Large deviations for ideal quantum systems

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We consider a general d-dimensional quantum system of non-interacting particles in a very large (formally infinite) container. We prove that, in equilibrium, the fluctuations in the density of particles in a subdomain Λ of the container are described by a large deviation function related to the pressure of the system. That is, untypical densities occur with a probability exponentially small in the volume of Λ, with the coefficient in the exponent given by the appropriate thermodynamic potential. Furthermore, small fluctuations satisfy the central limit theorem. © *2000 American Institute of Physics.* [S0022-2488(00)01803-X]

I. INTRODUCTION

Statistical mechanics is the bridge between the microscopic world of atoms and the macroscopic world of bulk matter. In particular it provides a prescription for obtaining macroscopic properties of systems in thermal equilibrium from a knowledge of the microscopic Hamiltonian. This prescription becomes mathematically precise and elegant in the limit in which the size of the system becomes very large on the microscopic scale (but not large enough for gravitational interactions between the particles to be relevant). Formally this corresponds to considering neutral or charged particles with effective translation invariant interactions inside a container and taking the infinite-volume or thermodynamic limit (TL). This is the limit in which the volume $|V|$ of the container V grows to infinity along some specified regular sequences of domains, say cubes or balls, while the particle and energy density approach some finite limiting value.[1–4] This limit provides a precise way for eliminating ''finite size'' effects.

It is then an important result (a theorem, under suitable assumptions) of statistical mechanics that the bulk properties of a physical system, computed from the thermodynamic potentials via any of the commonly used Gibbs ensembles (microcanonical, canonical, grand canonical, etc.), have well-defined ''equivalent'' TLs.[1–3] These free energy densities are furthermore proven to be the same for a suitable class of ''boundary conditions'' (bc), describing the interaction of the system with the walls and the ''outside'' of its container. When this independence of bc is ''strong enough,'' the bulk free energies also yield information about normal fluctuations and large deviations (LD), in particle number and energy, inside regions Λ that are macroscopically large but significantly smaller than V. The restriction to $|\Lambda| \ll |V|$ means that we deal here with *semi-local*, rather than global, LD.

The purpose of this note is to study semi-local LD for quantum systems. This seems interesting since the real world is quantum mechanical, with the classical description being an essentially uncontrolled approximation, albeit a very good one in many circumstances. For classical systems,

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the theory of such semi-local LD is well developed.[5–7] In contrast, for quantum systems, only results on global LD are available,[1,8–10] which are augmented by a theory of normal and anomalous fluctuations for local observables.[11–13]

In this paper we consider the semi-local LD in the number of particles for the simplest continuum quantum systems, namely the ideal quantum fluids. Their normal fluctuations will be shown to be a corollary of our LD result.

A. Classical systems

We begin by considering a classical system of N particles of mass m in a domain, say a cubical box $V\subset\mathbb{R}^d$, interacting with each other through a sufficiently rapidly decaying pair potential $\phi(r)$, e.g., a Lennard-Jones potential. The Hamiltonian of the system is then given by

$$H(N,V;b)=\frac{1}{2m}\sum_{i=1}^{N}\mathbf{p}_i^2+\frac{1}{2}\sum_{1\leqslant\neq j\leqslant N}\phi(r_{ij})+\sum_{i=1}^{N}u_b(\mathbf{r}_i), \tag{I.1}$$

where $\mathbf{p}_i\in\mathbb{R}^d$, $\mathbf{r}_i\in V$, $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$, and $u_b(r_i)$ represents the interaction of the ith particle with the world outside of the boundary of V. This boundary interaction (indicated here and in the sequel by b) is in addition to the action of the implicitly assumed ''hard wall'' which keeps the particles confined to V. The dynamic effect of the latter is to reflect the normal component of the particle's momentum when it hits the wall. However, sometimes it is convenient to replace it with periodic boundary conditions,[14] dropping the boundary term u_b in (I.1).

For a macroscopic system in equilibrium at reciprocal temperature β and chemical potential μ, the grand canonical Gibbs ensemble then gives the probability density for finding exactly N particles inside $V\subset\mathbb{R}^d$ at the phase point $X_N=(\mathbf{r}_1,\mathbf{p}_1,\ldots,\mathbf{r}_N,\mathbf{p}_N)=(\mathbf{R}_N,\mathbf{P}_N)\in\Gamma_N=V^N\times\mathbb{R}^{dN}$ as

$$\nu(X_n|\beta,\mu,V,b)=\frac{(N!)^{-1}h^{-Nd}\exp[-\beta(H(N,V;b)-\mu N)]}{\Xi(\beta,\mu|V,b)}. \tag{I.2}$$

Here Ξ is the grand canonical partition function

$$\Xi=\sum_{N=0}^{\infty}(N!)^{-1}\lambda_{\mathrm{B}}^{-dN}e^{\beta\mu N}\int_{V^N}d\mathbf{r}_1\ldots d\mathbf{r}_N e^{-\beta/2\Sigma\phi(r_{ij})-\beta\Sigma u_b(\mathbf{r}_i)}=\sum_{N=0}^{\infty}e^{\beta\mu N}Q(\beta,N|V,b), \tag{I.3}$$

and $Q(\beta,N|V,b)$ is the canonical partition function. We use h^{dN}, h being Planck's constant, as the unit of volume in the phase space Γ_N, so $\lambda_{\mathrm{B}}=h\sqrt{\beta/(2\pi m)}$ is the de Broglie wave length. The finite-volume, boundary-condition dependent, grand canonical pressure is

$$p(\beta,\mu|V,b)=(\beta|V|)^{-1}\log\Xi(\beta,\mu|V,b). \tag{1.4}$$

Taking now the TL, $V\nearrow\mathbb{R}^d$, we obtain, for a suitable class of bc, an intrinsic (bc independent) grand canonical pressure $p(\beta,\mu)$. This is related to the Helmholtz free energy density $a(\beta,\rho)$ in the canonical ensemble, obtained when Ξ is replaced by $Q^{-1}(\beta,N|V,b)$ in (I.4) and the limit is taken in such a way that $N/|V|\to\rho$, a specified particle density. The relation between p and a is given by the usual thermodynamic formula involving the Legendre transform

$$p(\beta,\mu)=\sup_{\rho}[\rho\mu-a(\beta,\rho)]=\pi(\beta,\overline{\rho}), \tag{I.5}$$

where $\pi(\beta,\rho)$ is the TL of the canonical pressure

$$\pi(\beta,\rho)=-\rho^2\frac{\partial(a/\rho)}{\partial\rho} \tag{I.6}$$

and

$$\overline{\rho}(\beta,\mu)=\frac{\partial p}{\partial \mu}(\beta,\mu) \tag{I.7}$$

is the average density in the grand canonical ensemble.

At a first-order phase transition $\mu \mapsto \overline{\rho}(\beta,\mu)$ is discontinuous and the left/right limits of the derivative on the rhs of (I.7) give the density in the coexisting phases. In our discussion we shall restrict ourselves to values of the parameters β and μ for which the system is in a unique phase. We can of course also go from the grand canonical pressure to the Helmholtz free energy density by the inverse of (I.5),

$$a(\beta,\rho)=\sup_{\mu}[\rho\mu - p(\beta,\mu)]. \tag{I.8}$$

Let $P(N_V \in \Delta|V||\beta,\mu,V,b)$ be the probability of finding a total particle density in V (i.e., $N_V/|V|$) in the interval $\Delta=[n_1,n_2]$. Then, for b in the right class of bc, we have (almost by definition) that

$$\lim_{V\nearrow \mathbb{R}^d}(\beta|V|)^{-1}\log P(N_V\in\Delta|V||\beta,\mu,V,b)=\sup_{n\in\Delta}[a(\beta,\overline{\rho})-a(\beta,n)+\mu(\overline{\rho}-n)], \tag{I.9}$$

were $\overline{\rho}$ is given by (I.7). In probabilistic language, this means that, up to a vertical translation, $-a(\beta,n)-\mu n$ is the LD functional, or rate function, for density fluctuations. [Note that $a(\beta,\rho)$ may be infinite for some values of ρ, i.e., when $\phi(r)=\infty$, for $r<D$, and ρ is above the close-packing density of balls with diameter D.]

On the other hand, the fluctuations in all of V are clearly bc and ensemble dependent (they are nonexistent in the canonical ensemble) and therefore not an intrinsic property of the system. Physically more relevant are the fluctuations not in the whole volume V but in a region Λ inside V. Of particular interest is the case when Λ is very large on the microscopic scale but still very small compared to V. The proper idealization of this situation is to first take the TL, $V\nearrow\mathbb{R}^d$, and then let Λ itself become very large. We are thus interested in the probability $P(N_\Lambda \in\Delta|\Lambda||\beta,\mu)$, for Λ a large region in an infinite system obtained by taking the TL. This probability should now be an intrinsic property of a uniform single-phase macroscopic system characterized either by a chemical potential μ or by a density ρ.

