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Kinetics of a Model Weakly Ionized Plasma in the Presence of Multiple Equilibria

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Abstract

We study, globally in time, the velocity distribution $f(v,t)$ of a spatially homogeneous system that models a system of electrons in a weakly ionized plasma, subjected to a constant external electric field E. The density f satisfies a Boltzmann-type kinetic equation containing a fully nonlinear electron-electron collision term as well as linear terms representing collisions with reservoir particles having a specified Maxwellian distribution. We show that when the constant in front of the nonlinear collision kernel, thought of as a scaling parameter, is sufficiently strong, then the L^1 distance between f and a certain time-dependent Maxwellian stays small uniformly in t. Moreover, the mean and variance of this time-dependent Maxwellian satisfy a coupled set of nonlinear ordinary differential equations that constitute the "hydrodynamical" equations for this kinetic system. This remains true even when these ordinary differential equations have non-unique equilibria, thus proving the existence of multiple stable stationary solutions for the full kinetic model. Our approach relies on scale-independent estimates for the kinetic equation, and entropy production estimates. The novel aspects of this approach may be useful in other problems concerning the relation between the kinetic and hydrodynamic scales globally in time.

1. Introduction

The mathematical understanding of equilibrium phenomena has greatly advanced in the past few decades. The elegant and precise theory of Gibbs measures provides a direct bridge between the microscopic and macroscopic descriptions of such systems. This includes a general conceptual framework as well as nontrivial explicit examples of the coexistence of multiple equilibrium phases for certain values of the macroscopic control parameters.

There is no comparable general theory for nonequilibrium systems, and the microscopic study of phase transition phenomena in such situations appears to be far beyond our mathematical grasp at the present time. Our mathematical understanding of the great variety of nonequilibrium phase transitions observed in fluids, plasmas, lasers, etc., therefore depends entirely on the study of bifurcations and other singular phenomena occurring in the nonlinear equations describing the *macroscopic* time evolution of such systems.

While there has been much progress recently in deriving such equations from simple microscopic and even realistic mesoscopic model evolutions, the passage to the macroscopic scale is well understood *only over time intervals in which the solutions of the macroscopic equations stay smooth*. This is true for example in the passage from kinetic theory, where the evolution is described by the Boltzmann equation, to hydrodynamics, where it is described by either the Euler or Navier-Stokes equations for compressible fluids, depending on how we choose our macroscopic time scale [1–6]. These derivations, which are based on Chapman-Enskog-type expansions, require for their validity the uniqueness and smoothness of the solutions of the hydrodynamic equations because the control of the error terms in the estimates depends on *a-priori* smoothness estimates for solutions of the macroscopic equations. Thus, they shed no light on the actual behavior of the mesoscopic description when the solution of the hydrodynamical equations develop singular behavior.

To overcome this problem it is clearly desirable to develop methods in which one does not use any a-priori smoothness estimates for solutions of the macroscopic equations, but instead uses scale-independent estimates on the mesoscopic equation. This is what we do here for a simple model inspired by plasma physics [7–9].

Our starting point is a description of the system by kinetic theory. Grave difficulties are posed by the fact that, as of yet, not very much is known in the way of *a-priori* regularity estimates for solutions of the spatially inhomogeneous Boltzmann equation. This is quite different, however, from the lack of estimates for the macroscopic equations — there it is clear that in the interesting cases the desired estimates just do not exist. Shock waves do form. In the Boltzmann case, however, it is likely that *a-priori* regularity estimates in the velocity variables, say, invariant under the Euler scaling, are there, but simply have not yet been discovered. Still, the lack of such estimates is a grave difficulty in the way of rigorous investigation of the problem at hand.

We sidestep this difficulty by considering a spatially homogeneous system, but one that is driven by an electric field, and coupled to heat reservoirs. In this case the usual hydrodynamic moments are not conserved and the system has nonequilibrium stationary states. We prove then in a certain simplified, but still recognizable physical situations, that the kinetic description closely tracks the macroscopic discription even when the driving is sufficiently strong for the latter to undergo phase transitions. More precisely, we show that the velocity distribution function is close to a Maxwellian parametrized by a temperature T and mean velocity u which satisfy certain non-linear equations, which are the macroscopic equations for this system. Moreover, it does so globally in time, even when the stationary solutions of these macroscopic equations are nonunique.

We are in fact particularly concerned with the stability of these stationary solutions — the existence of multiple stationary states being analogous to the coexistence of phases in equilibrium systems. For such questions we need results that guarantee that a solution of the kinetic equation stays near a solution of the macroscopic equations globally in time. This seems to be difficult to accomplish by standard expansion methods, at least in the range of driving field strengths where the macroscopic equations have the most interesting behavior. Instead of expansion methods, we use entropy production [10, 11] to show that the solution of the kinetic equations must stay close to *some* Maxwellian, globally in time. Then, we show that the moments of this Maxwellian must nearly satisfy the macroscopic equations. In this way we get our results. The next section specifies the model more closely, and states our main results. A preliminary account of this work in which the Boltzmann collisions were modeled by a Born-Kirkwood-Green (BKG) collision kernel was presented in [12].

2. The Model and the Results

Our formal setup is as follows: We consider a weakly ionized gas in $\mathbb{R}^3$ in the presence of an externally imposed constant electric field E. The density of the gas, the degree of ionization and the strength of the field are assumed to be such that: (i) the interactions between the electrons can be described by some nonlinear, Boltzmann type collision operator, and (ii) collisions between the electrons and the heavy components of the plasma, ions and neutrals, are adequately described by assuming the latter ones to have a spatially homogeneous time-independent Maxwellian distribution with an *a-priori* given temperature [7]. Under these conditions the time evolution of the spatially homogeneous velocity distribution function $f(v,t)$ satisfies a Boltzmann-type equation

$$\frac{\partial f(v,t)}{\partial t} = -E \cdot \nabla f + Lf + \varepsilon^{-1} Q(f), \tag{2.1}$$

where ∇ is the gradient with respect to v in $\mathbb{R}^3$, E is a constant force field and Q is a nonlinear collision term which takes either the form of the Boltzmann collision kernel for Maxwellian molecules, or the one corresponding to the BGK model. We treat both cases here because it is possible to provide a little more detail concerning the nature of the equilibria in the BGK case. The parameter $\varepsilon > 0$ is thought of as a scaling parameter that goes to zero in the hydrodynamical limit. The linear operator L represents the effect of collisions with reservoir particles. It is assumed to have the form

$$Lf(v) = L_1 f(v) + L_2 f(v), \tag{2.2}$$

with

$$L_1 f(v) = \nabla \cdot \left(D(v) M(v) \nabla \left(\frac{f(v)}{M(v)} \right) \right), \tag{2.3}$$

a Fokker-Planck operator, representing energy exchanges with the reservoir assumed to be at temperature $T = 1$, so that

$$M(v) = (2\pi)^{-3/2} \exp\left(-\tfrac{1}{2}|v|^2\right),$$

$$D(v) = a \exp\left(-\tfrac{1}{2}b|v|^2\right) + c \tag{2.4}$$

for some strictly positive constants a, b and c: the symbols a, b, c henceforth always refer to these parameters wherever they appear. The specific form (2.4) of the velocity-space diffusion coefficient is not important. We specify it for the sake of concreteness. The properties we really need for $D(v)$ will be clear from the proofs. The operator L_2 represents momentum exchanges with the heavy reservoir particles and is given by

$$L_2 f(v) = \nu[\bar{f}(v) - f(v)], \tag{2.5}$$

with ν a positive constant and $\bar{f}(v)$ the sphericalized average of $f(v)$.

