqrt{\lambda}$ in (A.1). By direct inspection one can verify that the function $g(q)$ is a positive and strictly monotonically decreasing function on the interval $0 < q < \infty$. We complete the proof by setting $c = g(b)$. \square

Lemma A.3. *Let $\lambda_0 > 0$ and $a > 0$. Then for all sufficiently large $L > 0$ and every Poisson random variable K with parameter $\lambda \geq \lambda_0$*

$$\mathbf{P}(|K - \lambda| > a\sqrt{\lambda} \log L) \leq L^{-d \log \log L},$$

where $d = a\sqrt{\lambda_0}/2$.

Proof. Using (A.1) with $B = a \log L$ gives

$$\mathbf{P}(|K - \lambda| > a\sqrt{\lambda} \log L) \leq 2e^{-(a \log L)^2 g(a \log L / \sqrt{\lambda_0})},$$

with $g(q)$ defined by (A.2). We note that for large L

$$g\left(\frac{a \log L}{\sqrt{\lambda_0}}\right) \sim \frac{\sqrt{\lambda_0} \log \log L}{a \log L}.$$

This completes the proof. \square

Remark. In most of our applications λ is large, of order $\lambda \sim L^b$ with some $b > 0$. Hence, the factor $\log L$ is small compared with $\sqrt{\lambda}$.

Corollary A.4. *Let $\lambda_0 > 0$ and $a > 0$. Let K be a Poisson random variable with parameter $\lambda \leq \lambda_0$. Then for all sufficiently large $L > 0$*

$$\mathbb{P}(K > a \log L) \leq L^{-d \log \log L},$$

where $d = a/2$.

Proof. The case $\lambda = \lambda_0$ follows easily from the previous lemma. And if $\lambda < \lambda_0$, then the event $K > a \log L$ is even less likely than for $\lambda = \lambda_0$. \square

Next, we need to study another random variable related to a Poisson process. For any domain D in the (x, v) -plane we consider the sum of the velocities

$$Z = Z_D = \sum_{(x,v) \in D} v$$

of the particles in D at time 0. We assume that $D \subset \{v_{\min} < v < v_{\max}\}$ (the case $D \subset \{-v_{\max} < v < -v_{\min}\}$ is completely symmetric and analogous). By projecting D on the v axis we obtain a Poisson process on the interval

$$I = (v_{\min}, v_{\max})$$

with density

$$\pi(v) = L^2 \int_{D \cap \{u=v\}} p_L(x, u) dx.$$

The random variable Z can now be described as follows.

We consider a one-dimensional Poisson process with density $\pi(v)$ on the interval I . This means that for any subinterval $J \subset I$ the number N_J of points in J is a Poisson random variable with mean

$$\mathbb{E}(N_J) = \int_J \pi(v) dv.$$

Each realization ω of this process is a finite subset of I . Let us consider the random variable

$$Z(\omega) = \sum_{v \in \omega} v.$$

We call Z an *integrated Poisson random variable*.

If we fix a large $n \geq 1$ and partition I into small intervals

$$\Delta_i = I \cap \left[\frac{i}{n}, \frac{i+1}{n} \right),$$

$i = 0, 1, 2, \dots$, then we can obviously bound Z by

$$\sum_i \frac{i}{n} N_{\Delta_i} \leq Z < \sum_i \frac{i+1}{n} N_{\Delta_i}, \quad (\text{A.3})$$

where N_{Δ_i} is the number of points of the process in the interval Δ_i . We note that the N_{Δ_i} are independent Poisson random variables with parameters

$$\lambda_i = \mathbf{E}(N_{\Delta_i}) = \int_{\Delta_i} \pi(v) dv.$$

Therefore, the moment generating function $\varphi_Z(t) = \mathbf{E}(e^{tZ})$ of Z is bounded by

$$\exp\left[\sum_i \lambda_i(e^{t\frac{i}{n}} - 1)\right] \leq \mathbf{E}(e^{tZ}) < \exp\left[\sum_i \lambda_i(e^{t\frac{i+1}{n}} - 1)\right].$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\varphi_Z(t) = \mathbf{E}(e^{tZ}) = \exp\left[\int_I (e^{tv} - 1)\pi(v) dv\right]. \tag{A.4}$$

By using (A.3), it is also easy to find the mean value

$$\mathbf{E}(Z) = \mu_Z = \int_I v\pi(v) dv \tag{A.5}$$

and the variance

$$\mathbf{D}(Z) = \sigma_Z^2 = \int_I v^2\pi(v) dv. \tag{A.6}$$

We note that Z is connected with the Poisson random variable $K = N_I$ with parameter

$$\lambda_Z = \int_I \pi(v) dv.$$

In particular, we have

$$v_{\min}K \leq Z \leq v_{\max}K, \tag{A.7}$$

and hence

$$v_{\min}\lambda_Z \leq \mu_Z \leq v_{\max}\lambda_Z. \tag{A.8}$$

We also have

$$v_{\min}^2\lambda_Z \leq \sigma_Z^2 \leq v_{\max}^2\lambda_Z. \tag{A.9}$$

The random variable Z admits bounds for large deviations similar to the ones we found for Poisson random variables.