A little thought shows that this probability corresponds to considering the grand canonical ensemble of a system of particles in a domain Λ with boundary interactions of the type

$$u_b(\mathbf{r}_i)=\sum_{k=1}^{\infty}\phi(|\mathbf{r}_i-\mathbf{x}_k|), \quad \mathbf{r}_i\in\Lambda, \mathbf{x}_k\in\Lambda^c, \tag{I.10}$$

i.e., we imagine that the boundary interactions come from particles of the same type as those inside Λ, specified to be at positions $\mathbf{x}_1,\mathbf{x}_2,\ldots$ outside Λ. These positions must then be averaged over according to the infinite-volume Gibbs measure. It follows then, from the independence of the bulk properties of the system of the boundary conditions, that Eq. (I.9) is still correct, that is,

$$\lim_{\Lambda\nearrow \mathbb{R}^d}(\beta|\Lambda|)^{-1}\log P(N_\Lambda\in\Delta|\Lambda||\beta,\mu)=\sup_{n\in\Delta}[a(\beta,\overline{\rho})-a(\beta,n)+\mu(\overline{\rho}-n)]. \tag{I.11}$$

This relation is indeed a theorem for classical systems, under fairly general conditions.[3,15,6]

B. Quantum systems

It is Eq. (I.11) and similar formulas for fluctuations in the energy density which we want to generalize to quantum systems. To do this, we begin by considering the boundary conditions imposed on the N-particle wave functions $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N|V)$ for a quantum system in the domain V. Usually this is done by requiring that whenever any $\mathbf{r}_i$ is at the boundary of V, $\mathbf{r}_i\in\partial V$, then Ψ is equal to α times its normal derivative

$$\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N|V)=\alpha\mathbf{n}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N|V) \tag{I.12}$$

with $\alpha=0$ corresponding to Dirichlet and $\alpha=+\infty$ to Neumann boundary conditions.

Denote by b_α the elastic boundary condition (I.12). The existence of the TL of the grand canonical pressure $p(\beta,\mu|V,b_0)$ has been proven for quantum systems with stable potentials,[1] and for positive potentials it is established that the pressure does not depend on α.[16] But, as far as we are aware, the dependence on $u_b(\mathbf{r}_i)$ has not been studied systematically, with the exception of the regime covered by the low-density expansion of Ginibre.[17,18] This only shows that the dependence on the boundary is not so well understood for continuous quantum systems.

To investigate the density fluctuations in quantum systems we note that the momentum variables did not play any role in the derivation of (I.9) and (I.11) for classical systems. The only thing relevant, when considering particle number fluctuations, is the probability density in the configuration space. This is given for a classical system by integrating ν in (I.2) over the momentum variables, whose distribution is always a product of Gaussians (Maxwellians). For a quantum system, where the analog of (I.2) is the density matrix $\hat{\nu}$, the configuration probability density is given by the diagonal elements of $\hat{\nu}$ in the position representation. For the grand canonical ensemble this can be written as

$$\hat{W}(\mathbf{R}_N|\beta,\mu,V,b_\alpha)=\frac{e^{\beta\mu N}\Sigma_\gamma|\Psi_\gamma(\mathbf{R}_N|V,b_\alpha)|^2e^{-\beta E_\gamma}}{\hat{\Xi}}, \tag{I.13}$$

where Ψ_γ and E_γ are the eigenstates and eigenvalues of H_N with the suitable statistics and b_α bc.[1,19]

It is clear from the derivation of the TL[1,2] that, when $\phi(r)$ is superstable, the TL for the canonical ensemble exists for all $\rho\in[n_1,n_2]$ with bc b_α. Then (I.9) carries over to quantum systems. The real problem is how to prove (I.11) for these systems. $\hat{W}$ is no longer a Gibbs measure with a pair potential as interaction and there is no good reason to expect it to be a Gibbs measure for any other ‘‘reasonable’’ many-body potential.[15] [Even if the latter were the case, this potential would almost certainly depend on the density and temperature of the system and would therefore not carry directly any information on (I.11).] It might in fact appear that there is no strong reason why (I.11) should hold for quantum systems. The reason for expecting it to be true is that it is a thermodynamic-type relation and such relations are in general unaffected by the transition from the classical to the quantum formalism. More explicitly, we see the difference between (I.9) and (I.11) as involving only boundary-type quantities which should become irrelevant when Λ is of macroscopic size. The proof of such a statement is however far from obvious (to us) and we therefore devote the rest of this note to proving it in the (technically) simplest case where there are no interactions between the particles, i.e., the ideal gas with either Bose–Einstein or Fermi–Dirac statistics. It turns out that even in this case the proof requires a certain amount of work.

II. MAIN RESULTS

We consider a d-dimensional square box $V=[-l/2,l/2]^d$. For computational convenience we choose periodic boundary conditions, but we do not expect our results to depend on this particular choice. (In fact, we will restrict the thermodynamic parameters to the one-phase region.) In V there is an ideal fluid (either Fermi or Bose) in thermal equilibrium, as described by the grand canonical ensemble. We label the Bose fluid, shorthand BE, with the index $+$, and the Fermi fluid, shorthand FD, with the index $-$, and introduce the Fock space

$$\mathcal{F}_{\pm}^{V} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L_{\pm}^{2}(V^n), \tag{II.1}$$

where $L_{\pm}^{2}(V^n)$ is the n-particle space of all symmetric, resp. antisymmetric, square-integrable functions on V^n. Of course, for $n=1$, $L_{\pm}^{2}(V)=L^2(V)$. In the sequel, in order to keep the notation light, we will often drop sub- or superscripts whenever there is no ambiguity.

Particles do not interact. Therefore the many-particle Hamiltonian in the box V can be written conveniently in the form

$$H_V = \bigoplus_{n=0}^{\infty} \sum_{i=1}^{n} \underbrace{1 \otimes \cdots \otimes h_V \otimes \cdots \otimes 1}_{i^{\text{th}} \text{ position out of } n} \tag{II.2}$$

where h_V, the one-particle Hamiltonian on $L^2(V)$, is defined through the one-particle energy $\epsilon(k)$ in momentum space. This means that, if $|k\rangle$ denotes the momentum eigenvector [represented in $L^2(V)$ as $\psi_V^{(k)}(x)=e^{ik\cdot x}$], then $h_V|k\rangle=\epsilon(k)|k\rangle$ with $k\in V'=(2\pi\mathbb{Z}/l)^d$, the dual of V.

We assume $\epsilon(k)$ to be continuous, $\epsilon(0)=0$ as a normalization, and $\epsilon(k)>0$ for $k\neq 0$. Also $\epsilon(k)\approx|k|^{\gamma}$ for small k and $\epsilon(k)\geqslant|k|^{\alpha}$ for large k, with α, $\gamma>0$. Furthermore, we require

$$\int d^d x \left| \int d^d k \, e^{ik\cdot x} \frac{1}{e^{\beta\epsilon(k)-\beta\mu}-\varepsilon} \right| < \infty \tag{II.3}$$

for $\varepsilon=\pm 1$, $\beta>0$, and suitable μ. [One might note the similarities between (II.3) and the space-clustering condition of Refs. 11 and 12, which ensures that generic observables there have normal fluctuations.]

The standard example of a nonrelativistic, resp. relativistic, kinetic energy for a particle of mass m is $\epsilon(k)=k^2/(2m)$, resp. $\epsilon(k)=\sqrt{m^2c^4+k^2c^2}-mc^2$ (having set Planck's constant $\hbar=1$). Both functions satisfy the above conditions. The relativistic case includes $m=0$, although this is not immediately obvious—cf. Appendix A 1 for details.

We observe that H_V may be rewritten as a quadratic form in the creation and annihilation operators on the Fock space $\mathcal{F}$. Let a_k^* be the operator that creates a particle in the state $|k\rangle$ and a_k the corresponding annihilator. Then

$$H_V = \sum_k \epsilon(k) a_k^* a_k = \sum_{j,k} \langle j|h_V|k\rangle a_j^* a_k = \langle \mathbf{a}|h_V|\mathbf{a}\rangle. \tag{II.4}$$

We fix $\beta>0$ and $\mu\in\mathbb{R}$ for FD, resp. $\mu<0$ for BE. The grand canonical state in the volume V is defined by

$$\langle A\rangle_{\pm,\mu}^{V} = \frac{\mathrm{Tr}_{\mathcal{F}_{\pm}^{V}}(Ae^{-\beta H_V+\beta\mu N})}{\Xi_{\pm}^{V}(\mu)} \tag{II.5}$$

for every bounded operator A on $\mathcal{F}_{\pm}^{V}$. $N=N_V$ is the operator for the number of particles in the box V, $N|_{L_{\pm}^{2}(V^n)}=n1_{L_{\pm}^{2}(V^n)}$, and $\Xi_{\pm}^{V}(\mu)=\mathrm{Tr}_{\mathcal{F}_{\pm}^{V}}(e^{-\beta H_V+\beta\mu N})$ denotes the partition function. As is well known (see, for example, Ref. 19) we have

$$\Xi_{+}^{V}(\mu) = \prod_k (1-e^{-\beta\epsilon(k)+\beta\mu})^{-1}, \tag{II.6}$$

$$\Xi_{-}^{V}(\mu) = \prod_k (1+e^{-\beta\epsilon(k)+\beta\mu}). \tag{II.7}$$

The infinite-volume thermal state is defined through the limit

$$\langle \cdot \rangle = \lim_{V \nearrow \mathbb{R}^d} \langle \cdot \rangle^V \tag{II.8}$$

when taking averages of local observables (Ref. 18, Sec. 2.6).

Taking the infinite volume limit of (II.6) and (II.7) one obtains the grand canonical pressure

$$p_\varepsilon(\mu) = \lim_{V \nearrow \mathbb{R}^d} \frac{\log \Xi_\varepsilon^V(\mu)}{\beta |V|} = -\frac{\varepsilon}{\beta (2\pi)^d} \int d^d k \log(1 - \varepsilon e^{-\beta \epsilon(k) + \beta \mu}) \tag{II.9}$$

and the average density

$$\rho_\varepsilon(\mu) = \frac{d p_\varepsilon}{d\mu}(\mu) = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{e^{\beta \epsilon(k) - \beta\mu} - \varepsilon}. \tag{II.10}$$

Here p_- is real analytic on the whole axis, whereas p_+ is real analytic only for $\mu < 0$ and has a finite limit as $\mu \to 0_-$. For convenience, we define $p_+(\mu) = \infty$ for $\mu > 0$. The slope of p_+ at 0_- is related to the Bose–Einstein condensation. We set

$$\rho_c = \rho_+(0_-) = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{e^{\beta \epsilon(k)} - 1}. \tag{II.11}$$

By the properties of $\epsilon(k)$, $\rho_c = \infty$ for $d \leqslant \gamma$, and is finite otherwise. Here ρ_c is the maximal density of the normal fluid and any surplus density is condensed into the $k=0$ ground state. To simplify the notation we use ρ_c also in the case of an ideal Fermi fluid, setting it equal to ∞.