For any probability density f, we let M_f denote the Maxwellian density with the same first and second moments as f. Explicitly,

$$M_f = (2\pi T)^{-3/2} \exp\left[-\frac{(v-u)^2}{2T}\right], \tag{2.6}$$

with

$$u := \int_{\mathbb{R}^3} v f(v)\, \mathrm{d}^3 v, \tag{2.7}$$

$$e := \frac{1}{2} \int_{\mathbb{R}^3} v^2 f(v)\, \mathrm{d}^3 v, \tag{2.8}$$

and $T = \frac{2}{3}(e - \frac{1}{2}u^2)$. In the BGK model the collision kernel is

$$Q_{BGK}(f) = M_f - f.$$

The Boltzmann collision kernel is given by

$$Q_B(f)(v) = \int_{\mathbb{R}^3} dv_* \int_{S_2^+} d\omega\, B(|v-v_*|, \omega)[f(v')f(v'_*) - f(v)f(v_*)]. \tag{2.9}$$

Here $S_2^+ = \{\omega \in \mathbb{R}^3 : \omega^2 = 1, \omega \cdot (v - v_*) \geqq 0\}$, and

$$\begin{aligned} v' &= v - \omega \cdot (v - v_*)\omega, \\ v'_* &= v_* + \omega \cdot (v - v_*)\omega \end{aligned} \tag{2.10}$$

are the outgoing velocities in a collision with incoming velocities v and v_* and impact parameter ω, $B(|v - v_*|, \omega)$ is the collision cross section, depending on the intermolecular interactions. For Maxwellian molecules, with a Grad angular cut-off

$$B(|v - v_*|, \omega) = h(\theta), \tag{2.11}$$

with θ the azimuthal angle of the spherical coordinates in S_2 with polar axis along $v - v_*$ and $h(\theta)$ a smooth non-negative bounded function. Thus, for any normalized f we can write

$$Q_B(f) = \ell(f \circ f - f) \tag{2.12}$$

with

$$\begin{aligned} f \circ f(v) &= \frac{1}{2\ell} \int_{\mathbb{R}^3} dv_* \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta h(\theta) |\sin\theta| f(v') f(v'_*), \\ \ell &= \pi \int_0^{\pi} d\theta h(\theta) |\sin\theta| > 0. \end{aligned} \tag{2.13}$$

With either Q_{BGK} or Q_B for the collision kernel in (2.1), this term tends to keep f close to M_f, and one could certainly expect this effect to dominate for small values of ε. Thus, formally in the limit as ε vanishes, f actually equals M_f for all time f, and to keep track of its evolution, we need only keep track of $u(t)$ and $e(t)$.

Using the prescription $f = M_f$, in the right side of (2.1), we easily evaluate the time derivative of the first two moments of the so modified (2.1), to obtain formally

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \tilde{u}(t) \\ \tilde{e}(t) \end{pmatrix} = \begin{pmatrix} F(\tilde{u}(t), \tilde{e}(t)) \\ G(\tilde{u}(t), \tilde{e}(t)) \end{pmatrix}. \tag{2.14}$$

The functions f and G are given explicitly by

$$F(u, e) = E - u \left[\nu + c + a \exp(-\omega) \frac{1 + b}{(1 + b/\beta)^{5/2}} \right], \tag{2.15}$$

$$\begin{aligned} G(u, e) = Eu - c &\left[2e(1 - \beta) + \beta u^2 \right] \\ &- \frac{a \exp(-w)}{(1 + b/\beta)^{5/2}} \left[2e(1 - \beta) + u^2 \left(\beta - b \frac{1 - \beta}{b + \beta} \right) \right], \end{aligned} \tag{2.16}$$

where $\beta = T^{-1}$ and $w = bu^2/2(1 + bT)$. The tildes in (2.14) are to remind us that the equation is valid only when $f = M_f$.

Equations (2.14) represent the hydrodynamical description of the gas. Our primary goal here is to show that such a description actually does hold for small, but positive, values of ε, i.e., that the interaction between the hydrodynamic and the non-hydrodynamic modes does not destroy the picture involving only the hydrodynamic modes. The following theorem enables us to do this.

Theorem 2.1. *Let f be a solution of* (2.1) *with*

$$f(\cdot, 0) = M_{f(\cdot,0)}.$$

Then, for any fixed integer $k_0 > 0$, there is an $\varepsilon_0 > 0$ and functions $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, going to zero as $\varepsilon \to 0$, depending only on a, b, c, $|E|$, and $e(0)$, such that for $\varepsilon < \varepsilon_0$ the solution of (2.1) *satisfies*

$$\sup_{t \in \mathbb{R}^+} \| f(\cdot, t) - M_{f(\cdot,t)} \|_{L^1(\mathbb{R}^3)} \leqq \delta_1(\varepsilon), \tag{2.17}$$

$$\sup_{t \in \mathbb{R}^+} \| \nabla^k \left(f(\cdot, t) - M_{f(\cdot,t)} \right) \|_{L^2(\mathbb{R}^3)} \leqq \delta_2(\varepsilon) \tag{2.18}$$

for $k \leqq k_0$.

Remark. The assumption on $f(\cdot, 0)$ is not essential: our methods allow us to easily modify the result to take into account an initial layer.

We shall use Theorem 2.1, in the proof of Theorem 2.2 below, to show that if we compute $u(t)$ and $e(t)$ for a solution f of (2.1) satisfying the conditions of Theorem 2.1, then the moments of f satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} F(u(t), e(t)) \\ G(u(t), e(t)) \end{pmatrix} + \delta(\varepsilon)^{1/2}\begin{pmatrix} \gamma(t) \\ \eta(t) \end{pmatrix} \tag{2.19}$$

where F and G are the non-linear functions of u and e given by (2.15) and (2.16), which arise in the $\varepsilon = 0$ limit, and $\gamma(t)$ and $\eta(t)$ are bounded uniformly in t with a bound independent of ε. Of course they depend on the full solution f of (2.1), but such estimates and a simple comparison argument will then lead to the following theorem, which says that the system (2.14) obtained in the $\varepsilon \to 0$ limit does give an accurate picture of the small ε regime.

Theorem 2.2. *Let (u^*, e^*) be a stable fixed point of the system* (2.14) *and let $M_{(u^*,\varepsilon^*)}$ be the corresponding Maxwellian density, with moments u^* and e^*. Then given any $\delta > 0$, there is an ε greater than zero such that if $f_\varepsilon(\cdot, t)$ satisfies* (2.1) *with this value of ε and if*

$$\|f_\varepsilon(\cdot, 0) - M_{(u^*,e^*)}(\cdot)\|_{L^1(\mathbb{R}^3)} \leqq \varepsilon, \tag{2.20}$$

then

$$\|f_\varepsilon(\cdot, t) - M_{(u^*,e^*)}(\cdot)\|_{L^1(\mathbb{R}^3)} \leqq \delta \tag{2.21}$$

for all $t \geqq 0$.

If, however, (u^*, e^*) is not stable, then there exists a $\delta > 0$ so that for every $\varepsilon > 0$, there is a solution $f_\varepsilon(\cdot, t)$ of (2.1) *with Maxwellian initial data satisfying*

$$\|f_\varepsilon(\cdot, 0) - M_{(u^*,e^*)}(\cdot)\|_{L^1(\mathbb{R}^3)} \leqq \varepsilon, \tag{2.22}$$

but such that for some finite $t > 0$

$$\|f_\varepsilon(\cdot, t) - M_{(u^*,e^*)}(\cdot)\|_{L^1\mathbb{R}^3)} \geqq \delta. \tag{2.23}$$

The proof of these theorems, which are fairly complicated even for the simple BGK model, will be given in the next sections.

The above theorems allow us to rigorously prove that our kinetic system has multiple equilibria in certain ranges of the parameters that specify it. This is because of the following result concerning the "hydrodynamic" system (2.14).

Proposition 2.3. *There are choices of the parameters a, b, c and ν for which there are nonempty intervals (E_0, E_1) such that, if $|E|$ is outside of the closed interval $[E_0, E_1]$, then there is a unique stable fixed point for the system* (2.14)*, while, if $|E| \in (E_0, E_1)$, then there are three fixed points for the system* (2.14)*. Moreover, two of them are stable and one is unstable.*

Stability here is meant in the sense that the eigenvalues of the differential have a strictly negative real part.

The proof of Proposition 2.3 is an explicit calculation which we omit (see [7]). However, to gain an intuitive understanding of why there should be multiple stable equilibria for certain parameter ranges, think of the $\varepsilon = 0$ limit of (2.1) as a constrained motion on the "manifold of Maxwellians". Without this constraint, which is generated by the collision kernel, the evolution would be the one ruled by the electric field and the linear operator L. Clearly, this evolution has a unique attracting fixed point, which, for $a > 0$, is not Maxwellian. What happens is that there are one or more places on the *non-linear* constraint manifold that are *locally closest* to the attracting point of the unconstrained system. Each of these is a stable equilibrium for the constrained evolution. As the parameters are varied, the position of the unconstrained fixed point relative to the manifold of Maxwellians varies, and with this variation in geometry, the number of locally closest points varies.

The main physical issues regarding this model are settled at this point: We have proved the existence of the multiple stable equilibria at the kinetic level — ε small, but positive — that had been found and investigated in [7] at $\varepsilon = 0$. Moreover, we remind the reader that we do not know how to establish such a result using conventional expansion methods, the difficulty being that if E is not small, and hence possibly out of the range where multiple equilibria exist for (2.14), we only know how to prove (2.21) *locally* in time. This is insufficient to show that for ε small enough, one *never* wanders far from any $M_{(a^*,e^*)}$ with (u^*, e^*) stable for (2.14). While the entropy methods we use do let us do this for arbitrary E, there are several finer questions that one could ask, but for which we have only incomplete answers.

First, one can ask whether or not there is an actual stationary solution inside the invariant neighborhoods of the $M_{(u^*,e^*)}$ that we have found, and second, once one knows that stationary solutions exist, one can ask whether or not solutions actually tend to converge to one of these stationary solutions as t tends to infinity.

