Letters in Mathematical Physics 42: 43–56, 1997.

The Micro-Canonical Point Vortex Ensemble: Beyond Equivalence

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(Received: 25 November 1996; revised version: 25 January 1997)

Abstract. The fluid limit $N \to \infty$ is constructed for a sequence of ensembles of N classical point vortices in a finite domain $\Lambda \subset \mathbb{R}^2$ whose ensemble densities (w.r.t. Liouville measure) are Gaussian approximations to $\delta(E - H)$. Letting the variance $\to 0$ after $N \to \infty$ has been taken, one recovers the special class of nonlinear stationary Euler flows that is expected from the micro-canonical ensemble. The construction improves over previous ones which either had to regularize the logarithmic singularities of the point vortex Hamiltonian or had to assume equivalence of ensembles. In particular, nonequivalence between micro-canonical and canonical ensemble prevails for certain geometries where conditionally stable configurations with negative 'global vortex pair-specific heat' can and do exist in the micro-canonical but not in the canonical ensemble.

Mathematics Subject Classification (1991). 82B21, 76F99, 35A15.

Key words: point vortices, Onsager's theory, continuum limit, nonequivalence of ensembles.

1. Introduction

The motion of N classical point vortices in a two-dimensional, incompressible Euler flow is governed [1–4] by the Hamiltonian

$$H^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{1 \leqslant i < j \leqslant N} c_i c_j G(\mathbf{r}_i, \mathbf{r}_j) + \sum_{1 \leqslant i \leqslant N} (c_i^2 F(\mathbf{r}_i) + c_i \psi^{ap}(\mathbf{r}_i)). \tag{1.1}$$

In (1.1), $\mathbf{r}_i = (x_i, y_i)$ is the position of the ith vortex in an open, connected domain $\Lambda \subset \mathbb{R}^2$ with finite area $|\Lambda|$ and piecewise regular Lipschitz boundary $\partial\Lambda$. Up to a trivial factor $\sqrt{|c_i|}$, the Cartesian components x_i and y_i are the canonically conjugate variables. The circulations c_i are expressed as dimensionless multiples of a suitable reference unit. The pair interaction $G(\mathbf{r}_i, \mathbf{r}_j) = G(\mathbf{r}_j, \mathbf{r}_i)$ is Green's function for $-\Delta$ on Λ, with 0 Dirichlet boundary conditions on $\partial\Lambda$. The interaction of a vortex with its own images, mediated by the boundary $\partial\Lambda$, is given by the regular part of Green's function,

$$F(\mathbf{r}) = \lim_{\tilde{\mathbf{r}} \to \mathbf{r}} \frac{1}{2} \left[G(\tilde{\mathbf{r}}, \mathbf{r}) + \frac{1}{2\pi} \ln|\tilde{\mathbf{r}} - \mathbf{r}| \right]. \tag{1.2}$$

Finally, ψ^{ap} is the stream function of an applied continuous background vorticity supported in Λ.

In a pioneering paper [5] on turbulent flows, Onsager studied the microcanonical ensemble of (1.1). The normalized phase space volume of the set $\{H^{(N)} < E\}$ in the N vortices phase space Λ^N, given by

$$\Phi(E) = |\Lambda|^{-N} \int_{\{H^{(N)}<E\}} \mathrm{d}^N\tau, \tag{1.3}$$

with $\mathrm{d}^N\tau = \prod_{i=1}^N \mathrm{d}\tau_i$ and $\mathrm{d}\tau_i = \mathrm{d}x_i\,\mathrm{d}y_i$ the two-dimensional Lebesgue measure, is a monotonically increasing, bounded function. Onsager noted that, as a consequence of this, Boltzmann's entropy, $S(E) = \ln \Phi'(E)$, must reach a maximum at a particular value E_m of energy, such that ([5], p. 281): "negative 'temperatures' ... will occur if $E > E_m$, ... [and] then vortices of the same sign will tend to cluster, ... It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; ... "

