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# Ionization of a model atom by perturbations of the potential

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We study the time evolution of the wave function of a particle bound by an attractive $\delta$-function potential when it is subjected to time-dependent variations of the binding strength (parametric excitation). The simplicity of this model permits certain nonperturbative calculations to be carried out analytically both in one and three dimensions. Thus the survival probability of bound state $|\theta(t)|^2$, following a pulse of strength $r$ and duration $t$, behaves as $|\theta(t)|^2-|\theta(\infty)|^2\sim t^{-\alpha}$, with both $\theta(\infty)$ and $\alpha$ depending on $r$. On the other hand, a sequence of short pulses produces an exponential decay over an intermediate time scale. © *2000 American Institute of Physics.* [S0022-2488(00)02306-9]

## I. INTRODUCTION

While there has been much progress in our understanding of the processes leading to the ionization of atoms and/or the dissociation of molecules subjected to time-dependent fields, the mathematical difficulties presented are such that there are no explicitly solvable models for transitions from a bound state into the continuum.[1–15] This motivates us to investigate here the ionization probability of a particle bound by an attractive point $\delta$-function potential in one dimension[5,6,15] and a spherically smeared out $\delta$ function in three dimensions. We obtain explicit expressions for the ionization probability and for the energy distribution of the ejected electrons for certain time-dependent parametric excitations, i.e., when we suddenly change the value of the coupling constant for a time interval $t$. For such changes the survival probability of the bound state shows no regime of exponential decay but approaches its asymptotic value as a power law.[11–13] The situation is different for periodic forcing with short pulses that is also treated here and more generally in Ref. 16. The survival probabilities now include intermediate exponential regimes followed by power law asymptotics.

### A. General formulation

We consider first the one-dimensional system with an unperturbed Hamiltonian,[5,6,15]

$$H_0=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-g\,\delta(x),\quad g>0,\quad -\infty<x<\infty. \tag{1}$$

$H_0$ has a single bound state,

$$u_b(p,x)=\sqrt{p}e^{-p|x|},\quad p=\frac{m}{\hbar^2}g, \tag{2}$$

with energy $-E_0=-\hbar\omega_0=-\hbar^2p^2/2m$ and a continuous uniform spectrum on the positive real line, with generalized eigenfunctions

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$$u(p,k,x)=\frac{1}{\sqrt{2\pi}}\left(e^{ikx}-\frac{p}{p+i|k|}e^{i|kx|}\right),\quad -\infty<k<\infty, \tag{3}$$

and energies $\hbar^2k^2/2m$ (with multiplicity two for $k\neq 0$). Here $u_b$ is normalized to 1 and $u(k,x)$ to $\delta(k-k')$.

Beginning at some initial time, say $t=0$, a perturbing potential $V(x,t)=-R(t)\,\delta(x)$ is applied to the system, i.e., we change the parameter $g$ in $H_0$,

$$g\to g+R(t),\quad t\geqslant 0. \tag{4}$$

We note here that the matrix elements, $|\langle u_b|V|k\rangle|^2=R^2(t)(p/2\pi)k^2/(p^2+k^2)$, which vanish as $k\to 0$ and approach $R^2(t)(p/2\pi)$ as $|k|\to\infty$. This implies, in particular, that the integral of the transition matrix over all $k$ is infinite.

To solve the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=H_0\psi(x,t)-R(t)\,\delta(x)\,\psi(x,t),\quad t\geqslant 0, \tag{5}$$

we expand $\psi(x,t)$ for $t\geqslant 0$ in the complete set of functions $u$:

$$\psi(x,t)=\theta(t)u_b(p,x)e^{i(\hbar^2/2m)t}+\int_{-\infty}^{\infty}\Theta(k,t)u(p,k,x)e^{-i(\hbar k^2/2m)t}\,dk,\quad t\geqslant 0, \tag{6}$$

and monitor the evolution of $\theta(t)$ and $\Theta(k,t)$ starting from the initial bound state $\theta(0)=1$, $\Theta(k,0)=0$.

The ionization probability at time $t$ caused by a pulse, which coincides with $R(t')$ for $t'<t$ and vanishes for $t'\geqslant t$, is given by

$$P(t)=1-|\theta(t)|^2=\int_{-\infty}^{\infty}|\Theta(k,t)|^2\,dk, \tag{7}$$

while $|\theta(t)|^2$ is the survival probability.

This model can be extended to a three-dimensional shell-like delta function potential. The Hamiltonian,

$$H_0=-\frac{\hbar^2}{2m}\Delta-g\,\delta(r-a),\quad a>0,\quad r=|\mathbf{r}|,\quad \mathbf{r}\in\mathbb{R}^3, \tag{8}$$

has bound states with angular momentum $l$ for all $l=0,1,\ldots$, such that $l<mga/\hbar^2-\frac{1}{2}$. The time-dependent perturbation is now of the form $V(r,t)=-R(t)\,\delta(r-a)$. The results for three dimensions, which are similar to those in one dimension, are described in Sec. V. This follows calculations of $\theta(t)$ and $\Theta(k,t)$ in one dimension.