The first question we can answer completely only in the BGK case. The positive answer is given by

Theorem 2.4. *Let $Q = Q_{BGK}$ and $a > 0$. Then for each fixed point of* (2.14) *there is exactly one stationary solution of equation* (2.1)*. This solution lies in a suitably small neighborhood of M_{u^*,e^*}, the Maxwellian corresponding to (u^*, e^*), and it inherits the stability properties of the hydrodynamical fixed point.*

For the Boltzmann kernel we have only a partial result that is reported in Section 8. Also on the question of convergence we have only very partial results. These are reported in Section 9, where the difficulties are explained as well. But though it would be desirable to have a more complete resolution of these issues, they are not central to establishing that the kinetic systems do actually have the several stable regimes that one sees in the limiting "hydrodynamic" equations.

The proofs are organized as follows. In Section 3 we prove moment bounds. Section 4 contains the proof of two "interpolation inequalities". The first of these will be used to obtain *a-priori* smoothness bounds in Section 5. The second will be used to transform the smoothness bounds of Section 5 into a lower bound on the variance of our density. The smoothness requires Sobolev estimates for the collision kernel, which are straightforward for Q_{BGK}, while for Q_B they rely on some recent results [13], which we simply state here. Having assembled these moment, smoothness and interpolation bounds, we can use a key entropy production inequality for Q_B proved in [11]. The analogous inequality for Q_{BGK} is proven in a simpler way in Section 6. This is used to get quantitative bounds on the tendency of the collision operator to keep the density nearly Maxwellian. What we obtain directly is L^1 control on the difference between f and M_f, but the smoothness bounds together with the interpolation bounds allow us to obtain control in stronger norms. Section 7 contains the proofs of the Theorems 2.1 and 2.2, which, given the lemmas, are quite short. Section 8 is devoted to the proof of the existence of stationary solutions. In Section 9 we discuss the tendency toward these stationary solutions.

3. Moment Bounds

In this section we esatblish *a-priori* moment bounds for solutions of (2.1). In estimating the evolution of the moments, we shall use one set of methods to treat the effects of collisions, and another set to treat everything else. Thus it is natural to rewrite (2.1) as

$$\frac{\partial}{\partial t} f(v,t) = \mathscr{L} f(v,t) + \frac{1}{\varepsilon} Q(F)(v,t) \tag{3.1}$$

where $\mathscr{L} f = -E \cdot \nabla f + Lf$.

We shall use the standard "bracket notation" for averages: $\langle \phi \rangle_t$ denotes $\int_{\mathbb{R}^3} \phi(v) f(v,t)\, \mathrm{d}^3 v$ for any positive or integrable function ϕ.

Throughout this paper, K denotes a computable constant that depends at most on the electric field E, the parameters $a,\ b,\ c,$ and ν specified in (2.4), and where indicated, also on the fourth moment of the initial distribution: $\langle |v|^4 \rangle_0$. The constant will, however, change from line to line.

In these terms, the main result of this section is

Theorem 3.1. *Let f denote a solution to* (2.1). *Then for $Q = Q_B$ or Q_{BGK},*

$$\langle |v|^4 \rangle_t \leqq \langle |v|^4 \rangle_0 + K.$$

Note that by Jensen's inequality, this immediately controls all lower-order moments as well. The first step, however, is to directly control the second moments.

Lemma 3.2. *Let f denote a asolution to* (2.1). *Then for both choices of the collision kernel Q,*

$$\langle |v|^2 \rangle_t \leqq \langle |v|^2 \rangle_0 + K. \tag{3.2}$$

Proof. Since $\langle |v|^2 \rangle_t$ is a collision invariant, and $\int v^2 L_2 f \mathrm{d}^3 v = 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle |v|^2 \rangle_t &= \int_{\mathbb{R}^3} |v|^2 \mathscr{L} f(v,t)\, \mathrm{d}^3 v \\ &\quad + 2E \cdot \langle v \rangle_t - 2 \int_{\mathbb{R}^3} D(v) M(v) v \cdot \nabla \left(\frac{f(v,t)}{M(v)} \right) \mathrm{d}^3 v \\ &= 2E \cdot \langle v \rangle_t + 6 \langle D \rangle_t + 2(v \cdot \nabla D) \langle (v \cdot \nabla D) \rangle_t - 2 \langle D |v|^2 \rangle_t . \end{aligned}$$

Now observe that

$$(v \cdot \nabla D) \leqq 0 \tag{3.3}$$

and that

$$c \leqq D \leqq (a+c) \tag{3.4}$$

for all v. Finally, by Jensen's inequality,

$$|\langle v\rangle_t| \leqq \big(\langle |v|^2\rangle_t\big)^{1/2}. \tag{3.5}$$

These facts, combined with the previous calculation, yield the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |v|^2\rangle_t \leqq -2c\langle |v|^2\rangle_t + 6(a+c) + 2|E|\big(\langle |v|^2\rangle_t\big)^{1/2}.$$

Straightforward estimation now leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |v|^2\rangle_t \leqq -c\langle |v|^2\rangle_t + K.$$

Then (3.2) in turn follows from the fact that any solution of the differential inequality $\dot{x}(t) \leqq -cx(t) + K$ satisfies $x(t) \leqq x(0) + K/c$. □

We next parlay these bounds into bounds on the forth moments; i.e., $\langle |v|^4\rangle_t$. Since $\langle |v|^4\rangle_t$ is not a collision invariant, these depend on the particular collision kernel under consideration.

Lemma 3.3. *Let $Q = Q_{BGK}$. Then for any density f,*

$$\int_{\mathbb{R}^3} |v|^4 Q(f)\,\mathrm{d}^3v \leqq \frac{160}{3}\big(\langle |v|^2\rangle_t\big)^2 - \langle |v|^4\rangle_t.$$

Proof. By an easy calculation,

$$\int_{\mathbb{R}^3} |v-\langle v\rangle_t|^4 M_f(v)\,\mathrm{d}^3v$$
$$= \tfrac{5}{3}\left(\int_{\mathbb{R}^3} |v-\langle v\rangle_t|^2 M_f(v)\,\mathrm{d}^3v\right)^2 = \tfrac{5}{3}\langle |v-\langle v\rangle_t|^2\rangle_t^2.$$

Next, note that $|v| \leqq |v-\langle v\rangle_t| + |\langle v\rangle_t|$, and thus, $|v|^4 \leqq 16(|v-\langle v\rangle_t|^4 + |\langle v\rangle_t|)^4$. Combining this and Jensen's inequality as in (3.5) with the above, we have

$$\int_{\mathbb{R}^3} |v|^4 M_f(v)\,\mathrm{d}^3v \leqq \tfrac{160}{3}\big(\langle |v|^2\rangle_t\big)^2.$$

The result now follows directly from the form of $Q(f)$. □

Lemma 3.4. *Let $Q = Q_B$. Then there are positive constants c_1 and c_2 such that for any density f,*

$$\int_{\mathbb{R}^3} |v|^4 Q(f)\,\mathrm{d}^3v \leqq c_1\big(\langle |v|^2\rangle_t\big)^2 - c_2\langle |v|^4\rangle_t.$$

Proof. The result follows once we prove that

$$\int d^3v|v|^4 f\circ f(v) \leqq (\ell - c_2]\langle|v|^4\rangle_t + c_1\big(\langle|v|^2\rangle_t\big)^2.$$

To check this, let π_ω and $\pi_\omega^\perp$ denote the projection in the direction ω and the complementary projection respectively. Then we can write v as

$$v = \pi_\omega v_* + \pi_o^\perp v_*',$$

$$|v|^2 = |\pi_\omega v_*|^2 + |\pi_o^\perp v_*'|^2.$$

Hence

$$|v|^4 = |\pi_\omega v_*|^4 + |\pi_o^\perp v_*'|^4 + 2|\pi_\omega v_*|^2|\pi_o^\perp v_*'|^2.$$

Averaging on S_2 we get

$$\begin{aligned}
&\int_{s_2} dwB(|v-v_*|,\omega)[|\pi_\omega v_*|^4 + |\pi_o^\perp v_*'|^4]\\
&\quad = |v|^4\pi\int_0^\pi d\theta|\sin\theta|h(\theta)[\cos^4\theta + \sin^4\theta]\\
&\quad = |v|^4\left(\ell - 2\pi\int_0^\pi d\theta|\sin\theta|h(\theta)\cos^2\theta\sin^2\theta\right) \equiv |v|^4(\ell - c_2). \quad \square
\end{aligned}$$