Onsager's insight suggests the following question: Given N point vortices distributed according to the micro-canonical measure, which vorticity structures are obtained in a suitable continuum (Euler fluid) limit $N \to \infty$? Considering for simplicity the one-species micro-canonical point vortex ensemble, the conjectured answer (cf., [6]) is the following. Fixing Λ and $\varepsilon = E/N^2 > 0$ as $N \to \infty$, the Boltzmann entropy per vortex converges to a continuous function of ε,

$$\lim_{N\to\infty} \frac{1}{N} \ln \Phi'(N^2\varepsilon) = s(\varepsilon). \tag{1.4}$$

The micro-canonical equilibrium measure converges, for $\varepsilon > 0$ and in a suitable topology, to a convex linear superposition of infinite products of one-vortex measures. The superposition measure is concentrated on those absolutely continuous one-vortex measures with density ρ_ε that have entropy $-\int_\Lambda \rho_\varepsilon \ln[|\Lambda|\rho_\varepsilon]\,\mathrm{d}\tau = s(\varepsilon)$, and which satisfy (the integral form of) Liouville's conformal PDE [7]

$$\rho_\varepsilon(\mathbf{r}) = \exp\left(\beta\left[\alpha - \int_\Lambda G(\mathbf{r},\mathbf{r}')\rho_\varepsilon(\mathbf{r}')\mathrm{d}\tau'\right]\right), \tag{1.5}$$

with constants β and α chosen to satisfy $\int_\Lambda \rho_\varepsilon\,\mathrm{d}\tau = 1$ and the energy constraint $\|\rho\|^2_{H^{-1}} = 2\varepsilon$. The solutions of (1.5) can then be interpreted as continuum vorticities satisfying the stationary Euler equations for incompressible flows, cf. [8].

While heuristic derivations of (1.5) have been proposed in [9–13], a rigorous proof of the conjectured answer has not yet been achieved. In rigorous works on this problem [6, 14] the micro-canonical point vortex ensemble is replaced by a regularized measure, the regularization being removed after the limit $N \to \infty$ has been taken. In addition, in [6] the logarithmic singularities in H are regularized, while in [14] the limit $N \to \infty$ is restricted to situations in which the micro-canonical and canonical ensembles are equivalent. The Euler fluid limit of the canonical point vortex ensemble in turn has been constructed without approximation in [15, 16], following earlier work [17] on regularized interactions.

Our interest here is in extending the results of [6, 14]. While we, too, regularize $\delta(H-E)$, our work is not based on equivalence of ensembles, and we keep the logarithmic singularities in (1.1).

The only limit results for $\delta(H-E)$, obtained for neutral point vortex systems, seem to be [18, 19], which however concern a different limit $N \to \infty$. The construction of the Euler fluid limit $N \to \infty$ starting directly from $\delta(H-E)$ remains open for future work, both for one and for neutral two-species systems. The latter have the curious feature that the reciprocal temperature, obtained from the micro-canonical ensemble in the Euler fluid limit $N \to \infty$, is bounded away from zero [20].

2. Statement of Results

We consider a one-species vortex Hamiltonian (1.1) with $c_i = 1$ and $\psi^{ap} \equiv 0$, for simplicity. For finite N, our regularized micro-canonical equilibrium probability measure on Λ^N is of the form

$$\mu^{(N,\varepsilon,\sigma)}(\mathrm{d}^N\tau) = \frac{1}{Z}\exp\Big[-N\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N)}\Big)^2\Big]\,\mathrm{d}^N\tau, \tag{2.1}$$

where $\sigma > 0$ and ε are fixed real numbers, and

$$Z(N,\varepsilon,\sigma) = \int_{\Lambda^N}\exp\Big[-N\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N)}\Big)^2\Big]\,\mathrm{d}^N\tau. \tag{2.2}$$

The micro-canonical ensemble is obtained in the limit $\sigma \to 0$ at fixed N in (2.1), giving a delta measure concentrated on $\{H^{(N)} = E\}$ with $E = N^2\varepsilon$, as can be easily verified using geometric measure theory [21]. We are interested in the asymptotic evaluation, as $N \to \infty$, of (2.1) for fixed ε and σ, which is the continuum Euler fluid scaling for H.