## II. INTEGRAL EQUATION FOR THE ONE-DIMENSIONAL (1-D) CASE

Using the orthonormality of the eigenfunctions (2), (3) and substituting (6) into (5) yields the following set of equations for the time-dependent amplitudes at $t\geqslant 0$:

$$i\hbar\frac{d\theta}{dt}=-\sqrt{p}T(t),\quad i\hbar\frac{\partial\Theta}{\partial t}=\frac{i|k|}{\sqrt{2\pi}(p-i|k|)}e^{i(\hbar/2m)(p^2+k^2)t}T(t), \tag{9}$$

where

$$T(t)=\left[\sqrt{p}\,\theta(t)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i|k|}{p+i|k|}e^{-i(\hbar/2m)(p^2+k^2)t}\Theta(k,t)dk\right]R(t), \tag{10}$$

determines both $\theta$ and $\Theta$:

$$\theta(t)=1+i\frac{\sqrt{p}}{\hbar}\int_0^t T(t')dt', \tag{11}$$

$$\Theta(k,t)=\frac{|k|}{\sqrt{2\pi}(p-i|k|)\hbar}\int_0^t T(t')e^{i(\hbar/2m)(p^2+k^2)t'}\,dt'. \tag{12}$$

Substituting (11) and (12) into (10) yields an integral equation for $T(t)$, which, using dimensionless variables obtained by setting $\hbar=2m=g/2=1$ (implying $p=1$, $\omega_0=1$), yields

$$\theta(t)=1+2i\int_0^t Y(t')dt', \quad \Theta(k,t)=\sqrt{\frac{2}{\pi}}\frac{|k|}{1-i|k|}\int_0^t Y(t')e^{i(1+k^2)t'}\,dt', \tag{13}$$

where $Y(t)$ is to be found from the integral equation

$$Y(t)=\eta(t)\left\{1+\int_0^t[2i+M(t-t')]Y(t')dt'\right\}, \tag{14}$$

and $\eta(t)=R(t)/g$. The function $M(s)$ in (14) is given by

$$M(s)=\frac{2i}{\pi}\int_0^\infty\frac{u^2e^{-is(1+u^2)}}{1+u^2}du=\frac{1}{2}\sqrt{\frac{i}{\pi}}\int_s^\infty\frac{e^{-iu}}{u^{3/2}}du. \tag{15}$$

$M(s)$ behaves as

$$M(s)=\begin{cases}\dfrac{1-i}{\sqrt{8\pi s^3}}e^{-is}+O(s^{-5/2}), & \text{when } s\to\infty,\\[2ex] \dfrac{1+i}{\sqrt{2\pi s}}-i+O(s^{1/2}), & \text{when } s\to 0.\end{cases} \tag{16}$$

## III. IONIZATION BY A RECTANGULAR PULSE

We study perturbations having the form of a step function, $\eta(t)=r$ for $t\geqslant 0$. The calculation of $P$ at any time $t$ will then correspond to the ionization probability caused by a pulse of amplitude $r$ and duration $t$. Substituting the $\eta(x)$ into (14) and taking the Laplace transform, we find, cf. Ref. 16,

$$\tilde{Y}(s)=\int_0^\infty Y(x)e^{-sx}\,dx=\frac{r}{s-r(i+\sqrt{is-1})}, \tag{17}$$

where

$$\mathfrak{I}\sqrt{is-1}>0. \tag{18}$$

Using the inverse transform, Eq. (13) has the form

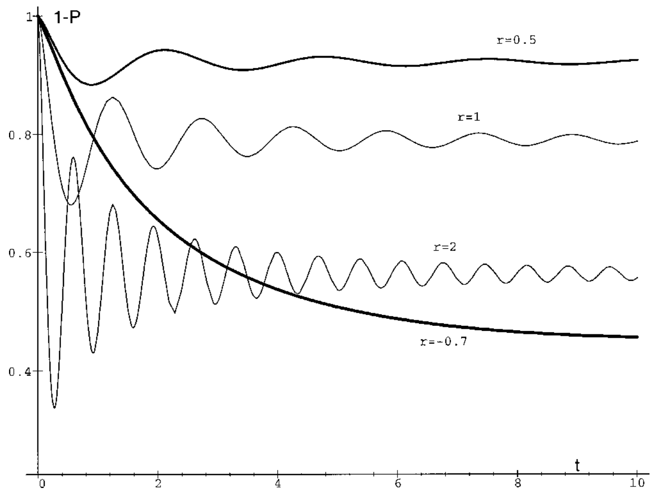


FIG. 1. The survival probability of the bound state following the imposition of rectangular pulses of duration $t$ for different relative amplitudes $r>-1$; see Eqs. (23), (24).