Proof of Theorem 3.1. Calculating as before, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle|v|^4\rangle_t &= \int_{\mathbb{R}^3}|v|^4\mathscr{L}f(v,t)\,\mathrm{d}^3v + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|v|^4Q(f)(v,t)\,\mathrm{d}^3v\\
&= 4E\cdot\langle|v|^2v\rangle_t - 4\int_{\mathbb{R}^3}D(v)M(v)|v|^2v\cdot\nabla\left(\frac{f(v,t)}{M(v)}\right)\mathrm{d}^3v\\
&\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|v|^4Q(f)\,\mathrm{d}^3v\\
&= 4E\cdot\langle|v|^2v\rangle_t + 12\langle|v|^2D\rangle_t + 4(|v|^2v\cdot\nabla D)\langle(v\cdot\nabla D)\rangle_t - 2\langle D|v|^4\rangle_t\\
&\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|v|^4Q(f)\,\mathrm{d}^3v.
\end{aligned}$$

We now estimate using (3.3) and (3.4) just as in the proof of Lemma 3.2, together with Lemmas 3.3 and 3.4, to control the collision term. (Note that once again L_2 makes no contribution.) The result is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle|v|^4\rangle_t &\leqq 4|E|\big(\langle|v|^4\rangle_t\big)^{3/4} + 12(a+c)\langle|v|^2\rangle_t - 4c\langle|v|^4\rangle_t\\
&\quad + \frac{1}{\varepsilon}\big(c_1(\langle|v|^2\rangle_t)^2 - c_2\langle|v|^4\rangle_t\big).
\end{aligned}$$

Next, by Lemma 3.1 together with Jensen's inequality, we know that $12(a+c)\langle|v|^2\rangle_t$ is bounded above by a universal constant plus $(\langle|v|^4\rangle_0)^{1/2}$. Thus, if we introduce the scaled time parameter

$$\tau = \frac{1}{\varepsilon}t,$$

and define $x(\tau) := \langle|v|^4\rangle_\tau$, we have that $x(\tau)$ satisfies a differential inequality of the form

$$\dot{x}(\tau) \leqq -x(\tau) + K$$

from which the result follows. □

4. Interpolation Inequalities

The lemmas in this section are several interpolation inequalities related to the familiar Gagliardo-Nirenberg inequalities, but with some special features adapted to our applications. The inequality of Lemma 4.2 is the most novel and interesting of these.

Lemma 4.1. *Let $f \in L^1(\mathbb{R}^3)$. Then there is a universal constant C such that if f has a square-integrable distributional Laplacian, then f has a square integrable gradient, and*

$$\|\nabla f\|_2 \leqq C\|f\|_1^{2/7}\|\Delta f\|_2^{5/7}. \tag{4.1}$$

Similarly, there is a universal constant C such that if $\Delta^2 f$ is square-integrable, then Δf is also square-integrable, and

$$\|\Delta f\|_2 \leqq C\|f\|_1^{4/11}\|\Delta^2 f\|_2^{7/11}. \tag{4.2}$$

Proof. Taking Fourier transforms, we have

$$\begin{aligned}
\|\nabla f\|_2^2 &= \int_{\mathbb{R}^3} |p|^2|\hat{f}(p)|^2\,\mathrm{d}^3p \\
&= \int_{|p|\leqq R} |p|^2|\hat{f}(p)|^2\,\mathrm{d}^3p + \int_{|p|\geqq R} |p|^2|\hat{f}(p)|^2\,\mathrm{d}^3p \\
&\leqq CR^5\|f\|_1^2 + R^{-2}\int_{|p|\geqq R} |p|^2|\hat{f}(p)|^4\,\mathrm{d}^3p \\
&\leqq CR^5\|f\|_1^2 + R^{-2}\|\Delta f\|_2^2,
\end{aligned}$$

where the computable constant C changes from line to line. Optimizing over R now yields (4.1). The proof of (4.2) is done in the same way. □

The next inequality is similar in effect to an "uncertainty principle". We shall use it to obtain uniform lower bounds on the variance of our density f.

Lemma 4.2. *Let $f \in L^1(\mathbb{R}^3)$, and suppose that f has a square-integrable distributional gradient. Then there is a universal constant C such that*

$$\int_{\mathbb{R}^3} f(v)\,\mathrm{d}^3v \leqq C\|\nabla f\|_2^{4/9} \left(\int_{\mathbb{R}^3} \left|v - \int_{\mathbb{R}^3} vf(v)\,\mathrm{d}^3v\right|^2 f(v)\,\mathrm{d}^3v\right)^{5/9}. \quad (4.3)$$

Proof. The right side of (4.3) is decreased when we replace f by its spherically symmetric decreasing rearrangement, while the left side is unchanged. We may therefore assume without loss of generality that f is spherically symmetric and radially decreasing. Now fix $R > 0$, and define $g(v) := f(v)$ for $|v| \leqq R$, and $g(v) = 0$ otherwise. Define h by $f = g + h$. Now clearly,

$$\int_{\mathbb{R}^3} h(v)\,\mathrm{d}^3v \leqq R^{-2} \int_{|v|>R} |v|^2 f(v)\,\mathrm{d}^3v.$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}^3} g(v)\,\mathrm{d}^3v \\
&\quad = \frac{4\pi}{3}R^3 f(R) + \frac{4\pi}{3}R^3 \left(\left(\frac{4\pi}{3}\right)^{-1} R^{-3} \int_{|v|<R} \Big(f(v) - f(R)\Big)\,\mathrm{d}^3v\right) \\
&\quad \leqq \frac{4\pi}{3}R^3 f(R) + \frac{4\pi}{3}R^3 \left(\left(\frac{4\pi}{3}\right)^{-1} R^{-3} \int_{|v|<R} \Big(f(v) - f(R)\Big)^2\,\mathrm{d}^3v\right)^{1/2}.
\end{aligned}$$

Now, since f is monotone,

$$\int_{|v|<R} |v|^2 f(v)\,\mathrm{d}^3v \geqq f(R) \int_{|v|<R} |v|^2 f\,\mathrm{d}^3v = f(R)\frac{4\pi}{5}R^5.$$

Then, with λ denoting the principal eigenvalue for the Dirichlet Laplacian in the unit ball, we have

$$\begin{aligned}
\int_{|v|<R} (f(v) - f(R))^2\,\mathrm{d}^3v &\leqq R^2\lambda^{-1} \int_{|v|<R} |\nabla(f(v) - f(R))|^2\,\mathrm{d}^3v \\
&\leqq R^2\lambda^{-1}\|\nabla f\|_2^2.
\end{aligned}$$

Combining the above, we have

$$\int_{\mathbb{R}^3} f(v)\,\mathrm{d}^3v \leqq C\left(\|\nabla f\|_2 R^{5/2} + R^{-2}\int_{\mathbb{R}^3} |v|^2 f(v)\,\mathrm{d}^3v\right).$$

Optimizing over R yields the result. □

5. Smothness Bounds

The purpose of this section is to establish *a-priori* smoothness bounds for solutions of (2.1). The main result is:

Theorem 5.1. *Let f be a solution of* (2.1) *such that $\|\nabla f(\cdot,0)\|_2$ is finite. If $Q = Q_{BGK}$, then there is a constant K such that*

$$\|\nabla f(\cdot,t)\|_2 \leqq K(1 + \|\nabla f(\cdot,0)\|_2) \tag{5.1}$$

for all $t > 0$. Similarly, suppose that $\|\Delta f(\cdot,0)\|_2$ is finite. Then there is a constant K such that

$$\|\Delta f(\cdot,t)\|_2 \leqq K(1 + \|\Delta f(\cdot,0)\|_2) \tag{5.2}$$

for all $t > 0$.

If $Q = Q_B$, the same results hold provided $\|f(\cdot,0) - M_{f(\cdot,0)}\|_1$ and ε are both sufficiently small.

Remark. The smallness condition on $\|f(\cdot,0) - M_{f(\cdot,0)}\|_1$ poses no problem here since we are avoiding an initial layer by assuming that $f(\cdot,0) = M_{f(\cdot,0)}$. However, it seems likely that it would be straightforward to include an initial-layer analysis, and to dispense with this condition — even the present proof does not require $\|f(\cdot,0) - M_{f(\cdot,0)}\|_1$ to be particularly small.

Proof. Once again, we write (2.1) in the form (3.1). Then differentiating, and integrating by parts, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f(\cdot,t)\|^2 &= -2\int_{\mathbb{R}^3} \Delta f(v,t)\left(\frac{\partial f(v,t)}{\partial t}\right)\mathrm{d}^3v\\ &= -2\int_{\mathbb{R}^3} \Delta f(v,t)\mathscr{L}f(v,t)\,\mathrm{d}^3v - 2\int_{\mathbb{R}^3} \Delta f(v,t)Q(f)(v,t)\,\mathrm{d}^3v.\end{aligned}$$

As with the moment bounds, we begin by estimating the individual contributions to the $\mathscr{L}$ term.