The infinite system is assumed to be in a pure thermal state, obtained through the limit (II.8) at the reference chemical potential μ. In this state the average density is $\overline{\rho} = \rho(\mu) < \rho_c$. We define the *translated pressure* by

$$g_{\varepsilon,\mu}(\lambda) = g_\varepsilon(\lambda) = p_\varepsilon(\mu + \lambda) - p_\varepsilon(\mu). \tag{II.12}$$

Here g_ε is convex up, increasing, $g(0) = 0$, and $g_\varepsilon'(0) = \overline{\rho}$. For large negative values we have

$$\lim_{\lambda \to \infty} g_\varepsilon(\lambda) = -p_\varepsilon(\mu), \qquad \lim_{\lambda \to -\infty} g_\varepsilon'(\lambda) = 0, \tag{II.13}$$

whereas for positive values

$$\lim_{\lambda \to \infty} g_-(\lambda) = \infty, \qquad \lim_{\lambda \to \infty} g_-'(\lambda) = \infty \tag{II.14}$$

in the case of fermions and

$$\lim_{\lambda \to -\mu} g_+(\lambda) = p_+(0) - p_+(\mu), \qquad \lim_{\lambda \to -\mu} g_+'(\lambda) = \rho_c \tag{II.15}$$

for bosons, with $g_+(\lambda) = \infty$ for $\lambda > -\mu$.

We define the *rate function* f_ε as the Legendre transform of g_ε, i.e.,

$$f_{\varepsilon,\mu}(x) = f_\varepsilon(x) = \inf_{\lambda \in \mathbb{R}} (g_\varepsilon(\lambda) - \lambda x) = g_\varepsilon(\lambda_o) - \lambda_o x. \tag{II.16}$$

Here $\lambda_o = \lambda_o(x)$ is the minimizer of $g(\lambda) - \lambda x$, which is unique by convexity. For $x \leqslant 0$ we have $\lambda_o = -\infty$. For $0 < x < \rho_c$, it is determined by $g'(\lambda_o) = x$, while for $x \geqslant \rho_c$ we have $\lambda_o = -\mu$. This shows that $f(x) = -\infty$ on the half-line $\{x < 0\}$ and finite elsewhere. In particular, f is convex down, strictly convex for $0 < x < \rho_c$, and $f_+(x) = p(0) - p(\mu) + \mu x$, for $x \geqslant \rho_c$, as a trace of the Bose–Einstein condensation.

Let us now consider a small subvolume Λ of our (already infinite) container V. The precise shape of Λ plays no role, only the ''surface area'' should be small compared to its volume $|\Lambda|$. Thus, by $\Lambda\nearrow\mathbb{R}^d$ we mean a sequence of subdomains such that for each Λ there exists a subset Λ' of Λ with $|\Lambda'|/|\Lambda|\to 1$ and $\operatorname{dist}(\Lambda',\mathbb{R}^d\setminus\Lambda)\to\infty$.

Let N_Λ be the number operator for the particles in Λ. With respect to $\langle\cdot\rangle$, N_Λ has some probability distribution. We follow the usual practice and use the same symbol N_Λ to denote also the corresponding random variable. Its distribution is indicated by $\mathbb{P}$, averages again by $\langle\cdot\rangle$.

We are now in a position to state the main result.

Theorem II.1: *Let $\beta>0$ and $\mu<0$ for BE, resp. $\mu\in\mathbb{R}$ for FD. Then, for any interval $I=[a,b]$,*

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{1}{\beta|\Lambda|}\log\mathbb{P}(\{N_\Lambda\in|\Lambda|I\})=\sup_{x\in I}f_{\varepsilon,\mu}(x).$$

III. LARGE DEVIATIONS IN THE DENSITY

In this section we explain how Theorem II.1 follows from the asymptotic behavior of the generating function $\langle e^{\beta\lambda N_\Lambda}\rangle_\varepsilon$.

Lemma III.1: There exists a $\lambda_{\max}(\Lambda)$ such that $\langle e^{\beta\lambda N_\Lambda}\rangle_\varepsilon<\infty$ for all $\lambda<\lambda_{\max}(\Lambda)$ and $\langle e^{\beta\lambda N_\Lambda}\rangle_\varepsilon=\infty$ for all $\lambda\geqslant\lambda_{\max}(\Lambda)$. For FD we have $\lambda_{\max}(\Lambda)=\infty$, whereas for BE $\lambda_{\max}(\Lambda)<\infty$ with $\lambda_{\max}(\Lambda)\searrow-\mu$, as $\Lambda\nearrow\mathbb{R}^d$.

Theorem III.2: *The limit*

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\log\langle e^{\beta\lambda N_\Lambda}\rangle_\varepsilon}{\beta|\Lambda|}=g_{\varepsilon,\mu}(\Lambda), \tag{III.1}$$

including any finite number of derivatives, exists uniformly on compacts of $\mathbb{R}$ for FD, resp. of $(-\infty,-\mu)$ for BE.

These results are proved in Secs. IV and V.

Inferring Theorem II.1 from our information on the generating function is a standard argument from the theory of LD[7,6] (at least for subcritical densities $a<\rho_c$; the case $a\geqslant\rho_c$ requires more effort).

The probability of the event in question can be rewritten as

$$Q_\Lambda=\mathbb{P}(\{N_\Lambda\in|\Lambda|I\})=\langle\chi_{|\Lambda|I}(N_\Lambda)\rangle, \tag{III.2}$$

where χ_A is the indicator function of the set $A\subseteq\mathbb{R}$. To make this event typical we introduce the modified average

$$\langle\cdot\rangle_\lambda=\frac{1}{Z_\lambda}\langle\cdot\, e^{\beta\lambda N_\Lambda}\rangle, \tag{III.3}$$

where $\lambda<\lambda_{\max}$ and the partition function $Z_\lambda=\langle e^{\beta\lambda N_\Lambda}\rangle$. With respect to this new state, (III.2) can be expressed as

$$Q_\Lambda=Z_\lambda\langle e^{-\beta\lambda N_\Lambda}\chi_{|\Lambda|I}(N_\Lambda)\rangle_\lambda. \tag{III.4}$$

The upper bound for Q_Λ comes from the exponential Chebychev inequality,

$$Q_\Lambda\leqslant\langle e^{\beta\lambda(N_\Lambda-a|\Lambda|)}\rangle=Z_\lambda e^{-\beta\lambda a|\Lambda|} \tag{III.5}$$

for any $0<\lambda<\lambda_{\max}$. As regards the lower bound we have to distinguish between two cases.

Case 1: $a<\rho_c$. One uses (II.13)–(II.15) to show that there exists a $\lambda_o<\lambda_{\max}$ such that $g'(\lambda_o)=a$. Differentiating (III.1) twice w.r.t. λ we obtain

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\langle N_\lambda\rangle_{\lambda_o}}{|\Lambda|}=g'(\lambda_o)=\rho(\mu+\lambda_o)=a, \tag{III.6}$$

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\beta}{|\Lambda|}[\langle N_\Lambda^2\rangle_{\lambda_o}-(\langle N_\Lambda\rangle_{\lambda_o})^2]=\frac{d\rho}{d\mu}(\mu+\lambda_o), \tag{III.7}$$

which is finite. This means that the event $\{N_\Lambda\approx|\Lambda|a\}$ is typical for the new state and a law of large numbers holds. Notice that $\lambda_o>0$, since ρ is strictly increasing in μ. From (III.4), $\forall c\in(a,b)$,

$$Q_\Lambda\geqslant Z_{\lambda_o}\langle e^{-\beta\lambda_o N_\Lambda}\chi_{|\Lambda|[a,c]}(N_\Lambda)\rangle_{\lambda_o}\geqslant Z_{\lambda_o}e^{-\beta\lambda_o c|\Lambda|}\langle\chi_{|\Lambda|[a,c]}(N_\Lambda)\rangle_{\lambda_o}\geqslant\alpha Z_{\lambda_o}e^{-\beta\lambda_o c|\lambda|} \tag{III.8}$$

for some $\alpha\in(0,1)$ and $|\Lambda|$ large. In fact, $\langle\chi_{|\Lambda|[a,c]}(N_\Lambda)\rangle_{\lambda_0}\rightarrow\frac{1}{2}$, as $\Lambda\nearrow\mathbb{R}^d$. Therefore we obtain from (III.17), (III.20) and Theorem III.2:

$$g(\lambda_o)-\lambda_o c+o(1)\leqslant\frac{\log Q_\Lambda}{\beta|\Lambda|}\leqslant g(\lambda_o)-\lambda_o a+o(1). \tag{III.9}$$

Since $c\in(a,b)$ is arbitrary, we conclude that

$$\lim_{\Lambda\nearrow\mathbb{R}_d}\frac{\log Q_\Lambda}{\beta|\Lambda|}=g(\lambda_o)-\lambda_o a=f(a)=\sup_{x\in[a,b]}f(x), \tag{III.10}$$

where f is the rate function defined in (II.16). λ_o is the same as in the definition of the Legendre transform (II.16), because of (III.6). The last equality comes from the convexity of f.

Case 2: $a\geqslant\rho_c$. In this case the problem is that one cannot find a *fixed* λ_o that verifies (III.6). As we will show later, it is nevertheless possible, for each finite Λ, to choose a λ_Λ such that the average density is a, i.e., $\langle N_\Lambda\rangle_{\lambda_\Lambda}=a|\Lambda|$. However, a might not correspond to the typical density in the limit $\Lambda\nearrow\mathbb{R}^d$, in the sense that no law of large numbers like (III.7) is guaranteed. Therefore, establishing a lower bound for Q_Λ is not so immediate in this case, and we need the following lemma.