Let $P(\Lambda^N)$ denote the probability measures on Λ^N. For any $N \in \mathbb{N}$, we define the entropy of $\varrho_N \in P(\Lambda^N)$ relative to the uniform measure $|\Lambda|^{-N}d^N\tau$ by

$$\mathbf{S}(\varrho_N) = -\int_{\Lambda^N}\rho_N \ln\Big(|\Lambda|^N\rho_N\Big)\,\mathrm{d}^N\tau, \tag{2.3}$$

if ϱ_N is absolutely continuous w.r.t. Lebesgue measure, having density ρ_N, and provided the integral on the r.h.s. of (2.3) exists; $\mathbf{S}(\varrho_N) = -\infty$ in all other cases. For $\varrho_1 = \varrho \in P(\Lambda)$, we define a one-vortex penalized entropy functional by

$$\mathbf{R}_{\varepsilon,\sigma}(\varrho) =: \mathbf{S}(\varrho) - \frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{2}\int_\Lambda \rho(\mathbf{r})(G*\rho)(\mathbf{r})\,\mathrm{d}\tau\Big)^2 \tag{2.4}$$

for those $\varrho(\mathrm{d}\tau) = \rho\,\mathrm{d}\tau$, with ρ a Lebesgue probability density, for which $\mathrm{S}(\varrho) > -\infty$, and $\mathbf{R}_{\varepsilon,\sigma}(\varrho) = -\infty$ otherwise. Here, $\mathrm{S}(\varrho) = \mathrm{S}(\varrho_1)$ is the one-vortex entropy as defined in (2.3), and

$$(G * \rho)(\mathbf{r}) = \int_\Lambda G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\tau'. \tag{2.5}$$

We write $M_{\varepsilon,\sigma}$ for the set $\{\varrho_{\varepsilon,\sigma}\}$ of maximizers of $\mathbf{R}_{\varepsilon,\sigma}$.

By $\Omega = \Lambda^{\mathbf{N}}$ we denote the Λ-valued infinite exchangeable sequences, and by $P^{\mathrm{ex}}(\Omega)$ the permutation-invariant probability measures on Ω. According to the theorem of de Finetti [22] and Dynkin [23] that every $\mu \in P^{\mathrm{ex}}(\Omega)$ is given by a convex linear superposition of product measures of the form

$$\mu_n(\mathrm{d}^n\tau) = \int_{P(\Lambda)} \nu(\mu|\mathrm{d}\varrho)\varrho^{\otimes n}(\mathrm{d}^n\tau), \tag{2.6}$$

where $\mu_n(\mathrm{d}^n\tau) \in P(\Lambda^n)$ is the nth marginal measure of μ. By a theorem of Hewitt and Savage [24], the product states $\varrho^{\otimes \mathbf{N}}$ are the extreme points of the convex set $P^{\mathrm{ex}}(\Omega)$. Hence, (2.6) is also the extremal decomposition of μ_n.

THEOREM 2.1. *For each $\varepsilon \in \mathbb{R}$ and $\sigma > 0$ fixed, we have*

$$\lim_{N\to\infty} \frac{1}{N} \ln\left[|\Lambda|^{-N} Z(N, \varepsilon, \sigma)\right] = \mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) \tag{2.7}$$

with $\varrho_{\varepsilon,\sigma} \in M_{\varepsilon,\sigma}$. Moreover, (2.1) *has at least one limit point, convergence understood in weak L^p sense, $p < \infty$, in the corresponding subset of $P^{\mathrm{ex}}(\Omega)$. The decomposition measure $\nu(\mu^{(\varepsilon,\sigma)}|\mathrm{d}\varrho)$ of any limit point $\mu^{(\varepsilon,\sigma)}$ is concentrated on $M_{\varepsilon,\sigma} \subset P(\Lambda)$.*