Lemma A.5. *For any $b > 0$ there is a $c > 0$ (determined by b , v_{\min} , and v_{\max}) such that for any integrated Poisson random variable Z and all $0 < B < b\sigma_Z$*

$$\mathbf{P}(|Z - \mu_Z| > B\sigma_Z) \leq 2e^{-cB^2}.$$

Proof. For brevity, we write $\mu = \mu_Z$ and $\sigma = \sigma_Z$. We claim that

$$\mathbf{P}(Z > \mu + B\sigma) \leq e^{-cB^2} \tag{A.10}$$

(the same bound for $\mathbf{P}(Z < \mu - B\sigma)$ can be proved in a similar way, as in Lemma A.1). For all $t > 0$ we have

$$\varphi_Z(t) \geq e^{(\mu+B\sigma)t} \mathbf{P}(Z > \mu + B\sigma),$$

and hence

$$\mathbf{P}(Z > \mu + B\sigma) \leq \exp \left[\int_I (e^{tv} - 1) \pi(v) dv - (\mu + B\sigma)t \right].$$

We substitute $t = Bs/\sigma$ with $s > 0$ to be chosen later and we expand e^{tv} in a Taylor series,

$$\mathbf{P}(Z > \mu + B\sigma) \leq \exp \left[\int_I \left(\sum_{n=1}^{\infty} \frac{(Bsv)^n}{\sigma^n n!} \right) \pi(v) dv - \frac{Bs\mu}{\sigma} - B^2 s \right].$$

The first two terms with $n = 1$ and $n = 2$ give

$$\int_I \frac{Bsv}{\sigma} \pi(v) dv = \frac{Bs\mu}{\sigma}$$

by (A.5) and

$$\int_I \frac{(Bsv)^2}{2\sigma^2} \pi(v) dv = \frac{B^2 s^2}{2}$$

by (A.6), respectively.

Therefore,

$$\mathbf{P}(Z > \mu + B\sigma) \leq \exp \left[-B^2 \left(s - \frac{s^2}{2} - s^2 \sum_{n=3}^{\infty} \frac{(Bs/\sigma)^{n-2} \int_I v^n \pi(v) dv}{n! \sigma^2} \right) \right].$$

Assuming that $B/\sigma < b$ and using (A.6) gives

$$\sum_{n=3}^{\infty} \frac{(Bs/\sigma)^{n-2} \int_I v^n \pi(v) dv}{n! \sigma^2} < \sum_{k=1}^{\infty} \frac{(sb)^k v_{\max}^{k+2}}{k! v_{\min}^2} < \frac{v_{\max}^2}{v_{\min}^2} (e^{sbv_{\max}} - 1).$$

For sufficiently small s we now choose

$$c := s - \frac{s^2}{2} - s^2 \frac{v_{\max}^2}{v_{\min}^2} (e^{sbv_{\max}} - 1) > 0.$$

This proves (A.10), and hence the lemma. \square

Lemma A.6. *Let $\lambda_0 > 0$. For all sufficiently large $L > 0$ and any integrated Poisson random variable Z with $\lambda_Z \geq \lambda_0$*

$$\mathbb{P}(|Z - \mu_Z| > \sigma_Z \log L) \leq L^{-d \log \log L},$$

where $d > 0$ is a constant determined by λ_0 , v_{\min} , and v_{\max} .

Proof. Let

$$b = \frac{2v_{\max}}{v_{\min}^2}.$$

If $\log L < b\sigma_Z$, then the result follows easily from the previous lemma.

We now assume that

$$\log L \geq b\sigma_Z. \tag{A.11}$$

Using the inequalities (A.8) and (A.9) gives

$$\lambda_Z \leq \frac{\sigma_Z^2}{v_{\min}^2} \leq \frac{(\log L)^2}{b^2 v_{\min}^2},$$

and hence

$$\mu_Z \leq v_{\max} \lambda_Z \leq \frac{v_{\max} (\log L)^2}{b^2 v_{\min}^2} \tag{A.12}$$

and

$$\mu_Z \leq v_{\max} \lambda_Z \leq \frac{v_{\max}}{v_{\min}^2} \sigma_Z^2. \tag{A.13}$$

Multiplying (A.12) and (A.13) and taking the square root gives

$$\mu_Z \leq \frac{v_{\max}}{bv_{\min}^2} \sigma_Z \log L = \frac{1}{2} \sigma_Z \log L. \tag{A.14}$$

Therefore, since Z is a positive random variable, we have

$$\mathbb{P}(|Z - \mu_Z| > \sigma_Z \log L) = \mathbb{P}(Z > \mu_Z + \sigma_Z \log L) \leq \mathbb{P}(Z > \sigma_Z \log L).$$

Moreover, combining (A.14) with (A.11) and (A.9) gives

$$\frac{1}{2} \sigma_Z \log L = \frac{v_{\max}}{bv_{\min}^2} \sigma_Z \log L \geq \frac{v_{\max}}{v_{\min}^2} \sigma_Z^2 \geq v_{\max} \lambda_Z$$

and, by (A.9),

$$\frac{1}{2} \sigma_Z \log L \geq \frac{1}{2} v_{\min} \sqrt{\lambda_Z} \log L.$$

Since $Z \leq v_{\max} K$ by (A.7), we have

$$\begin{aligned} \mathbb{P}(Z > \sigma_Z \log L) &\leq \mathbb{P}\left(Z > v_{\max} \lambda_Z + \frac{1}{2} v_{\min} \sqrt{\lambda_Z} \log L\right) \\ &\leq \mathbb{P}\left(K > \lambda_Z + \frac{v_{\min}}{2v_{\max}} \sqrt{\lambda_Z} \log L\right). \end{aligned}$$

The result now follows from Lemma A.3. \square

Corollary A.7. *Let $\lambda_0 > 0$, and let Z be an integrated Poisson random variable with $\lambda_Z \leq \lambda_0$. Then for all sufficiently large $L > 0$*

$$\mathbb{P}(Z > \log L) \leq L^{-d \log \log L},$$

where $d > 0$ is determined by λ_0 , v_{\min} , and v_{\max} .

This follows immediately from Corollary A.4.

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