Lemma III.3: For every subdomain $\Lambda\subset\mathbb{R}^d$ *and every* $a>0$, *there exists a unique* $\lambda_\Lambda=\lambda_\Lambda(a)$ *such that* $\langle N_\Lambda\rangle_{\lambda_\Lambda}=a|\Lambda|$. *If* $a\geqslant\rho_c$, *then*

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\lambda_\Lambda=-\mu; \tag{III.11}$$

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\log Z_{\lambda_\Lambda}}{\beta|\Lambda|}=g(-\mu); \tag{III.12}$$

$$\liminf_{\Lambda\nearrow\mathbb{R}^d}\frac{1}{|\Lambda|}\ln\langle\chi_{[a|\Lambda|,a|\Lambda|+1)}(N_\Lambda)\rangle_{\lambda_\Lambda}=0. \tag{III.13}$$

The above is proven in Sec. VI.

Now, the first two lines of (III.8) are still valid, with λ_Λ replacing λ_0. Taking the log and dividing by $|\Lambda|$, one obtains, via (III.13), the first inequality in (III.9), again for λ_Λ. The second inequality comes for free from Case 1. Finally, (III.11) and (III.12) are used to show that

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\log Q_\Lambda}{\beta|\Lambda|}=g(-\mu)+\mu a=f(a)=\sup_{x\in[a,b]}f(x), \tag{III.14}$$

yielding the linear part of the BE rate function. Q.E.D.

Theorem III.2 also implies the central limit theorem for the density in Λ.

Corollary III.4: Under the assumptions of Theorem II.1, the moments of the variable $\xi_\Lambda = (N_\Lambda - \langle N_\Lambda \rangle)/|\Lambda|^{1/2}$ *converge, as* $\Lambda \nearrow \mathbb{R}^d$, *to those of a Gaussian with variance* $\beta^{-1}(d\rho/d\mu) \times (\mu)$.

Proof: The kth cumulant of ξ_Λ is given by

$$C_\Lambda(k) = \frac{1}{\beta^k |\Lambda|^{k/2}} \left[\frac{d^k}{d\lambda^k} \log \langle e^{\beta\lambda N_\Lambda} \rangle \right]_{\lambda=0}, \tag{III.15}$$

$k \geqslant 2$. From Theorem III.2, $C_\Lambda(2) \to \beta^{-1} g''(0) = \beta^{-1}(d\rho/d\mu)(\mu)$, whereas, for $k>2$, $C_\Lambda(k) \to 0$. Also $C_\Lambda(1)=0$. These limits are the cumulants of a centered Gaussian variable with the specified variance. Q.E.D.

IV. GENERATING FUNCTION

We derive a determinant formula for the generating function $\langle e^{\beta\lambda N_\Lambda} \rangle_\varepsilon$. With its help we prove the claims of Lemma III.1. We will see in the next section that it is convenient to introduce the variables $\zeta = e^{\beta\lambda}$ and $\tilde{\zeta} = \zeta - 1$.

By (II.4), we have

$$-\beta H_V + \beta\mu N_V = \langle \mathbf{a} | (-\beta h_V + \beta\mu 1_V) | \mathbf{a} \rangle = \langle \mathbf{a} | A_V | \mathbf{a} \rangle, \tag{IV.1}$$

$$\beta\lambda N_\Lambda = \langle \mathbf{a} | \beta\lambda \chi_\Lambda | \mathbf{a} \rangle = \langle \mathbf{a} | B_\Lambda | \mathbf{a} \rangle, \tag{IV.2}$$

which define A_V and B_Λ as linear operators on $L^2(V)$. Here and in the sequel the indicator function χ_Λ stands for the corresponding multiplication operator, i.e., the projector onto $L^2(\Lambda)$. We use the following identity.

Lemma IV.1: Let A, B be self-adjoint and bounded from above. Then there exists a self-adjoint operator C such that $e^A e^B e^A = e^C$ *and*

$$e^{\langle \mathbf{a}|A|\mathbf{a}\rangle} e^{\langle \mathbf{a}|B|\mathbf{a}\rangle} e^{\langle \mathbf{a}|A|\mathbf{a}\rangle} = e^{\langle \mathbf{a}|C|\mathbf{a}\rangle}$$

for both BE and FD.

Proof: See the Appendix.

Let us apply Lemma IV.1 with $A = A_V/2$ and $B = B_\Lambda$, after a symmetrization of the density matrix in (II.5). This and definition (II.8) yield

$$\langle e^{\beta\lambda N_\Lambda} \rangle = \lim_{V \nearrow \mathbb{R}^d} \frac{\mathrm{Tr}_{\mathrm{F}_-^V}(e^{\langle \mathbf{a}|C|\mathbf{a}\rangle})}{\mathrm{Tr}_{\mathrm{F}_-^V}(e^{\langle \mathbf{a}|A_V|\mathbf{a}\rangle})}. \tag{IV.3}$$

Evaluating the trace of a quadratic form in a_i^*, a_j is a standard calculation for both BE and FD. Let us consider first the case of fermions. For a self-adjoint operator A on $L^2(V)$ such that e^A is trace-class, we have

$$\mathrm{Tr}_{\mathrm{F}_-^V}(e^{\langle \mathbf{a}|A|\mathbf{a}\rangle}) = \det{}_V(1_V + e^A) = \det(1 + \chi_V e^A \chi_V), \tag{IV.4}$$

where $\det_V$ is the determinant on $L^2(V)$ and det the determinant on $L^2(\mathbb{R}^d)$. We resort here to the theory of infinite determinants, as found, e.g., in Ref. 20, Sec. XIII.17. e^{A_V} is obviously trace-class, and so is e^C, since e^{B_Λ} is bounded. Using the definition of C, we obtain

$$\frac{\det_V(1_V+e^C)}{\det_V(1_V+e^{A_V})}=\det_V[(1_V+e^{A_V})^{-1}(1_V+e^{A_V/2}e^{B_\Lambda}e^{A_V/2})]$$
$$=\det_V[1_V+(1_V+e^{A_V})^{-1}e^{A_V/2}(e^{B_\Lambda}-1_V)e^{A_V/2}]$$
$$=\det[1+\tilde{\zeta}\chi_\Lambda D_{V,-}\chi_\Lambda], \tag{IV.5}$$

where $D_{V,-}=(1+e^{A_V})^{-1}e^{A_V}$. We used the fact that $e^{B_\Lambda}=(e^{\beta\lambda}-1)\chi_\Lambda+1=\tilde{\zeta}\chi_\Lambda+1$ and the cyclicity of the trace in the definition of the determinant. Finally, from (IV.3) and (IV.5),

$$\langle e^{\beta\lambda N_\Lambda}\rangle_-=\lim_{V\nearrow\mathbb{R}^d}\det[1+\tilde{\zeta}\chi_\Lambda D_{V,-}\chi_\Lambda]. \tag{IV.6}$$

One would like to take the limit on V inside the determinant by replacing $D_{V,-}$ with the corresponding operator on $L^2(\mathbb{R}^d)$ defined as

$$(\widehat{D_-\psi})(k)=\widehat{d_-}(k)\hat{\psi}(k),\quad (D_-\psi)(x)=\int dy\, d_-(y-x)\psi(y), \tag{IV.7}$$

where ˆ denotes the Fourier transform and

$$\widehat{d_-}(k)=\frac{1}{1+e^{\beta(\epsilon(k)-\mu)}}. \tag{IV.8}$$

Notice that $\widehat{d_-}\in L^1(\mathbb{R}^d)$ by our assumptions on $\epsilon(k)$ and so $d_-\in L^\infty(\mathbb{R}^d)$. Moreover, (II.3) ensures that $d_-\in L^1(\mathbb{R}^d)$.

By Ref. 20, Sec. XIII.17, Lemma 4(d), one has to establish that $\chi_\Lambda D_{V,-}\chi_\Lambda$ tends to $\chi_\Lambda D_-\chi_\Lambda$ in the trace norm.

Lemma IV.2: Let $\hat{d}$ be a continuous integrable function on $\mathbb{R}^d$. We define D through (IV.7) *as a linear operator acting on $L^2(\mathbb{R}^d)$. Furthermore, we define D_V by $D_V|k\rangle=\hat{d}(k)|k\rangle$ on $L^2(V)$ and by $D_V=0$ on the orthogonal complement $L^2(\mathbb{R}^d\setminus V)$. Then, for $\Lambda\subset V$, $\chi_\Lambda D_V\chi_\Lambda$ and $\chi_\Lambda D\chi_\Lambda$ are trace class, and*

$$\lim_{V\nearrow\mathbb{R}^d}\mathrm{Tr}|\chi_\Lambda(D_V-D)\chi_\Lambda|=0.$$

Proof: See the Appendix.

We conclude that

$$\langle\zeta^{N_\Lambda}\rangle_-=\det(1+\tilde{\zeta}\chi_\Lambda D_-\chi_\Lambda), \tag{IV.9}$$

with $\tilde{\zeta}=\zeta-1$.