Lemma 5.2. *There is a universal constant K so that for all solutions f of* (2.1),

$$-2\int_{\mathbb{R}^3} \Delta f(v,t)\mathscr{L}f(v,t)\,\mathrm{d}^3v \leqq K$$

for all $t \geqq 0$.

First, $2E \cdot \int_{\mathbb{R}^3} \Delta f \nabla f \mathrm{d}^3v = 0$. Next,

$$-2\int_{\mathbb{R}^3} \Delta f(\nabla D\cdot\nabla f)\,\mathrm{d}^3v = \int_{\mathbb{R}^3} |\nabla f|^2|\Delta D|\,\mathrm{d}^3v \leqq (a+c)\|\nabla f\|_2^2.$$

Now, repeatedly integrating by parts, we obtain

$$\begin{aligned}
-2\int_{\mathbb{R}^3} \Delta f(\nabla D\cdot v) f\, \mathrm{d}^3 v
&= 2\int_{\mathbb{R}^3} \nabla f (v(\Delta D) f + 3\nabla D f + (v\cdot\nabla D)\nabla f)\, \mathrm{d}^3 v\\
&\leqq -3\int_{\mathbb{R}^3} f^2(\Delta D)\, \mathrm{d}^3 v - \int_{\mathbb{R}^3} f^2\Big(v\cdot\nabla(\Delta D)\Big)\mathrm{d}^3 v
\end{aligned}$$

where we have once again used (3.3). This gives us a bound of the form

$$-2\int_{\mathbb{R}^3} \Delta f(\nabla D\cdot v) f\, \mathrm{d}^3 v \leqq K\|f\|_2^2.$$

We now use a standard interpolation inequality, the Nash inequality:

$$\|f\|_2 \leqq C\|\nabla f\|_2^{3/5}\|f\|_1^{2/5}.$$

This allows us to eliminate $\|f\|_2$ in favor of $\|\nabla f\|_2$, the quantity of interest, and $\|f\|_1$, the conserved quantity, so that

$$-2\int_{\mathbb{R}^3} \Delta f(\nabla D\cdot v) f\, \mathrm{d}^3 v \leqq K\|\nabla f\|_2^{6/5}\|f\|_1^{4/5}.$$

Next,

$$-2\int_{\mathbb{R}^3} \Delta f(Dv\cdot\nabla f)\, \mathrm{d}^3 v = \int_{\mathbb{R}^3} |\nabla f|^2\nabla\cdot(Dv)\, \mathrm{d}^3 v \leqq K\|\nabla f\|_2^2.$$

Apart from the collision term and the favorable dissipation term, the only other term to be bounded is $-2\int_{\mathbb{R}^3}\Delta f D f \mathrm{d}^3 v$. However, integrating by parts, we obtain terms identical to terms that we have already bounded.

The dissipation term is bounded using Lemma 4.1 as follows:

$$-2\int_{\mathbb{R}^3} D(\Delta f)^2\, \mathrm{d}^3 v \leqq -2c\|\Delta f\|_2^2 \leqq K\|\nabla f\|_2^{14/5}\|f\|_1^{-4/5} = K\|\nabla f|_2^{14/5}$$

since $\|f\|_1 = 1$ for all times t.

Thus, with $x(\tau) := \|f(\cdot,\tau)\|_2^2$, we have established that

$$\begin{aligned}
-2\int_{\mathbb{R}^3} \Delta f(v,t)\mathscr{L} f(v,t)\, \mathrm{d}^3 v&\\
&\leqq K(-x(t)^{7/5} + K(x(t)^{3/5} + x(t)^{3/10} + x(t))) \leqq K
\end{aligned}$$

since the largest power of $x(t)$ has a negative coefficient. □

To conclude the proof of (5.1), we need smoothness bounds for the collision kernel; in particular, we need an estimate on the smoothness of the gain term in the collision kernel.

For the BGK case, this is given by

Lemma 5.3. *For any positive integer n there is a constant K depending only on the second moment of f such that*

$$2\int_{\mathbb{R}^3}(-\Delta)^n f(v,t)Q(f)(v,t)\,\mathrm{d}^3v \leqq K - \|(-\Delta)^{n/2}f\|_2^2. \tag{5.3}$$

Proof. We only check the case $N = 1$; the rest are similar. We have

$$-2\int_{\mathbb{R}^3}\Delta f\,Q_{BGK}(f)\,\mathrm{d}^3v = -2\int_{\mathbb{R}^3}(\Delta M_f)f\,\mathrm{d}^3v - 2\|\nabla f\|_2^2. \tag{5.4}$$

Now, since

$$-2\int_{\mathbb{R}^3}(\Delta M_f)f\,\mathrm{d}^3v \leqq CT_f^{-1}\|\nabla f\|_2,$$

we can use Lemma 4.2 to get the bound

$$T_f^{-1} \leqq C\|\nabla f\|_2^{4/5} \tag{5.5}$$

and hence

$$-2\int_{\mathbb{R}^3}(\Delta M_f)f\,\mathrm{d}^3v \leqq CT_f^{-1}\|\nabla f\|_2 \leqq C\|\nabla f\|_2^{9/5} \leqq K + \|\nabla f\|_2^2.$$

Combining this with (5.4) we get the Lemma 5.3 for $n = 1$. □

The analog of Lemma 5.3 for Q_B is slightly more complicated:

Lemma 5.4. *For any positive integer n there are constants K and δ depending only on the a-priori bound on the second moments of f, such that*

$$2\int_{\mathbb{R}^3}(-\Delta)^n f(v,t)Q_B(f)(v,t)\,\mathrm{d}^3v \leqq K - \tfrac{1}{2}\ell\|(-\Delta)^{n/2}f\|_2^2 \tag{5.6}$$

whenever

$$\|f(\cdot,t) - M_{f(\cdot,t)}(\cdot)\|_1 \leqq \delta,$$

where ℓ is the constant in (2.13).

Proof. It is sufficient to note that

$$\begin{aligned}
2\int_{\mathbb{R}^3}&(-\Delta)^n f\,Q_B(f)\,\mathrm{d}^3v\\
&= 2\ell\int_{\mathbb{R}^3} f(-\Delta)^n f\circ f\,\mathrm{d}^3v - 2\ell\|(-\Delta)^{n/2}f\|_2^2\\
&\leqq 2\ell\big(\|(-\Delta)^{n/2}f\|_2\|(-\Delta)^{n/2}f\circ f\|_2 - \|(-\Delta)^{n/2}f\|_2^2\big).
\end{aligned}$$

What we need now is control over $\|(-\Delta)^{n/2}f\circ f\|_2$. This is provided by an inequality from [13], where it is shown that for any $\gamma > 0$ there are constants K and $\delta > 0$ such that

$$\|(-\Delta)^{n/2}f\circ f\|_2^2 \leqq K + \gamma\|(-\Delta)^{n/2}f\|_2^2 \tag{5.7}$$

whenever

$$\|f - M_f\|_1 \leqq \delta.$$

This inequality is proved in [13] under the assumption that f has zero mean and unit variance, and under these conditions, the constants K and δ are universal. Scaling the inequality, we have it holding with constants K and δ depending only on the second moments of f. (The inequality is applied in [13] to get strong exponential convergence estimates for the spatially homogeneous Boltzmann equation with physically realistic constants in the bounds.)

With this inequality, we need only take $\gamma = \frac{1}{2}$. □

Proof of Theorem 5.1. We begin with the proof of (5.1).

To put all of the lemmas together, let $\tau := (1/\varepsilon)t$ as before, and put $x(\tau) := \|f(\cdot,\tau)\|_2^2$. Combining Lemmas 5.2 and 5.3, we get in the BGK case that

$$\dot{x}(\tau) \leqq \varepsilon K + K - x(\tau)$$

and any solution of this differential inequality satisfies $x(\tau) \leqq x(0) + (1+\varepsilon)K$ for all τ. This establishes (5.1) in the BGK case. The proof of (5.2) in the BGK case is entirely analogous.

To handle the Maxwellian collision kernel case, first define

$$\bar{t} = \inf\{t \text{ such that } \gamma\|f(\cdot,t) - M_{f(\cdot,t)}\|_1^2 \geqq \tfrac{1}{2}\delta\} \tag{5.8}$$

where δ is the universal constant from Lemma 5.4. We take our initial condition so small that $\bar{t} > 0$. Then, for all $t \leqq \bar{t}$, we have the following differential inequality by combining Lemmas 5.2 and 5.4:

$$\dot{x}(\tau) \leqq \varepsilon K + K - \ell x(\tau)$$

and any solution of this differential inequality satisfies $x(\tau) \leqq x(0) + (1+\varepsilon)K/\ell$ for all $\tau \leqq (1/\varepsilon)\bar{t}$.