The subsequent limit $\sigma \to 0$ now gives the anticipated variational principle for the micro-canonical ensemble. We denote by $L_1^{1,+}(\Lambda)$ the subset of the positive cone of $L^1(\Lambda)$ whose elements satisfy $\int_\Lambda \rho\,\mathrm{d}\tau = 1$, and by $L_{1;\varepsilon}^{1,+}(\Lambda)$ the subset of $L_1^{1,+}(\Lambda)$ for which

$$\tfrac{1}{2}\|\rho\|^2_{H^{-1}} \equiv \frac{1}{2}\int_\Lambda\int_\Lambda \rho(\mathbf{r})G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\mathrm{d}\tau\,\mathrm{d}\tau' = \varepsilon. \tag{2.8}$$

Notice that Green's function G is the kernel of a positive operator.

THEOREM 2.2. *For fixed $\varepsilon \in \mathbb{R}$, define $\mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) \equiv s_\sigma(\varepsilon)$.*

Part 1. *Let $\varepsilon > 0$. Then the limit*

$$s(\varepsilon) = \lim_{\sigma\to 0} s_\sigma(\varepsilon) \tag{2.9}$$

exists and satisfies the variational principle

$$s(\varepsilon) = \max\left\{ \mathbf{S}(\varrho) |\ \varrho(\mathrm{d}\tau) = \rho\,\mathrm{d}\tau;\rho \in L^{1,+}_{1;\varepsilon} \right\}. \tag{2.10}$$

Any maximizer ρ_ε for (2.10) *satisfies the Euler–Lagrange equation*

$$\rho_\varepsilon(\mathbf{r}) = \exp\left(\beta\left[\alpha - \int_\Lambda G(\mathbf{r},\mathbf{r}')\rho_\varepsilon(\mathbf{r}')\mathrm{d}\tau'\right]\right), \tag{2.11}$$

where β and α are real numbers to be chosen to ensure $\rho \in L^{1,+}_{1;\varepsilon}$.

Furthermore, let M_ε denote the set of maximizers ρ_ε for (2.10)*. As $\sigma \to 0$, let $\mu^{(\varepsilon)}$ be a weak limit point of the measure $\mu^{(\varepsilon,\sigma)}$ on Ω. Then $\mu^{(\varepsilon)} \in P^{\mathrm{ex}}(\Omega)$, and its de Finetti–Dynkin decomposition measure $\nu(\mu^{(\varepsilon)}|\mathrm{d}\varrho)$ is concentrated on M_ε.*

Part 2. *Let $\varepsilon \leqslant 0$. In this case $\lim_{\sigma\to 0} s_\sigma(\varepsilon) = -\infty$.*

3. Penalized Entropy Functionals

We shall need the following properties of the penalized entropy (2.4).

PROPOSITION 3.1. *For each $\varepsilon \in \mathbb{R}$ and $\sigma > 0$ fixed, the functional $\mathbf{R}_{\varepsilon,\sigma}(\varrho)$ takes its maximum at an absolutely continuous ϱ. If ϱ is a maximizer of $\mathbf{R}_{\varepsilon,\sigma}(\varrho)$, then its density ρ solves the Euler–Lagrange equation*

$$\rho(\mathbf{r}) = \frac{\exp\left(\gamma(G*\rho)(\mathbf{r})\right)}{\int_\Lambda \exp\left(\gamma(G*\rho)(\mathbf{r}')\right)\mathrm{d}\tau'}, \tag{3.1}$$

where

$$\gamma = \frac{1}{\sigma^2}\left(\varepsilon - \frac{1}{2}\int_\Lambda \rho\, G*\rho\,\mathrm{d}\tau\right). \tag{3.2}$$