$$\theta(t)=1+\frac{r}{\pi}\int_{-i\infty+\alpha}^{i\infty+\alpha}\frac{e^{st}-1}{s[s-r(i+\sqrt{is-1})]}\,ds,\quad \alpha>0. \tag{19}$$

To evaluate (19) we make a cut in the complex plane of $s$ along the imaginary axis from $-i\infty$ to $-i$. In the left half-plane bounded by the left side of the cut and the vertical line from $-i+\alpha$ to $+i\infty+\alpha$, the integrand in (19) is analytic, except for a simple pole at $s=ir(r+2)$ when $r+1>0$. There are no poles if $r<-1$, i.e., when the coefficient of the $\delta$ function is positive and the potential for $t>0$ represents repulsion. The integral along the left half-circle of an infinitely large radius is clearly zero; therefore one may rewrite (19) as

$$\theta(t)=\frac{r}{\pi}\int\frac{e^{st}}{s[s-r(i+\sqrt{is-1})]}\,ds+2\,\frac{r+1+|r+1|}{(r+2)^2}\,e^{ir(r+2)t},$$

with counterclockwise integration around the cut. Straightforward manipulations with the integral term allow us to write $\theta$ finally in the form

$$\theta(t)=\frac{4r^2}{\pi}\int_0^\infty\frac{e^{-i(1+u^2)t}}{(1+u^2)^2[(r+1)^2+u^2]}\,u^2\,du+2\,\frac{r+1+|r+1|}{(r+2)^2}\,e^{ir(r+2)t}. \tag{20}$$

The integral in (20) can be expressed in terms of Fresnel's functions and the dependence of the survival probability on $r$ is shown in Figs. 1 and 2, where it seen that it is monotone for $r<-1$ but not for $r>1$ so we can have "atomic stabilization."[14,15]

Using (12) and (17) one can calculate $|\Theta(k,t)|^2$, which gives for $t\geqslant\tau$ the energy distribution of electrons kicked out from the bound state by a pulse of duration $\tau$. We find in the original units,

$$\begin{aligned}\Theta(k,t)=\sqrt{\frac{2p}{\pi}}\,\frac{i|k|(p-q)}{(p-i|k|)}\Bigg\{&\frac{(q+|q|)e^{i(\hbar q^2/2m)\tau}}{(q^2+k^2)(p+q)}-\frac{e^{-i(\hbar k^2/2m)\tau}}{(p+i|k|)(q-i|k|)}\\&+\frac{1}{p+q}\left[\frac{pe^{(\hbar p^2/2m)\tau}}{p^2+k^2}\mathrm{Erfc}\left(\frac{i+1}{2}\sqrt{\frac{\hbar p^2\tau}{m}}\right)-\frac{|q|e^{i(\hbar q^2/2m)\tau}}{p^2+k^2}\mathrm{Erfc}\left(\frac{i+1}{2}\sqrt{\frac{\hbar q^2\tau}{m}}\right)\right]\\&+\frac{i|k|(p-q)e^{-i(\hbar k^2/2m)\tau}}{(p^2+k^2)(q^2+k^2)}\mathrm{Erfc}\left(\frac{i-1}{2}\sqrt{\frac{\hbar k^2\tau}{m}}\right)\Bigg\},\quad t\geqslant\tau,\end{aligned} \tag{21}$$

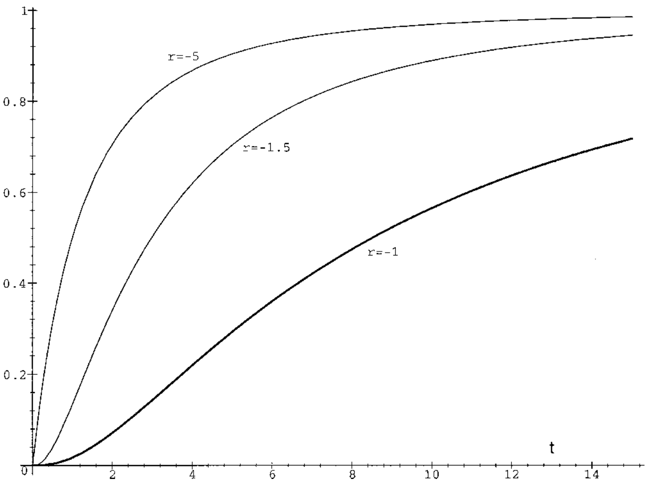


FIG. 2. The normalized survival probability following the imposition of repulsive rectangular pulses, $r<-1$. Plots of $|\theta(t)/\theta_{as}(t)|^2$ vs $t$, where $|\theta_{as}(t)|^2$ is $4/\pi t$ for $r=-1$ and $(r/r+1)^4/\pi t^3$ if $r<-1$; see Eqs. (24), (26).

where $q=(1+r)p$ and Erfc$(z)$ denotes the probability integral. For large $k,|\Theta(k,\tau)|^2$ decays like $k^{-4}$, giving a very long tail to the energy distribution of the emitted electrons. In Fig. 3 we plot $|\Theta(k,\tau)|^2$ vs $k$ for several values of $\tau$ when $r=-1$, i.e., when the pulse just destroys the attractive interaction. It is seen that the longer the pulse the more peaked is the distribution with the maximum moving toward small values of $k$.