It only remains to show that actually $\bar{t} = +\infty$. We shall do this in the next section using an entropy inequality. The entropy inequality requires the *a-priori* smoothness bounds from Theorem 5.1, and shows that as long as they hold; i.e., as long as $t \leqq \bar{t}$, we have an upper bound of the form

$$\|f(\cdot,t) - M_{f(\cdot,t)} \leqq \delta_{\text{entropy}}(\varepsilon).$$

All we have to do now is to take ε so small that

$$\delta_{\text{entropy}}(\varepsilon) \leqq \tfrac{1}{4}\delta$$

and then it is clear from the definition of $\bar{t}$ that $\bar{t} = \infty$. Thus, by borrowing the entropy bound from the next section, (5.1) is established for Q_B for all ε sufficiently small, and all initial data sufficiently close to a Maxwellian. Again, (5.2) for Q_B is handled in an entirely analogous way. □

6. Entropy Bounds

Let $h(\rho_1 \mid \rho_2)$ denote the relative entropy of two probability densities ρ_1 and ρ_2 on $\mathbb{R}^3$:

$$h(\rho_1 \mid \rho_2) = \int_{\mathbb{R}^3} \rho_1 \left(\frac{\rho_1}{\rho_2}\right) \ln \left(\frac{\rho_1}{\rho_2}\right) \mathrm{d}^3 v.$$

Here we are primarily interested in bounds on $h(f \mid M_f)$. It will be convenient, however, to first obtain bounds on $h(f \mid M)$, and to then relate the two relative entropies. We do this in the next two lemmas.

Lemma 6.1. *Let f be any solution of* (2.1) *with*

$$\langle |v|^4 \rangle_0 \leqq C, \quad \|\nabla f\|_2 < C$$

for all $t \geqq 0$. Then there is a constant K depending only on C such that

$$\frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot, t) \mid M) \leqq K + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \ln f(\cdot, t) Q(\cdot, t) \, \mathrm{d}^3 v. \tag{6.1}$$

Proof. Differentiating, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot, t) \mid M) &= \int_{\mathbb{R}^3} \frac{\partial f}{\partial t} (\ln f - \ln M) \, \mathrm{d}^3 v \\ &= \int_{\mathbb{R}^3} (-E \cdot \nabla f + Lf + \varepsilon^{-1} Q(f))(\ln f - \ln M) \, \mathrm{d}^3 v. \end{aligned}$$

Now, integration by parts reveals that

$$\int_{\mathbb{R}^3} Lf (\ln f - \ln M) \, \mathrm{d}^3 v \leqq 0$$

and since $\ln M$ is linear in $|v|^2$,

$$- \int_{\mathbb{R}^3} \ln M \, Q(f) = 0.$$

Finally,

$$\begin{aligned} &\int_{\mathbb{R}^3} (E \cdot \nabla f)(\ln f - \ln M) \, \mathrm{d}^3 v \\ &\qquad = \int_{\mathbb{R}^3} (E \cdot \nabla f) \ln f \, \mathrm{d}^3 v - \int_{\mathbb{R}^3} (E \cdot \nabla f) \ln M \mathrm{d}^3 v \\ &\qquad = E \cdot \langle v \rangle_t . \end{aligned}$$

Using Jensen's inequality to bound $|\langle v \rangle_t|$ in terms of the uniformly bounded $\langle |v|^2 \rangle_t$, we have the asserted result. □

Lemma 6.2. *Let f be any solution of* (2.1) *with*

$$\langle |v|^4\rangle_0 \leqq C, \quad \|\nabla f\|_2 < C$$

for all $t \leqq T$, for some $T > 0$. Then there is a constant K depending only on C such that

$$\frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot,t) \mid M_{f(\cdot,t)}) \leqq K + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \ln f(\cdot,t) Q(\cdot,t)\,\mathrm{d}^3 v \tag{6.2}$$

for all $t \leqq T$.

Proof. By the definitions, we have

$$\begin{aligned} &h(f(\cdot,t) \mid M_{f(\cdot,t)}) - h(f\cdot,t) \mid M) \\ &\quad = \langle \ln M\rangle_t - \langle \ln M_{f(\cdot,t)}\rangle_t \\ &\quad = \tfrac{3}{2}\ln\langle |v - \langle v\rangle_t|^2\rangle_t + \tfrac{3}{2} - \tfrac{1}{2}\langle |v|^2\rangle_t. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot,t) \mid M_{f(\cdot,t)}) = {} & \frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot,t) \mid M) \\ & + \tfrac{3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln\langle |v - \langle v\rangle_t|^2\rangle_t - \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle |v|^4\rangle_t. \end{aligned}$$

Computing further with the logarithmic derivative term, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\langle |v - \langle v\rangle_t|^2\rangle_t = (\langle |v - \langle v\rangle_t|^2\rangle_t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\langle |v|^2\rangle_t - \langle v\rangle_t^2).$$

To control this term, we need an upper bound on $(\langle |v - \langle v\rangle_t|^2\rangle_t)^{-1}$. But since $\langle |v - \langle v\rangle_t|^2\rangle_t \geqq C\|\nabla f\|_2^{-4/5} \geqq K$ by Lemma 4.2, we have on application of Lemma 5.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\langle |v - \langle v\rangle_t|^2\rangle_t \leqq K\frac{\mathrm{d}}{\mathrm{d}t}(\langle |v|^2\rangle_t - \langle v\rangle_t^2).$$

The lemmas of Section 3 provide uniform bounds on this last term and therefore provide uniform bounds on the derivative of $h(f(\cdot,t)|M_{f(\cdot,t)}) - h(f(\cdot,t)|M)$. The result now follows from the previous lemma. □

Lemma 6.3. *Let $Q = Q_{BGK}$. For any density f with finite second moments,*

$$\int_{\mathbb{R}^3} \ln f\, Q(f)\,\mathrm{d}^3 v \leqq -h(f|M_f). \tag{6.3}$$

Proof. Let $S(f) = -\int_{\mathbb{R}^3} f \ln f \mathrm{d}^3 v$ denote the entropy of f. For any density f and for any $\tau \in [0, 1]$, put $f(\tau) = (1-\tau)f + \tau M_f$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau) = M_f - f = Q(f)$$

and $f(0) = f$. Then

$$\int_{\mathbb{R}^3} \ln f \, Q(f) \, \mathrm{d}^3 v = -\frac{\mathrm{d}}{\mathrm{d}\tau} S(f(\tau)).$$

However, the entropy functional is concave, so that

$$S(f(\tau)) \geqq (1-\tau)S(f) + \tau S(M_f)$$

and hence

$$\frac{S(f(\tau)) - S(f(0))}{\tau} \geqq S(M_f) - S(f).$$

Since M_f and f share the same hydrodynamic moments, $S(M_f) - S(f) = h(f|M_f)$. □

Lemma 6.4. *Let $Q = Q_{BGK}$ and let f be any solution of* (2.1) *with*

$$\langle |v|^4 \rangle_0 \leqq C, \quad \|\nabla f\|_2 < C$$

for all t. Then there is a constant K depending only on C so that

$$\frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot,t)|M_{f(\cdot,t)}) \leqq K - \frac{1}{\varepsilon} h(f(\cdot,t)|M_{f(\cdot,t)}). \tag{6.4}$$

Proof. This follows immediately upon combining the last three lemmas. □

In the case of the Boltzmann collision kernel, Lemma 6.3 is replaced by the following proposition proved in [11]:

Proposition 6.5. *For all $C > 0$, there is a positive function $\Phi_C(r)$ strictly increasing in r, such that*

$$\int_{\mathbb{R}^3} \ln f \, Q_B(f) \, \mathrm{d}^3 v \leqq -\Phi_C[h(f|M_f)] \tag{6.5}$$

for all densities f with

$$\int_{\mathbb{R}^3} |v|^4 f(v) \, \mathrm{d}^3 v \leqq C, \quad \|\nabla f\|_2 \leqq C.$$

Consequently, for $Q = Q_B$, (6.4) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} h(f(\cdot, t)|M_{f(\cdot,t)}) \leqq K - \frac{1}{\varepsilon} \Phi_C[h(f(\cdot, t)|M_{f(\cdot,t)})]. \tag{6.6}$$

Now if $f(\cdot, 0) = M_{f(\cdot,0)}$, so that $h(f(\cdot, 0)|M_{f(\cdot,0)}) = 0$, then it is evident from this

$$\Phi_C[h(f(\cdot, t)|M_{f(\cdot,t)})] \leqq \varepsilon K$$

for all $t \geqq 0$. This together with Kullback's inequality

$$\|f(\cdot, t) - M_{f(\cdot,t)}\|_1^2 \leqq 2h(f(\cdot, t)|M_{f(\cdot,t)}) \tag{6.7}$$

clearly implies that there is a function $\delta_{\text{entropy}}(\varepsilon)$, decreasing to zero with ε so that

$$\|f(\cdot, t) - M_{f(\cdot,t)}\|_1 \leqq \delta_{\text{entropy}}(\varepsilon)$$

for all $t \leqq \bar{t}$, the times for which we know that $f(\cdot, t)$ satisfies the bounds in the hypothesis of Proposition 6.5. But as explained at the end of the proof of Theorem 5.1, this is enough to show that $\bar{t} = \infty$ for all sufficiently small ε. Thus, the entropy bound provides the information needed to complete the proof of Theorem 5.1 as claimed, and moreover, (5.8) holds globally in time.

7. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Let us consider first the case $Q = Q_{BGK}$. Since Maxwellian initial data satisfy the hypotheses of Lemma 6.4, we have that (6.4) holds for the solution $f(\cdot, t)$ of (2.1) under consideration in Theorem 2.1. But then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/\varepsilon} h(f(\cdot, t)|M_{f(\cdot,t)})\right) \leqq K e^{t/\varepsilon}.$$

Since by hypothesis, $h(f(\cdot, 0)|M_{f(\cdot,0)}) = 0$, we have

$$h(f(\cdot, t)|M_{f(\cdot,t)}) \leqq \varepsilon K$$

for all $t \geqq 0$. This together with Kullback's inequality (6.7) yields the first inequality asserted in Theorem 2.1, with $\delta_1(\varepsilon) = \varepsilon^{1/2}$.

To obtain the second, note that since $f(\cdot, 0)$ is Maxwellian, there is a bound on $\|\Delta f\|_2$ depending only on $\langle |v|^2 \rangle_0$. Inequality (5.2) of Lemma 5.1 now gives us a uniform bound on $\|\Delta f(\cdot, t)\|_2$. Combining this with the interpolation inequality (4.1) finally yields the second inequality of Theorem 2.1 with $\delta_2(\varepsilon) = \varepsilon^{1/7}$ for $Q = Q_{BGK}$.

The proof for $Q = Q_B$ is only slightly more involved, but in fact we have already given the proof of the first part of Theorem 2.1 in our "back and forth" proof of the smoothness bounds and entropy bounds for this case. As observed at the end of Section 6,

$$\|f(\cdot, t) - M_{f(\cdot,t)}\|_1 \leqq \delta_{\text{entropy}}(\varepsilon)$$

for all $t \geqq 0$, and $\delta_{\text{entropy}}(\varepsilon)$ does decrease to 0 with ε as required. The second part follows in an entirely similar way. □

Proof of Theorem 2.2. This follows from Theorem 2.1, and it is now no longer necessary to separate the cases $Q = Q_{BGK}$ and $Q = Q_B$.

As we have computed in the proof of Lemma 3.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |v|^2\rangle_t = -2E \cdot \langle v\rangle_t + 6\langle D\rangle_t + 2(v \cdot \nabla D)\langle (v \cdot \nabla D)\rangle_t - 2\langle D|v|^2\rangle_t .$$

If we replace the density $f(\cdot, t)$ everywhere on the right by $M_{f(\cdot,t)}$, by definition we obtain the function $G(u(t), e(t))$ where $u(t)$ and $e(t)$ are the moments of $f(\cdot, t)$ figuring in Theorem 2.2. The error we make has to be estimated term by term. The least trivial of these terms concerns the contribution from $\langle D|v|^2\rangle_t$, and is estimated as

$$\begin{aligned}
&|\langle D|v|^2\rangle_t - \int_{\mathbb{R}^3} D(v)|v|^2 M_{f(\cdot,t)}(v)\,\mathrm{d}^3v| \\
&\quad \leqq \int_{\mathbb{R}^3} D(v)|v|^2 |f(v,t) - M_{f(\cdot,t)}(v)|\,\mathrm{d}^3v \\
&\quad = \int_{|v|\leqq R} D(v)|v|^2 |f(v,t) - M_{f(\cdot,t)}(v)|\,\mathrm{d}^3v \\
&\qquad + \int_{|v|\geqq R} D(v)|v|^2 |f(v,t) - M_{f(\cdot,t)}(v)|\,\mathrm{d}^3v \\
&\quad \leqq C\left(R^2\|f(\cdot,t) - M_{f(\cdot,t)}\|_1 + R^{-2}\int_{\mathbb{R}^3} |v|^4 |f(v,t) - M_{f(\cdot,t)}(v)|\,\mathrm{d}^3v\right) \\
&\quad \leqq C\left(R^2\|f(\cdot,t) - M_{f(\cdot,t)}\|_1 + R^{-2}K\left(\langle |v|^2\rangle_t + 1\right)\right).
\end{aligned}$$

Optimizing over R now yields the result

$$|\langle D|v|^2\rangle_t - \int_{\mathbb{R}^3} D(v)|v|^2 M_{f(\cdot,t)}(v)\,\mathrm{d}^3v| \leqq K\|f(\cdot,t) - M_{f(\cdot,t)}\|_1^{1/2}.$$

Theorem 2.1 yields a bound of the size $\delta_1(\varepsilon)^{1/2}$. The other error terms in the time derivatives of $\langle |v|^2\rangle_t$ and $\langle v\rangle_t$ are bounded by a direct application of Theorem 2.1 (and hence yield errors of order $\delta_1(\varepsilon)$ instead of $\delta_1(\varepsilon)^{1/2}$). □

8. Stationary Solutions

In this section we discuss stationary solutions of (2.1), and prove Theorem 2.4. The stationary solutions of (2.1) are the positive normalized solutions of

$$\varepsilon \mathscr{L} f + Q(f) = 0. \tag{8.1}$$

The existence of such solutions relies on a simple fixed-point argument. Recalling the expression of $Q(f)$, we put

$$J(f) = \begin{cases} M_f & \text{in the BGK case,} \\ f \circ f & \text{in the Boltzmann case.} \end{cases}$$

Equation (8.1) can be rewritten as

$$f = J(f) + \frac{\varepsilon}{\ell} \mathscr{L} f, \tag{8.2}$$

where $\ell = 1$ in the BGK case. The explicit form of the linear operator $\mathscr{L}$ implies that, for ε sufficiently small, the operator

$$\mathscr{G} = \Big(1 - \frac{\varepsilon}{\ell} \mathscr{L}\Big)^{-1} \tag{8.3}$$

preserves positivity and normalization on $L_1(\mathbb{R}^3)$. (That is, it is a Markovian operator.) Moreover, it is clear from the smoothing properties of $\mathscr{G}$ that it is compact on $L_1(\mathbb{R}^3)$. Also, J is a positivity- and normalization-preserving map of $L_1(\mathbb{R}^3)$ into itself. Then since we can rewrite (8.2) as

$$f = \mathscr{G} J(f), \tag{8.4}$$

we see that the solutions of (8.1) that we seek are the fixed points of the map $f \mapsto \mathscr{G} J(f)$. The properties of this map listed above prove the existence of fixed points (see [14]). Let us denote by f_* one of them. It is then easy to check that it satisfies (8.1) pointwise.

Next we note that the same arguments used in previous sections imply that, if f satisfies (8.1), then

$$\|f_* - M_{f_*}\| \leqq \delta_{\text{entropy}}(\varepsilon).$$

This concludes the first part of Theorem 2.4.

Everything done so far would apply in the Boltzmann case as well as the BGK case. What we do not know at this point is: Are there fixed points in *each* of the stable neighborhoods, and if so, is there *exactly one* in each stable neighborhood?

These questions can be positively answered in the BGK case in a simpler way. It seems likely that one could also provide a positive answer for at least the first of them for the Boltzmann kernel, but we have not done more than sketch a lengthy argument, and so confine ourselves to the BGK case.

In fact, since $\mathscr{G} = I + \varepsilon \mathscr{L}\mathscr{G}$ with I the identity map and $J(f) = M_f$, we can write (8.4) as

$$f = M_\varepsilon + \varepsilon \mathscr{L}\mathscr{G} M_\varepsilon. \tag{8.5}$$

Note that the right-hand side of (8.5) depends only on u and e, the first and second moments of f. If we multiply (8.5) by v or by v^2 and integrate, we get

$$F_\varepsilon(u, e) = 0, \quad G_\varepsilon(u, e) = 0, \tag{8.6}$$

because f and M_f have the same first two moments. The functions F_e and G_e are quite complicated, but for $\varepsilon = 0$ they reduce to the functions F and G in the right-hand side of (2.14). Then, by Proposition 2.3 we know there are solutions (u_*, e_*) to (8.6) for $\varepsilon = 0$. Moreover the differential of the map $(u, e) \to (F_\varepsilon(u, e), G(u, e))$ has eigenvalues with non-vanishing real part, in $\varepsilon = 0$ and $(u, e) = (u_*, e_*)$, when E is in the appropriate range. Therefore, by the implicit function theorem, for ε sufficiently small, we have a unique solution $(u_\varepsilon, e_\varepsilon)$ in a neighborhood of (u_*, e_*) to (8.6). Let M_ε be the Maxwellian with moments $(u_\varepsilon, e_\varepsilon)$. Then it is easy to check that

$$f = M_\varepsilon + \mathscr{L}\mathscr{G} M_\varepsilon$$

is the solution to (8.1). This concludes Theorem 2.4.