Proof. Let $\rho \in L^{1,+}_1(\Lambda) \cap \{\mathbf{S}(\varrho) > -\infty\}$ and $0 < \lambda \leqslant 8\pi$. It was shown in [16] and [15] that the Gibbs variational principle together with a simple bound on the configurational integral, based on convexity and an explicit computation, imply that there is a constant $C > -\infty$ that may depend on Λ and λ but not on ρ, such that

$$\inf_\rho\left(-\frac{\lambda^2}{2}\int_\Lambda\int_\Lambda \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\mathrm{d}\tau\,\mathrm{d}\tau' + \lambda\int_\Lambda \rho\ln(|\Lambda|\rho)\,\mathrm{d}\tau\right) \geqslant C. \tag{3.3}$$

Whence, for $\rho \in L^{1,+}_1(\Lambda) \cap \{\mathbf{S}(\varrho) > -\infty\}$ we have that $\rho \in H^{-1}(\Lambda)$, where $H^{-1}(\Lambda)$ is the $\|.\|_{H^{-1}}$ norm closure of $C^\infty(\Lambda)$. We remark, cf. [25], that (3.3) is related by convex duality to Moser's [26] corollary

$$\inf_u\left[\frac{1}{2}\int_\Lambda |\nabla u|^2\,\mathrm{d}\tau - \lambda\ln\frac{1}{|\Lambda|}\int_\Lambda e^u d\tau\right] \geqslant C \text{ for } u \in H^1_0 \tag{3.4}$$

of the Trudinger [27] – Moser [26] inequality (see also [28, 29]), which implies the embedding $H_0^1(\Lambda) \to L^{\phi}$, where L^{ϕ} is the Orlicz space with defining function $\phi = \exp(t^2) - 1$; see [27]. Here, $H_0^1(\Lambda)$ is the closure of $C_0^{\infty}(\Lambda)$ w.r.t. the norm induced by Dirichlet's form. By (3.3), we see that the set $\{\rho : \mathrm{S}(\varrho) \geqslant s\}$, with $s > -\infty$, is H^{-1} norm bounded, therefore it is H^{-1} weakly closed and compact. It now follows [30] that $\mathrm{S}(\varrho)$ is H^{-1} weakly upper semi-continuous. Therefore $\mathbf{R}_{\varepsilon,\sigma}$ takes its maximum on any subset of $P(\Lambda)$ of the form $\{\varrho(\mathrm{d}\tau) = \rho\,\mathrm{d}\tau : \rho \in L_1^{1,+}(\Lambda); \mathrm{S}(\varrho) \geqslant s > -\infty\}$.

Next, since $\mathrm{S}(\varrho) \leqslant 0$, we have that $\mathbf{R}_{\varepsilon,\sigma} \to -\infty$ for $\|\rho\|_{H^{-1}} \to \infty$ at least as $\|\rho\|_{H^{-1}}^4$, so $\mathbf{R}_{\varepsilon,\sigma}$ is coercive in H^{-1} topology. Whence, the maximum of $\mathbf{R}_{\varepsilon,\sigma}(\varrho)$ over $P(\Lambda)$ is taken in the interior of $H^{-1}(\Lambda)$ so that γ, (3.2), is well defined for any maximizer ρ. In addition, the entropy functional guarantees that a maximizer is also in the interior of $L_1^{1;+}$. We thus see that the maximum is at a stationary point and satisfies the Euler–Lagrange equation. Taking the Gateaux derivative and applying the Fundamental Lemma of Variational Calculus gives (3.1). □

COROLLARY 3.2. *For each $\varepsilon \in \mathbb{R}$ and $\sigma > 0$, the maximizers of $\mathbf{R}_{\varepsilon,\sigma}(\varrho)$ over $P(\Lambda)$ have density $\rho_{\varepsilon,\sigma} \in (L_+^{1;1} \cap L^{\infty})(\Lambda)$.*