The total energy of the electrons ejected by the pulse is given by

$$E(t)=\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}k^2|\Theta(k,\tau)|^2\,dk,\quad t\geqslant\tau.$$

For measurements made outside the range of the potential, this energy will be the same as the kinetic energy of the emitted electrons. An analytical evaluation of $E(t)$ yields a very long and not very illuminating formula. Instead, we present in Fig. 4 a numerical plot of $E(t)$ for $r=-1$. When the pulse length $\tau\rightarrow\infty$, $E(t)$ approaches the value

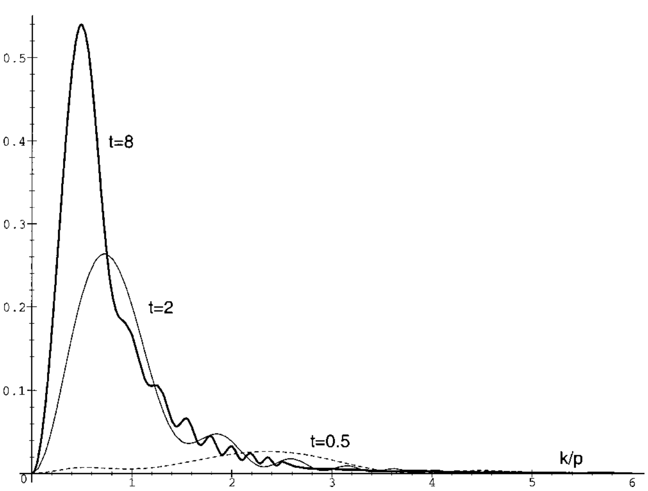


FIG. 3. The plot of $|\Theta(k,\tau)|^2$, which represents the energy distribution of electrons kicked out by the rectangular pulses of duration $t$; see Eq. (21).

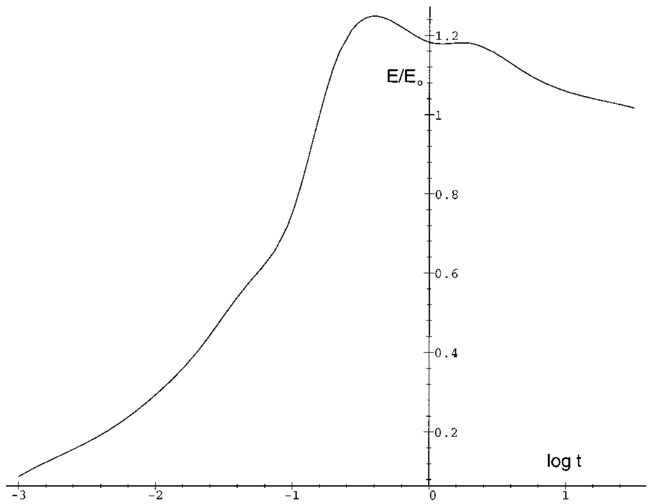


FIG. 4. The total energy of states of electrons in the continuum spectrum ejected by the rectangular pulse of duration $t$ for $r=-1$.

$$E(\infty)=E_0\left(\frac{r}{|r+1|+1}\right)^2\left[1+2|r+1|+2(r+1+|r+1|)\frac{(r+1)(r+3)}{(r+2)^2}\right],$$

which increases linearly with $|r|$ when $|r|\to\infty$:

$$E(\infty)\to 2E_0|2r+|r||.$$

Attractive long pulses, $r>0$, thus give three times as much energy to the ejected electrons than do the repulsive ones, $r<0$. This is shown in Fig. 5, where it is seen that $E(\infty)$ is monotone for both positive and negative $r$.