For bosons we proceed in the same way, except that (IV.4) is replaced by

$$\mathrm{Tr}_{\mathrm{F}_+^V}(e^{\langle\mathbf{a}|A|\mathbf{a}\rangle})=\det_V(1_V-e^A)^{-1}, \tag{IV.10}$$

requiring in addition $\|e^A\|<1$. In fact, for $\|e^A\|\geqslant 1$, the lhs of (IV.10) is ∞, whereas the rhs might be finite if 1 is not an eigenvalue of the trace-class operator e^A. In our case, by assumption $\|e^{A_V}\|<1$. As for e^C, the function

$$\lambda\mapsto\|e^C\|=\|(e^{\beta\lambda}-1)e^{A_V/2}\chi_\Lambda e^{A_V/2}+e^{A_V}\| \tag{IV.11}$$

is increasing and $\lambda_{\max}(\Lambda)$ is defined to be that λ which makes it equal to 1. Since the rhs of (IV.11) is increasing in Λ and its sup is $e^{\beta\lambda}\|e^{A_V}\|=e^{\beta(\lambda+\mu)}$, then one checks that $\lambda_{\max}(\Lambda)\searrow-\mu$, as $\Lambda\nearrow\mathbb{R}^d$. Therefore, following the computation for FD, we have

$$\langle \zeta^{N_\Lambda} \rangle_+ = \lim_{V \nearrow \mathbb{R}^d} \frac{\det(1-e^C)^{-1}}{\det(1-e^{A_V})^{-1}} = \det(1+\tilde{\zeta}\chi_\Lambda D_+ \chi_\Lambda)^{-1}, \tag{IV.12}$$

for $\lambda < \lambda_{\max}$ and ∞ otherwise. Here D_+, the limit of $D_{V,+} = (e^{A_V}-1)^{-1} e^{A_V}$, is defined as in (IV.7) with

$$\widehat{d_+}(k) = \frac{1}{1-e^{\beta(\epsilon(k)-\mu)}}. \tag{IV.13}$$

Equation (IV.12) is the analog of (IV.9) and proves Lemma III.1.

V. INFINITE VOLUME LIMIT

Instead of the chemical potential, in this section we use the fugacity $z = e^{\beta\mu}$, regarding it as a complex variable. This will come in handy for the proof of Theorem III.2. The variables ζ and $\tilde{\zeta}$, defined at the beginning of the previous section, will also be extended to the complex plane. In this setup the translated pressure (II.12) becomes

$$g_z(\zeta) = p(z\zeta) - p(z), \tag{V.1}$$

where, with a slight abuse of notation, we keep the same name for the pressure as a function of the fugacity.

Expressions (II.9) and (II.10) for the pressure and the average density define two analytic functions of μ in

$$E_+ = \{\mathrm{Re}\,\mu < 0\} \cup \{\mathrm{Re}\,\mu \geq 0, \mathrm{Im}\,\mu \neq 2\pi j/\beta, \forall j \in \mathbb{Z}\}, \tag{V.2}$$

$$E_- = \{\mathrm{Re}\,\mu < 0\} \cup \{\mathrm{Re}\,\mu \geq 0, \mathrm{Im}\,\mu \neq (2j+1)\pi/\beta, \forall j \in \mathbb{Z}\}. \tag{V.3}$$

Hence $g_\varepsilon(\zeta)$ is analytic in

$$G_+ = \mathbb{C} \backslash [z^{-1}, +\infty); \quad G_- = \mathbb{C} \backslash (-\infty, -z^{-1}]. \tag{V.4}$$

We proceed to give the proof of Theorem III.2. Let $K \subset G_\varepsilon$ be a compact set in the complex plane. We choose K such that $L = K \cap \mathbb{R}^+$ is also compact, since its image through the function $\zeta \mapsto \lambda$ verifies the hypotheses of the theorem. Our argument, however, is valid for any K. Without loss of generality, we can assume that $1 \in K$.

For ζ restricted to $G_\epsilon \cap \mathbb{R}^+$, let us define

$$\phi^\Lambda_{\varepsilon,z}(\zeta) = \frac{1}{|\Lambda|} \log \langle \zeta^{N_\Lambda} \rangle_{\varepsilon,z} = -\frac{\varepsilon}{|\Lambda|} \mathrm{Tr} \log(1+\tilde{\zeta}\chi_\Lambda D_\varepsilon \chi_\Lambda) \tag{V.5}$$

according to (IV.9) and (IV.12). The proof of Theorem III.2 will be subdivided into three steps.

(1) ϕ^Λ_ε can be analytically continued to G_ε.
(2) There is a positive r such that $\phi^\Lambda(\zeta)$ converges uniformly to $\beta g_z(\zeta)$ for $|\zeta - 1| \leq r$.
(3) $|\phi^\Lambda|$ is uniformly bounded on K. Therefore, by Vitali's lemma (Ref. 21, Sec. 5.21), ϕ^Λ and any finite number of its derivatives converge uniformly on K.

Step 1: We leave the proof of the following lemma for the Appendix.
Lemma V.1: The function $\phi^\Lambda_\varepsilon(\zeta)$, *as defined by the trace in* (V.5), *is analytic in* G_ε.
Step 2: Expanding the log in (V.5) one has, for $|\tilde{\zeta}| < \|D_\varepsilon\|^{-1}$,

$$\phi^\Lambda_\varepsilon(\tilde{\zeta}+1) = -\frac{\varepsilon}{|\Lambda|} \mathrm{Tr} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (\tilde{\zeta}\chi_\Lambda D_\varepsilon \chi_\Lambda)^m. \tag{V.6}$$

We would like to interchange the summation with the trace. To do so, we need dominated convergence for the series:

$$|\tilde{\zeta}\chi_\Lambda D_\varepsilon \chi_\Lambda|^m \leq |\tilde{\zeta}|^m \|D_\varepsilon\|^{m-1}(\chi_\Lambda D_\varepsilon \chi_\Lambda). \tag{V.7}$$

Since $|\Lambda|^{-1}\,\mathrm{Tr}\,(\chi_\Lambda D_\varepsilon \chi_\Lambda)=d_\varepsilon(0)$ (see proof of Lemma IV.2 in the Appendix), each term of (V.6) is bounded by a term of an integrable series independent of Λ. Therefore, for the same $\tilde{\zeta}$'s as above,

$$\phi_\varepsilon^\Lambda(\tilde{\zeta}+1)=-\varepsilon\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\tilde{\zeta}^m}{m}\frac{1}{|\Lambda|}\mathrm{Tr}\,(\chi_\Lambda D_\varepsilon \chi_\Lambda)^m. \tag{V.8}$$

Suppose that we are able to prove that

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{1}{|\Lambda|}\mathrm{Tr}\,(\chi_\Lambda D_\varepsilon \chi_\Lambda)^m=\int dk\,[\hat{d}_\varepsilon(k)]^m, \tag{V.9}$$

with a rest bounded above by mR^m for some positive constant R. Then, using (IV.8) and (IV.13), we would have that, for any $r<\min\{\|D_\epsilon\|^{-1},R^{-1}\}$, uniformly for $|\tilde{\zeta}|\leq r$,

$$\begin{aligned}\lim_{\Lambda\nearrow\mathbb{R}^d}\phi_\varepsilon^\Lambda(\tilde{\zeta}+1)&=-\varepsilon\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\tilde{\zeta}^m}{m}\int dk\left(\frac{1}{1-\varepsilon z^{-1}e^{\beta\varepsilon(k)}}\right)^m\\&=-\varepsilon\int dk\log\left(1+\frac{\zeta-1}{1-\varepsilon z^{-1}e^{\beta\varepsilon(k)}}\right)\\&=-\varepsilon\int dk\log\left(\frac{1-\varepsilon z\zeta e^{-\beta\varepsilon(k)}}{1-\varepsilon z e^{-\beta\varepsilon(k)}}\right)=\beta g_z(\tilde{\zeta}+1),\end{aligned} \tag{V.10}$$

the last equality coming from (II.9). This would complete Step 2.

Let us pursue this project. One sees that

$$\begin{aligned}\int dk[\hat{d}_\epsilon(k)]^m&=(\underbrace{d_\epsilon * d_\epsilon * \cdots d_\epsilon}_{m\ \text{times}})(0)\\&=\frac{1}{|\Lambda|}\int_\Lambda dx_1\int_{\mathbb{R}^d}dx_2\,d_\varepsilon(x_1-x_2)\cdots\int_{\mathbb{R}^d}dx_m d_\varepsilon(x_{m-1}-x_m)d_\varepsilon(x_m-x_1).\end{aligned} \tag{V.11}$$

The normalized integration over x_1 is harmless since, by translation invariance, the integrand does not depend on that variable. On the other hand, it is not hard to verify that

$$\begin{aligned}\frac{1}{|\Lambda|}\mathrm{Tr}\,(\chi_\Lambda D_\varepsilon \chi_\Lambda)^m&=\frac{1}{|\Lambda|}\int_\Lambda dx_1\,\langle x_1|(\chi_\Lambda D_\varepsilon \chi_\Lambda)^m|x_1\rangle\\&=\frac{1}{|\Lambda|}\int_\Lambda dx_1\int_\Lambda dx_2\,d_\varepsilon(x_1-x_2)\,\ldots\int_\Lambda dx_m\,d_\varepsilon(x_{m-1}-x_m)d_\varepsilon(x_m-x_1).\end{aligned} \tag{V.12}$$

In view of (V.9), we want to compare (V.11) with (V.12). We observe that

$$\underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}}_{m-1\ \text{times}} - \underbrace{\int_{\Lambda}\int_{\Lambda}\cdots\int_{\Lambda}}_{m-1\ \text{times}} = \sum_{i=1}^{m-1} \underbrace{\int_{\Lambda}\cdots\int_{\Lambda}}_{i-1\ \text{times}} \int_{\Lambda^c} \underbrace{\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}}_{m--i\ \text{times}}. \tag{V.13}$$