Proof. Solutions $\rho \in L_1^{1,+}$ of the Euler–Lagrange equation (3.1) with finite one particle entropy are in H^{-1}, by (3.3). Therefore, γ exists and so does

$$\psi(\mathbf{r}) = \gamma(G * \rho)(\mathbf{r}). \tag{3.5}$$

Moreover, $\|\nabla\psi\|_{L^2} < \infty$, by convex duality. By (3.3), this implies the regularity condition

$$\int_{\Lambda} \mathrm{e}^{\psi}\,\mathrm{d}\tau < \infty, \tag{3.6}$$

which implies that there exists some $\lambda \in \mathbb{R}$ such that ψ satisfies the conformal Liouville [7] equation in Λ,

$$-\Delta\psi = \lambda\,\mathrm{e}^{\psi}, \tag{3.7}$$

with 0 Dirichlet conditions on $\partial\Lambda$. By 3.6 the right-hand side of (3.7) is in L^1, whence by a theorem of Brezis and Merle, i.e., Corollary 2 on p. 1229 in [31], we have $\psi \in L^{\infty}$, whence by (3.5) and by (3.1) now $\rho \in L^{\infty}$. □

We need to generalize (2.4) to the N-vortices measures. Let ϱ_N be a symmetric probability measure on Λ^N. If ϱ_N is absolutely continuous w.r.t. Lebesgue measure, let $\rho_N \in L_1^{1,+}(\Lambda^N)$ denote its density. We define a penalty of ϱ_N by

$$\mathbf{P}_{\varepsilon,\sigma}(\varrho_N) = N\frac{1}{2\sigma^2}\int_{\Lambda^N}\Big(\varepsilon - \frac{1}{N^2}H^{(N)}\Big)^2 \varrho_N(\mathrm{d}^N\tau) \tag{3.8}$$

if the integral exists, and $\mathbf{P}_{\varepsilon,\sigma}(\varrho_N) = \infty$ otherwise. An N vortices penalized entropy functional is now defined by

$$\mathbf{S}_{\varepsilon,\sigma}(\varrho_N) = \mathbf{S}(\varrho_N) - \mathbf{P}_{\varepsilon,\sigma}(\varrho_N) \tag{3.9}$$

where $\mathbf{S}(\varrho_N)$ is given in (2.3), and $\mathbf{P}_{\varepsilon,\sigma}(\varrho_N)$ in (3.8). In particular, for $\mu \in P^{\text{ex}}(\Omega)$, let $\mu_n, n \in \mathbb{N}$, be the nth marginal measure. If μ_n is absolutely continuous w.r.t. Lebesgue measure, let ρ_n be its density. An n-vortices entropy of μ_n is given by $\mathbf{S}(\mu_n)$, see (2.3). In addition, we define $\mathbf{S}(\mu_{-k}) = 0$ for $k \in \mathbb{N} \cup \{0\}$. The penalty of μ_n is given by $\mathbf{P}_{\varepsilon,\sigma}(\mu_n)$, with $\mathbf{P}_{\varepsilon,\sigma}(.)$ as in (3.8).

We are now in the position to define a penalized mean entropy functional on $P^{\text{ex}}(\Omega)$. We restrict $P^{\text{ex}}(\Omega)$ to the subspace $P_p^{\text{ex}}(\Omega)$ for which the penalty exists for all marginals of μ,

$$P_p^{\text{ex}}(\Omega) = \{\mu \in P^{\text{ex}}(\Omega) : \mathbf{P}_{\varepsilon,\sigma}(\mu_n) < \infty \text{ for all } n < \infty\}. \tag{3.10}$$

For each $\mu \in P_p^{\text{ex}}(\Omega)$, a mean penalty of μ is now uniquely defined as the limit

$$\mathbf{p}_{\varepsilon,\sigma}(\mu) =: \lim_{n\to\infty} \frac{1}{n}\mathbf{P}_{\varepsilon,\sigma}(\mu_n). \tag{3.11}$$