We note that a rectangular pulse perturbation is a special case of a sudden jump from the initial Hamiltonian $H_0$ to a new time-independent Hamiltonian $H_1$, which in return jumps to $H_0$ when the perturbation ends. The amplitudes $\theta$ and $\Theta$ can thus also be calculated by projecting (twice) a new state onto the old one, which is just the evaluation of overlap integrals. The Laplace method, which gives $\theta(t)$ for the general time dependence of $R(t)$ in the form

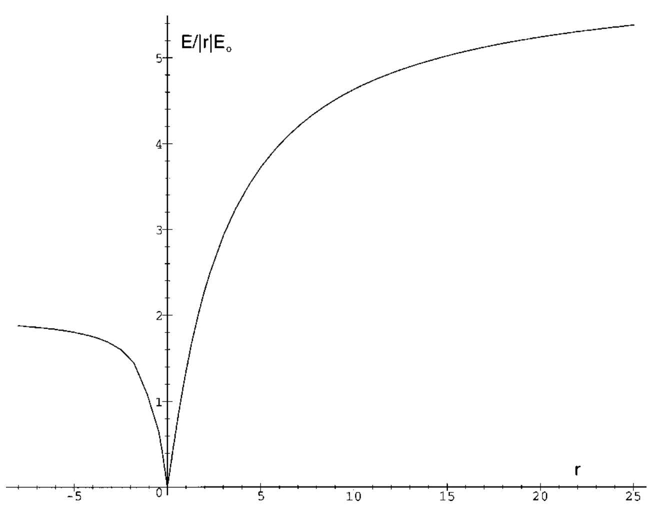


FIG. 5. The plot of the electron kinetic energy versus the amplitude of very long rectangular pulses.

$$\theta(t)=1+\frac{1}{\pi}\int_{-i\infty+\alpha}^{i\infty+\alpha}\frac{e^{st}-1}{s}\tilde{Y}(s)ds, \tag{22}$$

shows that $|\theta(t)|^2$ will have an exponential decay for $t\to\infty$ only when $\tilde{Y}(s)$ has poles in the left half of the complex plane $s$.

### A. Power law decay

When the pulse length $t$ goes to infinity, the integral in (20) vanishes, and in the limit we have

$$|\theta(\infty)|^2=\begin{cases}16(r+1)^2/(r+2)^4, & \text{if } r\geqslant -1,\\ 0, & \text{if } r<-1.\end{cases} \tag{23}$$

It is seen from (23) that any two very long pulses (at least one of them must be repulsive) produce the same ionization if their amplitudes $r$ and $r'$ satisfy the relation $1/r+1/r'=-1$.

For large $t$ the asymptotics of the integral term in (20) can be easily found. Using contour integration we can rewrite the integral as

$$\frac{1}{(r+1)^2 t\sqrt{it}}\int_0^\infty \frac{y^2 e^{-y^2}}{(1-iy^2/t)^2[1-iy^2/t(1+r)^2]}dy,$$

and integrate by expanding the integrand in powers of $y^2/t$. Let us first study the case $r=-1$, which corresponds to the perturbation removing the potential and making the electron evolve for $t>0$ like a free particle. The decay of the bound state in this case is rather slow:

$$|\theta(t)|^2=\frac{4}{\pi t}+O(t^{-2}),\quad r=-1. \tag{24}$$

When both $t$ and $t|r+1|$ are large we get

$$\theta(t)=2\frac{r+1+|r+1|}{(r+2)^2}e^{itr(r+2)}+\frac{r^2}{(r+1)^2 t\sqrt{i\pi t}}e^{-it}+O(t^{-5/2}). \tag{25}$$

For the survival probability of the bound state we have

$$|\theta(t)|^2\approx\begin{cases}|\theta(\infty)|^2+\dfrac{8r^2\cos[(r+1)^2 t]}{(r+1)(r+2)^2 t\sqrt{\pi t}}, & \text{if } r>-1,\\[2ex] \dfrac{r^4}{(r+1)^4\pi t^3}, & \text{if } r<-1.\end{cases} \tag{26}$$

Thus, for $r\leqslant -1$, when the evolution takes place with a repulsive $\delta$ function, the approach to zero of $|\theta(t)|^2$ is like $t^{-3}$, compared to the $t^{-1}$ decay given in (24) for the free evolution; see Fig. 2. Note that the coefficient of $t^{-3}$ becomes independent of $r$ for $|r|\gg 1$. For $r>-1$ the approach of $|\theta(t)|^2$ to its nonvanishing asymptotic value is oscillatory with an envelope that decays like $t^{-3/2}$. These oscillations are very rapid for large $r$ (Fig. 1), but their amplitude is small, of order $1/r$. These asymptotic power law decays are in agreement with general results for the decay of initially localized states; cf. Refs. 5–8.

## IV. IONIZATION BY PERIODIC SHORT PULSES

The behavior of $P(t)$ for short pulses of duration, $t\ll 1$, is very different for cases when $a=r\sqrt{t}$ is large or small comparing with 1. Writing $P(t)=P(t,a)$, we analyze Eq. (20) and have in the case of a single pulse,

$$P(t,a)=4\sqrt{\frac{2t}{\pi}}\begin{cases}2a^2/3, & \text{for } a\ll 1,\\ 1 & \text{for } a\gg 1.\end{cases} \tag{27}$$