Subtracting (V.11) from (V.12) leads then to $m-1$ terms of the form

$$\frac{1}{|\Lambda|}\int_{\Lambda} dx_1 \int_{\Lambda^c} dx_2\, d_\varepsilon(x_1-x_2)\ \ldots \int_{A_m} dx_m\, d_\varepsilon(x_{m-1}-x_m) d_\varepsilon(x_m-x_1), \tag{V.14}$$

where the sets $A_3,\ldots,A_m$ can be either $\mathbb{R}^d$ or Λ. Equation (V.14) holds because, due to the cyclicity of the integration variables, one can cyclically permute the order of integration without touching the integrand. We overestimate by switching to absolute values and integrating $x_3,\ldots,x_m$ over $\mathbb{R}^d$,

$$\frac{1}{|\Lambda|}\int_{\Lambda} dx_1\, (\chi_{\Lambda^c-x_1}|d_\varepsilon|)*|d_\varepsilon|*\ldots*|d_\varepsilon|)(x_1) = \frac{1}{|\Lambda|}\int_{\Lambda} dx_1\, u_\Lambda(x_1), \tag{V.15}$$

which defines $u_\Lambda(x_1)$. To estimate this function, we use recursively the relation $\|f*g\|_\infty \leqslant \|f\|_\infty \|g\|_1$ and obtain

$$u_\Lambda(x_1) \leqslant \|d_\varepsilon\|_1^{m-1} \sup_{\Lambda^c - x_1} |d_\varepsilon|. \tag{V.16}$$

Recalling now the definition of Λ' given before the statement of Theorem II.1, one sees that, if $x_1 \in \Lambda'$ and $y \in \Lambda^c - x_1$, then $|y| \to \infty$ as $\Lambda \nearrow \mathbb{R}^d$. Hence, from (V.16),

$$\sup_{x_1\in\Lambda'} \sup_{m\geqslant 1} \|d_\varepsilon\|_1^{-m+1} u_\Lambda(x_1) \to 0. \tag{V.17}$$

Also from (V.16), *pointwise in* x_1,

$$\sup_{m\geqslant 1} \|d_\varepsilon\|_1^{-m+1} u_\Lambda(x_1) \leqslant \|d_\varepsilon\|_\infty. \tag{V.18}$$

When we average over $x_1 \in \Lambda$, the last two relations and the properties of Λ' prove that

$$\lim_{\Lambda\nearrow\mathbb{R}^d} \sup_{m\geqslant 1} \|d_\varepsilon\|_1^{-m+1} \frac{1}{|\Lambda|}\int_{\Lambda} dx_1\, u_\Lambda(x_1) = 0. \tag{V.19}$$

This takes care of each term as in (V.14), and we have $m-1$ of these terms. Hence (V.9) holds with $R = \|d_\varepsilon\|_1$. This ends Step 2.

Step 3: Again we expand (V.5) in powers of $\tilde{\zeta}$, but this time about a generic $\tilde{\zeta}_0 \in G_\varepsilon - 1$ [see (V.4)]. We obtain

$$\frac{1}{|\Lambda|}\mathrm{Tr}\log(1+\tilde{\zeta}\chi_\Lambda D_\varepsilon \chi_\Lambda) = \frac{1}{|\Lambda|}\mathrm{Tr}\log(1+\tilde{\zeta}_0\chi_\Lambda D_\varepsilon\chi_\Lambda) + \frac{1}{|\Lambda|}\mathrm{Tr}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m} \times((1+\tilde{\zeta}_0\chi_\Lambda D_\varepsilon\chi_\Lambda)^{-1}\chi_\Lambda D_\varepsilon\chi_\Lambda)^m(\tilde{\zeta}-\tilde{\zeta}_0)^m. \tag{V.20}$$

Let us estimate this series. First of all, using some spectral theory (Ref. 22, Sec. 7.4),

$$\|(1+\tilde{\zeta}_0\chi_\Lambda D_\varepsilon\chi_\Lambda)^{-1}\| \leqslant [\mathrm{dist}(1,\sigma(-\tilde{\zeta}_0\chi_\Lambda D_\varepsilon\chi_\Lambda))]^{-1} \leqslant [\mathrm{dist}(1,\sigma(-\tilde{\zeta}_0 D_\varepsilon))]^{-1}, \tag{V.21}$$

since we know from definitions (IV.7), (IV.8), and (IV.13) that

$$\sigma(\chi_\Lambda D_- \chi_\Lambda) \subset [0,\|\chi_\Lambda D_- \chi_\Lambda\|] \subset [0,\|D_-\|] = \sigma(D_-) = [0,1/(1+z^{-1})], \tag{V.22}$$

$$\sigma(\chi_\Lambda D_+ \chi_\Lambda) \subset [-\|\chi_\Lambda D_+ \chi_\Lambda\|,0] \subset [-\|D_+\|,0] = \sigma(D_+) = [1/(1-z^{-1}),0]. \tag{V.23}$$

Repeating the same reasoning as in Step 2, we use the above to exchange the trace with the summation in (V.20)—which is legal for small $|\tilde{\zeta}-\tilde{\zeta}_0|$ as to be determined shortly. This yields a new series, whose mth term is bounded above by

$$[\mathrm{dist}\,(1,\sigma(-\tilde{\zeta}_0 D_\varepsilon))]^{-m} \|D_\varepsilon\|^{m-1} d_\varepsilon(0) |\tilde{\zeta}-\tilde{\zeta}_0|^m = a(b(\tilde{\zeta}_0)|\tilde{\zeta}-\tilde{\zeta}_0|)^m, \tag{V.24}$$

where $d_\varepsilon(0) = |\Lambda|^{-1}\,\mathrm{Tr}\,(\chi_\Lambda D_\varepsilon \chi_\Lambda)$. Hence, in view of (V.5), (V.20) implies

$$|\phi_\varepsilon^\Lambda(\tilde{\zeta}+1)| \leqslant |\phi_\varepsilon^\Lambda(\tilde{\zeta}_0+1)| + a\,\frac{b(\tilde{\zeta}_0)|\tilde{\zeta}-\tilde{\zeta}_0|}{1-b(\tilde{\zeta}_0)|\tilde{\zeta}-\tilde{\zeta}_0|} \leqslant |\phi_\varepsilon^\Lambda(\tilde{\zeta}_0+1)| + a, \tag{V.25}$$

for $|\tilde{\zeta}-\tilde{\zeta}_0| \leqslant (2b(\tilde{\zeta}_0))^{-1}$.

The crucial fact is that $b(\tilde{\zeta})^{-1}$ stays away from zero when $\tilde{\zeta}$ is away from the boundary of $G_\varepsilon - 1$. This can be seen via the following argument, exploiting (V.24) and (V.22) and (V.23). In the FD case $\sigma(-\tilde{\zeta}_0 D_-)$ is a segment that has one endpoint at the origin and the phase of $-\tilde{\zeta}_0$ is the angle it forms with the positive semi-axis. This means that, as long as $\tilde{\zeta}_0$ does not go anywhere near the negative semi-axis, we are safe. For $\tilde{\zeta}_0 \in (-z^{-1}-1,0)$ [see (V.4)], $\sigma(-\tilde{\zeta}_0 D_-)$ is contained in $\mathbb{R}_o^+$. However, notice from (V.22) that the other endpoint is located at $-\tilde{\zeta}_0/(1+z^{-1}) < 1$. For BE the reasoning is analogous, except that in this case the phase of $\tilde{\zeta}_0$ is the angle between $\sigma(-\tilde{\zeta}_0 D_+)$ and $\mathbb{R}_o^+$. Therefore the ''safe'' span is the complement of the positive semi-axis. Also, if $\tilde{\zeta}_0 \in (0,z^{-1}-1)$ [again see (V.4)], the ''floating'' endpoint of $\sigma(-\tilde{\zeta}_0 D_+)$ is found at $\tilde{\zeta}_0/(z^{-1}-1) < 1$.

With the above estimate we can use (V.25) recursively. If $|\tilde{\zeta}_0| \leqslant r$, from Step 2, $|\phi_\varepsilon^\Lambda(\tilde{\zeta}_0+1)| \leqslant M$, for some M, since ϕ_ε^Λ converges uniformly there. Then, from (V.25), we have that $|\phi_\varepsilon^\Lambda(\tilde{\zeta}_1+1)| \leqslant M+a$, for any $\tilde{\zeta}_1$ such that $|\tilde{\zeta}_1-\tilde{\zeta}_0| < (2b(\tilde{\zeta}_0))^{-1}$. Proceeding, we see that $|\phi_\varepsilon^\Lambda(\tilde{\zeta}_k+1)| \leqslant M+ka$, whenever $|\tilde{\zeta}_k-\tilde{\zeta}_{k-1}| < (2b(\tilde{\zeta}_{k-1}))^{-1}$. In this way we will cover K in finitely many steps since it keeps at a certain distance from the boundary of G_ε and the $(b(\tilde{\zeta}_k))^{-1}$ are bounded below. This completes Step 3, i.e., $\phi_\varepsilon^\Lambda(\zeta)$ is bounded on K and Vitali's lemma can be applied. Q.E.D.

VI. SUPERCRITICAL DENSITIES

For bosons in high dimension, Theorem III.2 is not enough to establish the LD result for supercritical densities. In this section we prove Lemma III.3. In particular, we derive a useful property of the distribution of N_Λ, w.r.t. the state $\langle\cdot\rangle_{\lambda_\Lambda}$, with λ_Λ chosen as stated in the lemma. $\varepsilon = +1$ will be understood in the reminder.

Letting the chemical potential go to zero in such a way that the average density remains constant (and bigger than ρ_c) is the usual way to proceed in the theory of Bose–Einstein condensation.[19,23,24] The limiting distribution of the global density $N_V/|V|$ is called *the Kac distribution*, and has been derived for several choices of V, $\epsilon(k)$, and d.[23–26,9] We do not go as far in this paper. It is safe to say, however, that there is no reason to expect the distribution of $N_\Lambda/|\Lambda|$ to become degenerate [which would make (III.13) trivial].