9. Long-Time Behavior

It is natural to ask whether the stationary solutions are the asymptotic limits as $t \to +\infty$ of the evolution starting in appropriate neighborhoods of the fixed-point Maxwellian. To this we have only a partial answer even in the BGK case:

Proposition 9.1. *Choose a stable fixed point* (u^*, e^*) *of* (2.14) *and let* f_* *be a stationary solution of* (2.1) *in the neighborhood of* $M_{(u^*,e^*)}$. *Assume that the solution* f_t *of the time-dependent problem starting near* $M_{(u^*,e^*)}$ *has moments* $u(t)$ *and* $e(t)$ *converging to* u^* *and* e^**, respectively. Then*

$$\lim_{t\to+\infty} \|f - f_*\|_2 = 0.$$

Unfortunately we do not have enough control on the time behavior of the solution to check the convergence of the moments. We expect however such convergence, and this can be proved for a modified model where we consider, instead of a diffusion coefficient D depending on the velocity, one depending only on the average e. A straightforward calculation then shows that one gets closed equations for the first two moments, and the long-time asymptotics is easily obtained. In this case all our results still apply and the conditions of Proposition 2.5 are fulfilled.

First we prove Proposition 2.5. Let f_* be a fixed point and f the distribution at time t. Calculating as in previous sections, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|f - f_*\|_2^2 \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3v\, (f(v) - f_*(v))\Big[\mathscr{L} f(v) + \frac{1}{\varepsilon} Q(f)(v)\Big] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3v\, (f(v) - f_*(v))\Big[\mathscr{L}(f(v) - f_*(v)) + \frac{1}{\varepsilon}(Q(f)(v) - Q(f_*)(v))\Big] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3v\, (f(v) - f_*(v))\mathscr{L}(f(v) - f_*(v)) \\
&\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \mathrm{d}^3v\, (f(v) - f_*(v))(M_f(v) - M_{f_*}(v)) - \frac{1}{\varepsilon}\|f - f_*\|_2^2.
\end{aligned}$$

It is easy to check that

$$\int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v)) L_2 (f(v) - f_*(v)) \leqq 0,$$

$$\int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v)) E \cdot \nabla_v (f(v) - f_*(v)) = 0.$$

On the other hand

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v)) L_1 (f(v) - f_*(v)) \\
&= - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, \nabla_v (f(v) - f_*(v)) D(v) M(v) \nabla_v \left(\frac{f(v) - f_*(v)}{M} \right) \\
&= - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, |\nabla_v (f(v) - f_*(v))|^2 D(v) \\
&\quad - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, v \cdot \nabla_v (f(v) - f_*(v)) D(v) (f(v) - f_*(v)) \\
&= - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, |\nabla_v (f(v) - f_*(v))|^2 D(v) - \tfrac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 v\, v \cdot \nabla_v (f(v) - f_*(v))^2 D(v) \\
&= - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, |\nabla_v (f(v) - f_*(v))|^2 D(v) + \tfrac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v))^2 \nabla_v \cdot (v D(v)) \\
&= - \int_{\mathbb{R}^3} \mathrm{d}^3 v\, |\nabla_v (f(v) - f_*(v))|^2 D(v) + \tfrac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v))^2 v \cdot \nabla_v D(v) \\
&\quad + \tfrac{3}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v))^2 D(v) \\
&\leqq \tfrac{3}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v))^2 D(v).
\end{aligned}$$

The last inequality is a consequence of the (3.3). Since

$$\int_{\mathbb{R}^3} \mathrm{d}^3 v\, (f(v) - f_*(v))(M_f(v) - M_{f_*}(v)) \leqq \tfrac{1}{2} \| f - f_* \|_2^2 + \tfrac{1}{2} \| M_f - M_{f_*} \|_2^2,$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \| f - f_* \|_2^2 \leqq \frac{1}{\varepsilon} \| M_f - M_{f_*} \|_2^2 - \left[\frac{1}{\varepsilon} - 3(a + c) \right] \| f - f_* \|_2^2.$$

With $x(t) = \| f - f_* \|_2^2$ and $a = \| M_f - M_{f_*} \|_2^2$ we have, for ε sufficiently small, that

$$\dot{x} + c x \leqq a$$

and hence

$$x(t) \leqq x_0 e^{-t/e} + \int_0^t \mathrm{d}s e^{-(t-s)/ea(s)}.$$

This implies that $x(t) \to 0$ as $t \to \infty$, provided that $a(t) \to 0$ as $t \to \infty$. This concludes the proof of Proposition 2.5.

The convergence of M_f to M_{f_*} is not easy to get for (2.1). A simple answer is obtained if one replaces the operator L_1 given by (2.3) with

$$L_1 f(v) = D(e)\nabla \cdot \left(M(v)\nabla f \left(\frac{f(v)}{M(v)} \right) \right). \tag{9.1}$$

This model is much simpler than the one already considered, but still has a non-trivial behavior on the hydrodynamical scale. In particular, it is easy to see that, if we write the equations for u and e, we get a closed system in those variables. Namely, (2.14)–(2.16) are replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} u(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} F(u(t), e(t)) \\ G(u(t), e(t)) \end{pmatrix}, \tag{9.2}$$

the functions F and G being given explicitly by

$$F(u, e) = E - u[\nu + D(e)], \tag{9.3}$$

$$G(u, e) = Eu + 3D(e) - 2eD(e). \tag{9.4}$$

Equation (9.2) is exact for this model independently of ε, and for suitable choices of the functions $D(e)$ has several critical points for E in an appropriate range. Moreover, the asymptotic behavior for large times is easy to establish. All our results apply to this model without substantial changes. In particular, in this case we can use Proposition 2.5 to obtain the convergence to stationary solutions for large times.

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References

1. T. Nishida, *Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation*, Comm. Math. Phys. **61**, 119–148 (1978).
2. R. E. Caflisch, *The fluid dynamic limit of the nonlinear Boltzmann equation*, Comm. Pure Appl. Math. **33**, 651–666 (1980).
3. S. Ukai & K. Asano, *The Euler limit and initial layer of the nonlinear Boltzmann equation*, Hokkaido Math. J. **12**, 303–324 (1983).
4. A. De Masi, R. Esposito & J. L. Lebowitz, *Incompressible Navier-Stokes and Euler limits of the Boltzmann equation*, Comm. Pure Appl. Math. **42**, 1189–1214 (1989).
5. R. Esposito, J. L. Lebowitz & R. Marra, *Hydrodynamical limit of the stationary Boltzmann equation in a slab*, Comm. Math. Phys. **160**, 49–80 (1994).
6. R. Esposito, J. L. Lebowitz & R. Marra, *Navier-Stokes behavior of stationary solutions of the Boltzmann equation*, J. Stat. Phys. **78**, 389–412 (1995).

7. A. ROKHLENKO, *Electron component of a plasma in a homogneous electron field*, Phys. Rev. A **43**, 4438 (1991); A. ROKHLENKO & J. L. LEBOWITZ, *Electron velocity distribution in a weakly ionized plasma with an external electric field*, Phys. Fluids B: Plasma Physics **5**, 1766 (1993).
8. R. BALESCU, *Transport Processes in Plasmas*, North-Holland (1988).
9. R. N. FRANKLIN, *Plasma Phenomena in Gas Discharges*, Clarendon Press, Oxford (1976): N. J. CARRON, *Energy-diffusion calculation of electron-swarm distribution function*, Phys. Rev. A **45**, 2499 (1992).
10. E. A. CARLEN & M.C. CARVALHO, *Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation*, J. Stat. Phys. **67**, 575–608 (1992).
11. E. A. CARLEN & M.C. CARVALHO, *Entropy production estimates for Boltzmann equations with physically realistic collision kernels*, J. Stat. Phys. **74**, 743–782 (1994).
12. E. CARLEN, R. ESPOSITO, J. L. LEBOWITZ, R. MARRA & A. ROKHLENKO, *Nonunique stationary states in driven collisional systems with application to plasmas*, Phys. Rev. E **52**, 40–43 (1995).
13. E. A. CARLEN, E. GABETTA & G. TOSCANI, *Propagation of smoothness in velocities and strong exponential convergence for Maxwellian molecules*, Commun. Math. Phys., in press.
14. M. KRASNOSEL'SKI, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan Press, New York (1964).

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