We turn now to the survival probability when we bombard our system with a whole train of short pulses of duration $\tau\ll 1$ repeated periodically with period $\sigma\sim 1$. Using (13) yields

$$\theta(n\sigma+\tau)=\theta_n=1+2i\sum_{k=0}^{n} J_k^0, \quad \theta_n=1 \quad \text{if } n<0, \tag{28}$$

where we have defined

$$J_n^m=\int_{n\sigma}^{n\sigma+\tau}(n\sigma+\tau-x)^m Y(x)dx. \tag{29}$$

By integrating Eq. (13) for $\theta(t)$ in $t$ from $t=n\sigma$ to $t=n\sigma+\tau$ and using (16), we obtain

$$r^{-1}J_n^m=\frac{\tau^{m+1}}{m+1}\left[1+\sum_{k=0}^{n-1}(2i+M_{n-k})J_k^0\right]+\sqrt{\frac{i}{\pi}}k_m J_n^{m+1/2}$$
$$+\frac{i}{m+1}J_n^{m+1}+\frac{3k_m}{\sqrt{i\pi}(2m+3)}J_n^{m+3/2}+O(\tau^{m+2}), \tag{30}$$

where $M_n=M(n\sigma)$, $k_m=\sqrt{\pi}\Gamma(m+1)/\Gamma(m+\frac{3}{2})$. The inequality $|\theta_n|\leq 1$ implies $|J_k^0|\leq 1$, and therefore by integration by parts we get

$$|J_k^m|\leq \tau^m. \tag{31}$$

Let us eliminate in (30) the term $J_n^{m+1/2}$ by using (30) with $m\to m+\frac{1}{2}$, which gives

$$r^{-1}J_n^m=\left(\frac{\tau^{m+1}}{m+1}+2k_m r\sqrt{\frac{i}{\pi}}\frac{\tau^{m+3/2}}{2m+3}\right)\left[1+\sum_{k=0}^{n-1}(2i+M_{n-k})J_k^0\right]$$
$$+i\left(\frac{1}{m+1}+\frac{rk_m k_{m+1/2}}{\pi}\right)J_n^{m+1}+O(\tau^{m+2}). \tag{32}$$

Combining (31) with the estimate $|M(s)|<\sqrt{\pi/2s^3}$, we have an upper bound on the sum in (32) in the form $|\Sigma_{k=0}^{n-1}M_{n-k}J_k^0|<2.4\sigma^{-3/2}\max_{j\in[0,n-1]}|J_j^0|$. Treating the amplitude $r$ as a quantity of order of unity, one can immediately improve the upper bounds (31) for $J_n^m$ to

$$|J_n^m|\sim\tau^{m+1}. \tag{33}$$

Equation (33) allows us to rewrite Eq. (32) for $m=0$ as a simple recurence,

$$J_n^0=\rho\left(1+2i\sum_{k=0}^{n-1}J_k^0\right)+\tau^2 f_n(\tau), \tag{34}$$

where, for $\sigma\geq 1$, $r\leq 1$ we have $|f_n(\tau)|<7$ uniformly in $n$ and

$$\rho=\tau\tau\left[1+\frac{4r\sqrt{\tau}(1+i)}{3\sqrt{2\pi}}\right].$$

Starting with $J_0^0=\rho\sim\tau$ one can find successively $J_n^0$ using (34). The terms of such a sequence will be close to the corresponding terms of the solution of the simplified equations,

$$\tilde{J}_n=\rho\left(1+2i\sum_{k=0}^{n-1}\tilde{J}_k\right), \tag{35}$$

until $|\tilde{J}_n|\gg\tau^2$ and $|\tilde{J}_n|\gg|\rho|n\tau^2$. It is easy to solve (35) to find

$$\tilde{J}_n=\rho(1+2i\rho)^n, \tag{36}$$

and therefore $|\tilde{J}_n|\approx|\rho|e^{-n\gamma}$, where $\gamma=8r^2\tau^{3/2}/3\sqrt{2\pi}\ll1$. Our condition for approximating $J_n^0$ by $\tilde{J}_n$ now has the form

$$e^{-n\gamma}\gg n\tau^2. \tag{37}$$

Using $\tilde{J}_n$ and (28), we find

$$\theta(t)\approx\exp\left(\frac{-\gamma+2ir\tau}{\sigma}t\right), \tag{38}$$

if the duration of a train of pulses $t=n\sigma$ is not too long and satisfies (37). One can obtain from (37) that the decay of survival probability $|\theta(x)|^2$ up to a value $\mu$ is accurately described by (38) if $\sqrt{\tau}<2r^2\sqrt{\mu}/\ln\mu^{-1}$; for $\mu=0.01$, $r\approx1$ this gives $\sqrt{\tau}<0.04$ and a train of about 300 pulses. For shorter $\tau$ the train can be longer and the ionization more complete.