Proof of Lemma III.3: Using a sloppy notation, let us write $\phi^\Lambda(\lambda)$ for $\phi^\Lambda(e^{\beta N_\Lambda})$ [see definition (V.5)]. Differentiating this function, we get the mean density in the modified state: For $\lambda < \lambda_{\max}$,

$$\rho^\Lambda(\lambda)=\frac{d\phi^\Lambda}{d\lambda}(\lambda)=\frac{1}{|\Lambda|}\frac{\langle N_\Lambda e^{\beta\lambda N_\Lambda}\rangle}{\langle e^{\beta\lambda N_\Lambda}\rangle}=\frac{\langle N_\Lambda\rangle_\lambda}{|\Lambda|}, \tag{VI.1}$$

since, by virtue of Lemma III.1, we can use dominated convergence to differentiate inside the average. Likewise,

$$v^\Lambda(\lambda)=\frac{d\rho^\Lambda}{d\lambda}(\lambda)=\frac{1}{|\Lambda|}[\langle N_\Lambda^2\rangle_\lambda-(\langle N_\Lambda\rangle_\lambda)^2]. \tag{VI.2}$$

Thus, $\rho^\Lambda(\lambda)$ is increasing. The proof of Lemma III.1 gives that $\lim_{\lambda\to\lambda_{\max}}\rho^\Lambda(\lambda)=+\infty$. It is also clear that $\lim_{\lambda\to-\infty}\rho^\Lambda(\lambda)=0$. The above implies the existence and uniqueness of λ_Λ such that $\rho^\Lambda(\lambda_\Lambda)=a$.

Let us fix $a\geqslant\rho_c$. Theorem III.2 states in particular that, if $\Lambda\nearrow\mathbb{R}^d$, then $\rho^\Lambda(\lambda)\to\rho(\lambda)<\rho_c$, for $\lambda<-\mu$. This and Lemma III.1 yield (III.11).

Continuing, let us choose some $\lambda_1<-\mu$. From (III.11), $\lambda_1\leqslant\lambda_\Lambda$, for Λ big enough. Using the monotonicity of ϕ^Λ and ρ^Λ,

$$\phi^\Lambda(\lambda_\Lambda)-\phi^\Lambda(\lambda_1)=\rho^\Lambda(\overline{\lambda})(\lambda_\Lambda-\lambda_1)\leqslant a(\lambda_\Lambda-\lambda_1), \tag{VI.3}$$

where $\overline{\lambda}\in(\lambda_1,\lambda_\Lambda)$ is given by Lagrange's mean value theorem. Due to λ_1 being arbitrary, Theorem III.2 and (VI.3) imply

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\phi^\Lambda(\lambda_\Lambda)=\lim_{\lambda_1\to-\mu}\beta g(\lambda_1)=\beta g(-\mu). \tag{VI.4}$$

This is precisely (III.22).

The proof of (III.13) is more elaborate. Looking at (IV.12) and the immediately following definition of $D_{V,+}$, we can write

$$\chi_\Lambda D_+\chi_\Lambda=(e^{-\beta(h'_\Lambda-\mu 1_\Lambda)}-1_\Lambda)^{-1}e^{-\beta(h'_\Lambda-\mu 1_\Lambda)}, \tag{VI.5}$$

which introduces a new one-particle Hamiltonian h'_Λ on $L^2(\Lambda)$. By Lemma IV.2, $\chi_\Lambda D\chi_\Lambda$ is trace-class, hence it has discrete spectrum. With the help of (V.23), we see that (VI.5) can be solved for h'_Λ. Thus, h'_Λ is well defined and has the same spectral decomposition as $\chi_\Lambda D\chi_\Lambda$. In particular, $\sigma(h'_\Lambda)$ is discrete. The eigenvalues of h'_Λ are indicated by $\epsilon'_j=\epsilon'_j(\Lambda)$, and are assumed to be in increasing order.

The idea behind this definition is to eliminate the cutoff χ_Λ in Eq. (IV.12), which is responsible for all the complications in the proof of Theorem III.2, and is the only manifestation that we are dealing with LD in Λ. The effective Hamiltonian h'_Λ allows one to think of a system of free bosons in the container Λ and apply the available results for global fluctuations.[23,27,9] The drawback is that in general we have no precise information about $\sigma(h'_\Lambda)$. Even so, it is possible to determine the ground state of h'_Λ. In fact, by (VI.5),

$$1_\Lambda+(e^{\beta\lambda}-1_\Lambda)\chi_\Lambda D\chi_\Lambda=(1_\Lambda-e^{-\beta(h'_\Lambda-\mu 1_\Lambda)})^{-1}(1_\Lambda-e^{-\beta(h'_\Lambda-(\mu+\lambda)1_\Lambda)}) \tag{VI.6}$$

(cf. Step 2 in Sec. V). Since $\chi_\Lambda D\chi_\Lambda$ is negative, the inf of the lhs of (VI.6) (in the sense of the quadratic form) is a decreasing function of λ, and attains zero at $\lambda_{\max}$. This is so by the very definition of $\lambda_{\max}$—see (IV.11) and (IV.12). On the other hand, the inf of the rhs of (VI.6) is zero if, and only if, $e^{-\beta(\epsilon'_0-\mu-\lambda)}=1$, where

$$\epsilon'_0(\Lambda)=\lambda_{\max}(\Lambda)+\mu>0. \tag{VI.7}$$

Let φ_j denote the eigenfunction relative to ϵ'_j, P_j the corresponding projector in $L^2(\Lambda)$, and a_j^* the creation operator on $\mathcal{F}^V$. We introduce $N_\Lambda^{(j)}=a_j^*a_j$, the operator for the number of particles in the state φ_j. $\{N_\Lambda^{(j)}\}$ is a commuting family and $N_\Lambda = \Sigma_{j=0}^\infty N_\Lambda^{(j)}$. We want to verify that these operators behave like independent random variables w.r.t. $\langle\cdot\rangle_\lambda$. We can study their joint generating function, employing the same techniques as in Sec. IV. In fact, for η_j bounded, define $B'_\Lambda = \Sigma_j \eta_j P_j + \beta\lambda\chi_\Lambda$ and replace (IV.2) with

$$\langle \mathbf{a}|B'_\Lambda|\mathbf{a}\rangle = \sum_{j=0}^{\infty} \eta_j N_\Lambda^{(j)} + \beta\lambda N_\Lambda . \tag{VI.8}$$

One verifies that, in $L^2(V)$ or in $L^2(\mathbb{R}^d), e^{B'_\Lambda} = \Sigma_j e^{\eta_j + \beta\lambda} P_j + 1 - \chi_\Lambda$. In particular $e^{B'_\Lambda} - 1 = \chi_\Lambda (e^{B'_\Lambda} - 1)\chi_\Lambda$. This allows us to proceed as in (IV.12), and write

$$Z_\lambda \langle e^{\Sigma_j \eta_j N_\Lambda^{(j)}}\rangle_\lambda = \langle e^{\langle \mathbf{a}|B'_\Lambda|\mathbf{a}\rangle}\rangle = \lim_{V\nearrow\mathbb{R}^d} \det[1 + (e^{B'_\Lambda} - 1)\chi_\Lambda D_V \chi_\Lambda]^{-1}. \tag{VI.9}$$

Taking the above limit is slightly more complicated than the corresponding computation in Sec. IV. Since $e^{B'_\Lambda} - 1$ is bounded,

$$\mathrm{Tr}|(e^{B'_\Lambda} - 1)\chi_\Lambda (D_V - D)\chi_\Lambda| \leqslant \|e^{B'_\Lambda} - 1\| \, \mathrm{Tr}|\chi_\Lambda (D_V - D)\chi_\Lambda|. \tag{VI.10}$$

One then applies Lemma IV.2 and Ref. 20, Sec. XIII.17, Lemma 4(d), so that (VI.9) gives

$$\begin{aligned}\langle e^{\Sigma_j \eta_j N_\Lambda^{(j)}}\rangle_\lambda &= Z_\lambda^{-1} \det[1 + (e^{B'_\Lambda} - 1)\chi_\Lambda D \chi_\Lambda]^{-1} \\ &= Z_\lambda^{-1} \det{}_\Lambda [1_\Lambda + (e^{B'_\Lambda} - 1_\Lambda)(e^{-\beta(h'_\Lambda - \mu 1_\Lambda)} - 1_\Lambda)^{-1} e^{-\beta(h'_\Lambda - \mu 1_\Lambda)}]^{-1} \\ &= \prod_{j=0}^{\infty} \frac{1 - e^{-\beta(\epsilon'_j - \mu - \lambda)}}{1 - e^{-\beta(\epsilon'_j - \mu - \lambda) + \eta_j}},\end{aligned} \tag{VI.11}$$

having used (IV.12) and (VI.6) to express Z_λ.

This shows that the $N_\Lambda^{(j)}$'s represent a set of independent, geometrically distributed random variables, with averages $\langle N_\Lambda^{(j)}\rangle_\lambda = (e^{\beta(\epsilon'_j - \mu - \lambda)} - 1)^{-1}$. At this point we can apply Ref. 27, Lemma 2, to $N_\Lambda^{(0)}$ and $N_\Lambda - N_\Lambda^{(0)}$. We obtain

$$\langle \chi_{[a|\Lambda|, a|\Lambda|+1)}(N_\Lambda)\rangle_{\lambda_\Lambda} \geqslant \frac{1}{a|\Lambda|} e^{-\beta(\epsilon'_0(\Lambda) - \mu - \lambda_\Lambda)(a|\Lambda|+2)}. \tag{VI.12}$$

Assertion (III.13) is derived from (VI.12) via (III.11), (VI.7) and Lemma III.1. Q.E.D.

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APPENDIX: PROOFS

1. Relativistic massless particles

We prove that the energy dispersion $\epsilon(k) = c|k|$ satisfies our assumptions. The only condition to be checked is (II.3), that is, the Fourier transform of $k \mapsto (e^{\beta(c|k| - \mu)} - \varepsilon)^{-1}$ is in $L^1(\mathbb{R}^d)$. This is a consequence of the following.