Also, for each $\mu \in P^{\text{ex}}(\Omega)$, a mean entropy of μ is uniquely defined as the limit

$$\mathbf{s}(\mu) = \lim_{n\to\infty} \frac{1}{n}\mathbf{S}(\mu_n). \tag{3.12}$$

A penalized mean entropy functional on $P_p^{\text{ex}}(\Omega)$ is finally defined by

$$\mathbf{s}_{\varepsilon,\sigma}(\mu) = \mathbf{s}(\mu) - \mathbf{p}_{\varepsilon,\sigma}(\mu). \tag{3.13}$$

To see that $\mathbf{s}_{\varepsilon,\sigma}(\mu)$ is well defined, recall first that the map $n \mapsto \mathbf{S}(\mu_n)$ is non-positive, $\mathbf{S}(\mu_n) \leqslant 0$ for all n, monotonic decreasing, $\mathbf{S}(\mu_{n'}) \leqslant \mathbf{S}(\mu_n)$ for $n' > n$, and strongly sub-additive, i.e. for $n', n'' \leqslant n$, and with $m = n - n' - n''$,

$$\mathbf{S}(\mu_n) \leqslant \mathbf{S}(\mu_{n'}) + \mathbf{S}(\mu_{n''}) + \mathbf{S}^{(m)}(\mu_m) - \mathbf{S}^{(-m)}(\mu_{-m}). \tag{3.14}$$

(See [32, 16] for the proofs.) Regarding (3.12), sub-additivity (3.14) implies (see Lemma IX.2.4 in [33]) that, if $s_0 = \inf_n\{n^{-1}\mathbf{S}(\mu_n)\} > -\infty$, then $\mathbf{s}(\mu) = s_0$. Otherwise $\mathbf{s}(\mu) = -\infty$. In particular, if $\mathbf{S}(\mu_n) = -\infty$ for some $n = n_0$, then by the monotonic decrease also $\mathbf{S}(\mu_n) = -\infty$ for all $n > n_0$. Second, regarding 3.11, for $\mu \in P_p^{\text{ex}}(\Omega)$, de Finetti–Dynkin decomposition and a simple computation yield

$$\begin{aligned}&\frac{1}{n}\mathbf{P}_{\varepsilon,\sigma}(\mu_n)\\ &= \int_{P(\Lambda)} \nu(\mu|\mathrm{d}\varrho)\left[\frac{1}{2\sigma^2}\left(\varepsilon - \frac{1}{2}\int_\Lambda \rho\, G * \rho\, \mathrm{d}\tau\right)^2 + \mathrm{O}\left(\frac{1}{n}\right)\right]\end{aligned} \tag{3.15}$$

so that (3.11) is given by the n-independent term on the r.h.s. of (3.15). Hence, (3.11) is well defined. It follows that (3.13) is well defined.

PROPOSITION 3.3. *For any* $\varepsilon \in \mathbb{R}$, *and* $\sigma > 0$, *the penalized mean entropy* $\mathrm{s}_{\varepsilon,\sigma}(\mu)$ *is affine, i.e. for* $0 \leqslant q \leqslant 1$, *and* $\mu, \mu' \in P_p^{\mathrm{ex}}(\Omega)$, *we have*

$$\mathrm{s}_{\varepsilon,\sigma}(q\mu + [1-q]\mu') = q\mathrm{s}_{\varepsilon,\sigma}(\mu) + (1-q)\mathrm{s}_{\varepsilon,\sigma}(\mu') \tag{3.16}$$

Proof. As in [32], one finds that the mean entropy functional is affine. Moreover, $\mathbf{p}_{\varepsilon,\sigma}(\mu)$ is clearly affine. Therefore, the penalized mean entropy of μ is affine. □