For longer trains of perturbation the term $\Sigma_{k=0}^{n-1}M_{n-k}J_k^0$ in (30) cannot be ignored, and it makes the eventual decay slower with strong oscillations due to interference with the eigenfrequency. We see that in the exponentially decaying regime the survival probability is independent of $\sigma$. This is very different from the case where the time-dependent $\eta(t)=r\sin\omega t$ considered in Ref. 16. In that case the exponential decay depends strongly on $\omega$. In our case $\tau\to0$, which means that $\eta(t)$ will contain all ranges of frequencies.

## V. THREE-DIMENSIONAL MODEL

The Hamiltonian (8) has eigenfunctions in the continuum spectrum,

$$\Psi_{l,m}(k,\mathbf{r})=Y_{l,m}(\theta,\varphi)R_l(k,r),\quad r\geq0,\quad 0\leq\theta\leq\pi,\quad 0\leq\varphi\leq2\pi, \tag{39}$$

where the radial functions are

$$R_l(k,r)=A_l\sqrt{\frac{k}{r}}J_{l+1/2}(kr)+A_l\sqrt{\frac{k}{r}}\begin{cases}0, & \text{if } r\leq a,\\ \frac{\pi i}{4}QJ_{l+1/2}(ka)[H_{l+1/2}^{(1)}(ka)H_{l+1/2}^{(2)}(kr)-H_{l+1/2}^{(1)}(kr)H_{l+1/2}^{(2)}(ka)], & \text{if } r>a.\end{cases} \tag{40}$$

The dimensionless parameters $A_l$ normalize $R_l(r)$ to a $\delta$ function,

$$A_l(k)=\{1+Q\pi J_{l+1/2}(ka)N_{l+1/2}(ka)+Q^2\pi^2J_{l+1/2}^2(ka)[J_{l+1/2}^2(ka)+N_{l+1/2}^2(ka)]\}^{-1/2},$$

and the notations for normalized spherical harmonics $Y_{l,m}$ and Bessel functions are the usual ones. The energy corresponding $\psi_{l,m}(k,r)$ is $\hbar^2k^2/2m$.

The parameter $Q=2mga/\hbar^2$ plays a crucial role for the existence of the bound states,

$$QK_{l+1/2}(p_la)I_{l+1/2}(p_la)=1 \tag{41}$$

is the equation for the energy $-\hbar^2 p_l^2/2m$ of all bound $l$ states (they are of different axial symmetry). The left side of (41) is a monotonically decreasing function of its argument $\gamma = p_l a$ and it is equal to $Q(2l+1)^{-1}$ when $\gamma = 0$, therefore

$$Q > 2l+1$$

is the condition to have the bound states for all $l' \leqslant l$.

The radial normalized eigenfunctions can be written in the form

$$R_l^b(r) = \frac{B_l p_l}{\sqrt{r}} \begin{cases} I_{l+1/2}(p_l r), & \text{if } r \leqslant a, \\ I_{l+1/2}(p_l a) K_{l+1/2}(p_l r)/K_{l+1/2}(p_l a), & \text{if } r > a, \end{cases} \tag{42}$$

where

$$B_l = \frac{\sqrt{2} K_{l+1/2}(p_l a)}{\sqrt{1 - p_l a K_{l+1/2}(p_l a)[I_{l-1/2}(p_l a) + I_{l+3/2}(p_l a)]}},$$

and $I$, $K$ are the modified Bessel functions.

There are no transitions between states of different angular symmetry if both the potential in (8) and perturbation $V(t,r)$ are central. For simplicity we consider our three-dimensional model with $Q > 1$ in the $s$ state. Dropping the index $l = 0$, Eq. (41) for the energy of the bound state $-\hbar^2 p^2/2m$ is

$$Q = \frac{2ap}{1 - e^{-2ap}}.$$

The eigenfunctions (42) of the bound and the continuum states are, respectively,

$$\Psi_b(r) = \frac{p^{1/2}}{r\sqrt{\pi(e^{2pa} - 1 - 2pa)}} \begin{cases} \sinh pr, & \text{if } r \leqslant a, \\ e^{-p(r-a)} \sinh pa, & \text{if } r > a, \end{cases} \tag{43}$$

$$\Psi_{0,0}(k,r) = \frac{2^{-1/2}}{\pi r \sqrt{1 - Q\dfrac{\sin 2ka}{ka} + Q^2 \dfrac{\sin^2 ka}{k^2 a^2}}} \begin{cases} \sin kr, & \text{if } r \leqslant a, \\ \sin kr - Q\dfrac{\sin ka}{ka} \sin k(r-a), & \text{if } r > a. \end{cases} \tag{44}$$