Lemma A.1: Let $f:[0,+\infty) \to \mathbb{C}$ *be of Schwartz class. With the common abuse of notation, denote by* $\hat{f}(|\xi|)$ *the Fourier transform of* $f(|x|)$, *for x,* $\xi \in \mathbb{R}^d$. *Then, for some positive C,*

$$\hat{f}(|\xi|) \leqslant \frac{C}{|\xi|^{d+1}}.$$

Proof: For simplicity let us write $\xi = |\xi|$. The Fourier transform of a radial function is

$$\hat{f}(\xi) = \frac{(2\pi)^{d/2}}{\xi^{d/2-1}} \int_0^\infty dr\, f(r) r^{d/2} J_{d/2-1}(r\xi), \tag{A1}$$

cf. Ref. 28, Chap. IV, Thm. 3.3, where J_ν is the standard Bessel function of order ν.[29] One has

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \tag{A2}$$

for $x \to 0$, whereas

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{2\nu+1}{4}\pi \right) + g(x) \right], \tag{A3}$$

with $g(x) \to 0$ for $x \to \infty$. Using the relation

$$\int_0^x dt\, t^\nu J_{\nu-1}(t) = x^\nu J_\nu(x), \tag{A4}$$

we integrate (A1) by parts repeatedly, taking into account also (A2) and the hypothesis on f. After n integrations we get, up to constants, n terms of the form

$$\frac{1}{\xi^{d/2+n-1}} \int_0^\infty dr\, f^{(i)}(r) r^{d/2-n+i} J_{d/2+n-1}(r\xi), \tag{A5}$$

with $i = 1,...,n$. For our purposes it suffices to iterate up to $n \geqslant d/2+2$. In fact, if i is such that $d/2 - n + i > -d$, then in (A5) we can estimate the Bessel function by a constant. The integral converges by the rapid decay of $f^{(i)}$ and the whole term is of the order ξ^{-d-1} or better. For smaller values of i, the estimate uses (A2), for $x \in [0,a]$, and (A3) otherwise. Since $|f^{(i)}| \leqslant c$, (A5) is bounded by

$$\frac{A}{\xi^{d/2+n-1}} \int_0^{a/\xi} dr\, r^{d/2-n+i} (r\xi)^{d/2+n-1} + \frac{B}{\xi^{d/2+n-1}} \int_{a/\xi}^\infty dr\, r^{d/2-n+i} (r\xi)^{-1/2} \approx \frac{1}{\xi^{d+i}}, \tag{A6}$$

the second integral being convergent because of the choice of i. Q.E.D.

2. Proof of Lemma IV.1

As before, we set $\varepsilon = \pm 1$, according to either bosons or fermions. A general $f \in L^2(V)$ can be expanded in the Fourier basis as $f = \Sigma_k f_k |k\rangle$. The corresponding creation operator is then defined by

$$a(f)^* = \sum_{k \in V'} f_k a_k^*. \tag{A7}$$

For the sake of simplicity, we denote $\mathcal{A} = \langle \mathbf{a} | A | \mathbf{a} \rangle = \Sigma_{ij} A_{ij} a_i^* a_j$ (same for $\mathcal{B}$). Recalling the canonical (anti)commutation relations,

$$[a_i, a_j^*]_{-\varepsilon} = a_i a_j^* - \varepsilon a_j^* a_i = \delta_{ij}; \tag{A8}$$

$$[a_i, a_j]_{-\varepsilon} = a_i a_j - \varepsilon a_j a_i = 0,$$

one calculates that

$$[\mathcal{A}, a(f)^*] = a(Af)^*, \tag{A9}$$

and in exponential form

$$e^{tA} a(f)^* e^{-tA} = a(e^{tA} f)^*. \tag{A10}$$

Now, let $|0\rangle$ be the ground state of $\mathcal{F}^V$. For $n \in \mathbb{N}$, and $f_1, f_2, \ldots, f_n \in L^2(V)$, the finite linear combinations of the states

$$|f_1, f_2, \ldots, f_n\rangle = a(f_1)^* a(f_2)^* \ldots a(f_n)^* |0\rangle \tag{A11}$$

are dense in $\mathcal{F}$, which is another way of stating that $|0\rangle$ is cyclic w.r.t. the algebra generated by the creation operators. Therefore, we need only test our assertion on vectors of the type (A11). Using (A10) with $t=1$, and observing that $\mathcal{A}|0\rangle = 0$, we obtain

$$e^A |f_1, \ldots, f_n\rangle = e^A a(f_1)^* e^{-A} \ldots e^A a(f_n)^* e^{-A} |0\rangle = a(e^A f_1)^* \ldots a(e^A f_n)^* |0\rangle = |e^A f_1, \ldots, e^A f_n\rangle. \tag{A12}$$

The existence of C is a consequence of the spectral theorem. We call $\mathcal{C}$ the corresponding quadratic form in a_i^*, a_j. Through the repeated use of (A12), one checks that applying $e^A e^B e^A$ to the states (A11) is the same as applying e^C. The semiboundedness of A and B ensures that the domain of their exponentials is the whole $L^2(V)$ and all quantities are well defined. Q.E.D.

3. Proof of Lemma IV.2

For any symmetric operator $A, \chi'_\Lambda A \chi_\Lambda \leqslant \chi_\Lambda |A| \chi_\Lambda$. Hence $|\chi_\Lambda A \chi_\Lambda| \leqslant \chi_\Lambda |A| \chi_\Lambda$ and $\mathrm{Tr}\,|\chi_\Lambda A \chi_\Lambda| \leqslant \mathrm{Tr}_\Lambda |A|$. When $A = D_V$, the convergence of the trace is proven by writing the further estimate $\mathrm{Tr}_\Lambda |D_V| \leqslant \mathrm{Tr}_V |D_V|$ and then summing an integrable sequence of discrete eigenvalues. For $A = D$, one uses the Dirac-delta representation of the trace to find out that $\mathrm{Tr}_\Lambda |D| = |\Lambda| (2\pi)^{-d} \int dk |\hat{d}|$. The first assertion of the lemma has been proven.

As for the second part, let us write

$$\mathrm{Tr}\,(|\chi_\Lambda (D_V - D) \chi_\Lambda|) = \mathrm{Tr}\,(U \chi_\Lambda (D_V - D) \chi_\Lambda) = \mathrm{Tr}_V\,(U \chi_\Lambda D_V \chi_\Lambda) - \mathrm{Tr}\,(U \chi_\Lambda D \chi_\Lambda) = T_l - T, \tag{A13}$$

where U is the partial isometry $L^2(\Lambda) \to L^2(\Lambda)$ that realizes the spectral decomposition as in Ref. 20, Thm. IV.10. It is convenient to use the position representation for the bases. So, $\psi^{(k)}(x) = e^{ik \cdot x}$ and, as defined in Sec. II, $\psi_V^{(k)} = \psi^{(k)} \chi_V$. Let us work on T_l: using the cyclicity of the trace one obtains

$$\begin{aligned} T_l &= \frac{1}{l^d} \sum_{k \in V'} \langle \psi_V^{(k)} | \chi_\Lambda U \chi_\Lambda D_V | \psi_V^{(k)} \rangle \\ &= \frac{1}{l^d} \sum_{k \in V'} \hat{d}(k) \langle \psi_V^{(k)} | \chi_\Lambda U \chi_\Lambda | \psi_V^{(k)} \rangle \\ &= \frac{1}{l^d} \sum_{k \in V'} \hat{d}(k) \langle \psi^{(k)} | \chi_\Lambda U \chi_\Lambda | \psi^{(k)} \rangle, \end{aligned} \tag{A14}$$

the last equality being due to the presence of the indicator functions χ_Λ. In complete analogy with the above,

$$T = \frac{1}{(2\pi)^d} \int dk\, \hat{d}(k) \langle \psi^{(k)} | \chi_\Lambda U \chi_\Lambda | \psi^{(k)} \rangle. \tag{A15}$$

Since $|\langle \psi^{(k)} | \chi_\Lambda U \chi_\Lambda | \psi^{(k)} \rangle| \leq |\Lambda|$, it is obvious that (A14) tends to (A15) for $l \to \infty$. Q.E.D.

4. Proof of Lemma V.1

With regard to (IV.9) and (IV.12), $\det(1+\tilde{\zeta}\chi_\Lambda D_\varepsilon \chi_\Lambda)$ is entire in $\tilde{\zeta}$ (hence in ζ) by Ref. 20, Sec. XIII.17, Lemma 4(c). In order to evaluate its log (on the suitable Riemann surface) we need to avoid the zeros. Using Ref. 20, Thm. XIII. 106, we want to make sure that $\sigma(-\tilde{\zeta}\chi_\Lambda D_\epsilon \chi_\Lambda)$ does not hit 1. Step 3 in Sec. IV [see in particular formulas (V.22) and (V.23) and the last paragraphs] shows that this is never the case if $\tilde{\zeta} \notin (-\infty, -z^{-1}-1)$ for FD, or $\tilde{\zeta} \notin [z^{-1}-1, +\infty)$ for BE. Q.E.D.

Actually, we can say more. Consider FD, just to fix the ideas. We see from (V.22) that the ''floating'' endpoint of $\sigma(-\tilde{\zeta}\chi_\Lambda D_- \chi_\Lambda)$ is strictly contained in the segment $(0, -\tilde{\zeta}/(1+z^{-1}))$, which means that $\tilde{\zeta}$ is allowed to exceed slightly $G_- - 1$, as given by (V.4), without any vanishing of (IV.9). For bosons this is related to Lemma III.1. In this case the ''forbidden region'' is $[\tilde{\zeta}_{\max}, +\infty)$, where $\tilde{\zeta}_{\max} = e^{\beta\lambda_{\max}} - 1$ (see also the proof of Lemma III.3).

In conclusion, for each finite Λ, the domain of analyticity of $\phi_\epsilon^\Lambda(\zeta)$ is indeed strictly bigger than G_ε.

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