Assuming that the particle is in the bound state $\Psi_b(r)$ at $t = 0$ and the perturbation has the form $V(r,t) = -R(t) g \delta(r-a)$, we use the method of projections that was described in Sec. IV to find the ionization probability induced by the rectangular pulses $R(t) = rg$ for $t > 0$. After the end of pulse at $t = \tau$ we have for $\theta(\tau)$ an equation similar to (21),

$$\theta(\tau) = \frac{4pq}{(e^{2pa} - 1 - 2pa)(e^{2qa} - 1 - 2qa)} \left[ \frac{e^{(p+q)a}}{p+q} - \frac{pe^{(p-q)a} - qe^{(q-p)a}}{p^2 - q^2} \right]^2 e^{i(hq^2/2m)\tau}$$
$$+ 8p \frac{[(pa - Q_1)\sinh pa + pa \cosh pa]^2}{\pi a^2} \int_0^\infty \frac{e^{-i(hk^2/2m)\tau} \sin^2 ka}{(p^2 + k^2)^2 \left( 1 - Q_1 \dfrac{\sin 2ka}{ka} + Q_1^2 \dfrac{\sin^2 ka}{k^2 a^2} \right)} dk, \tag{45}$$

where $q$ is the solution of Eq. (41) with $Q_1 = (1+r)Q$ instead of $Q$ ($q$ gives the energy of the new bound state). If $Q_1 < 1$ the first term in (45) vanishes; otherwise the square of its absolute value represents the probability $1-P(\infty)$ of the electron to remain in the bound state when $\tau\to\infty$. Using the dimensionless time $\omega_0 t\to t$, the asymptotics of the decaying term in (45) when $t\to\infty$ is

$$\theta(t)=\theta(\infty)-\sqrt{2}\frac{[(pa-Q_1)\sinh pa+pa\cosh pa]^2}{(Q_1-1)^2}\frac{1+i}{t\sqrt{\pi t}}+O(t^{-5/2}),\quad t\to\infty, \tag{46}$$

or

$$|\theta(t)|^2\approx|\theta(\infty)|^2+\begin{cases}O(t^{-3/2}), & \text{if } Q_1>1,\\ O(t^{-3}), & \text{if } Q_1<1.\end{cases}$$

The dimensionality as one can see changes the character of asymptotics only of the free evolution ($t^{-3/2}$ vs $t^{-1/2}$). An interesting case is $Q_1=1$, when the perturbed Hamiltonian has a ''zero energy bound state.'' The asymptotic behavior of $\theta(t)$ is now given by

$$\theta(t)=\frac{4[(pa-Q_1)\sinh pa+pa\cosh pa]^2(1-i)}{p^2a^2\sqrt{2\pi t}}+O(t^{-3/2}),\quad t\to\infty, \tag{47}$$

which has the same character as for the free decay in the one-dimensional model.

In three dimensions the same technique as that used in Sec. III allows us to derive a one-dimensional integral equations similar to (14) for each pair of quantum numbers $l$, $m\leqslant l$:

$$T_{l,m}(t)=R(t)a^2\left[R_l^b(a)\theta_{l,m}(0)+\frac{i}{\hbar}\int_0^t K_l(t-t')T_{l,m}(t')dt'\right], \tag{48}$$

which determines the evolution. In particular, the amplitude of the bound state develops in time as

$$\theta_{l,m}(t)=\theta_{l,m}(0)+i\frac{R_l^b(a)}{\hbar}\int_0^t T_{l,m}(t')dt'. \tag{49}$$

The function $K_l$ in Eq. (48),

$$K_l(\vartheta)=[R_l^b(a)]^2+\int_0^\infty|R_l(k,a)|^2e^{-i\hbar(k^2+p_l^2)\vartheta/2m}\,dk, \tag{50}$$

is independent of the quantum number $m$. Each spherical harmonic evolves autonomously and if $\theta_{l,m}$ was zero at $t=0$ it does not change for our perturbation. Though the kernel of Eq. (48) even for $l=0$,

$$K_0(\vartheta)=\frac{4p\sinh^2 pa}{a^2(e^{2pa}-1-2pa)}+\frac{2}{\pi a^2}\int_0^\infty\frac{e^{-i\hbar[k^2+p^2/2m]\vartheta}\sin^2 ka}{1-Q\sin 2ka/ka+Q^2\sin^2 ka/k^2a^2}\,dk, \tag{51}$$

cannot be expressed in terms of standard functions numerical calculations are quite feasible.

## VI. CONCLUDING REMARKS

Some general features of our results with possible implications for realistic systems include the following.

(a) The ionization probability approaches its asymptotic value as $t^{-3/2}$ if the electron can be bound in the perturbed state, goes to zero as $t^{-1}$ if the perturbation makes the electron a free particle, and as $t^{-3}$ when the perturbation converts the attractive well into a repulsive one.

(b) A finite train of periodically repeated short pulses makes the survival probability of the bound state decay exponentially without oscillations. When the frequency of repetition is comparable with the eigenfrequency of the bound state or is lower, the decay scales in such a way that only the total number of pulses is important.

(c) The three-dimensional potential gives a similar behavior of the ionization. The free evolution in one dimension corresponds here to a marginal situation with the ''zero-energy'' bound state.

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