COROLLARY 3.4. *For any* ε *and* $\sigma > 0$, *the penalized mean entropy takes its finite global maximum on a subset of the extreme points of* $P_p^{\mathrm{ex}}(\Omega)$. *In particular,*

$$\max_{\mu \in P_p^{\mathrm{ex}}(\Omega)} \mathrm{s}_{\varepsilon,\sigma}(\mu) = \max_{\varrho \in P(\Lambda)} \mathbf{R}_{\varepsilon,\sigma}(\varrho). \tag{3.17}$$

Proof. It is well known that the supremum of an affine function over a convex space can be computed by a maximizing sequence restricted to the set of extreme points of the convex set, given here by all $\varrho^{\otimes \mathrm{N}} \in P_p^{\mathrm{ex}}$. But $\mathrm{s}_{\varepsilon,\sigma}(\varrho^{\otimes \mathrm{N}}) = \mathbf{R}_{\varepsilon,\sigma}(\varrho)$. By (3.1), the supremum of $\mathbf{R}_{\varepsilon,\sigma}$ is a maximum. Let $\varrho_{\varepsilon,\sigma}$ be a maximizer for $\mathbf{R}_{\varepsilon,\sigma}$. Clearly, the corresponding product measure $\varrho_{\varepsilon,\sigma}^{\otimes \mathrm{N}}$ is a maximizer of $\mathrm{s}_{\varepsilon,\sigma}(\mu)$. □

4. The Fluid Limit $N \to \infty$ for $\sigma > 0$

To prove our Theorem 2.1 we establish sharp lower and upper bounds on $\ln Z$ as $N \to \infty$.

PROPOSITION 4.1. *For* $\varepsilon \in \mathbb{R}$ *and* $\sigma > 0$, *we have*

$$\liminf_{N\to\infty} \frac{1}{N} \ln \left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right] \geqslant \max_{\rho \in L_1^{1,+}(\Lambda)} \mathbf{R}_{\varepsilon,\sigma}(\varrho). \tag{4.1}$$

Proof. We first note that $\ln Z$ satisfies the following variational principle,

$$\max_{\rho_N \in L_1^{1,+}(\Lambda^N)} \mathbf{S}_{\varepsilon,\sigma}(\varrho_N) = \ln \left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right], \tag{4.2}$$

and the maximum of $\mathbf{S}_{\varepsilon,\sigma}$ is taken if and only if ϱ_N equals the finite N probability measure (2.1). The proof of (4.2) is straightforward. In fact, when $\mathbf{r}_i \to \mathbf{r}_j$, or $\mathbf{r}_i \to \partial\Lambda$, the worst singularities in $H^{(N)}$ are logarithmic. Therefore, for each $\delta > 1$, $\mathbf{S}_{\varepsilon,\sigma}(\varrho_N)$ is well defined for $\rho_N \in (L_1^{1,+} \cap L^{1+\delta})(\Lambda^N)$. In particular, let $\mu^{(N)} := \mu^{(N,\varepsilon,\sigma)}$, then $\mathrm{d}\mu^{(N)}/\mathrm{d}^N\tau =: \eta^{(N)} \in (L_1^{1,+} \cap L^{\infty})(\Lambda^N)$, with supp $(\eta^{(N)}) = \overline{\Lambda}^N$. Whence, $\mathbf{S}_{\varepsilon,\sigma}(\mu^{(N)})$ exists. This justifies the computation

$$\mathbf{S}_{\varepsilon,\sigma}(\mu^{(N)}) = \ln \left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right]. \tag{4.3}$$

Taking now an arbitrary symmetric ϱ_N, either $\mathbf{S}_{\varepsilon,\sigma}(\varrho_N) = -\infty$, or, in case $\mathbf{S}_{\varepsilon,\sigma}(\varrho_N)$ does exist, we have from the convexity estimate $t\ln t \geqslant -1+t$, with equality holding only for $t = 1$, and from $\int \eta^{(N)}\,\mathrm{d}^N\tau = \int \rho_N\,\mathrm{d}^N\tau$, that

$$\mathbf{S}_{\varepsilon,\sigma}(\mu^{(N)}) - \mathbf{S}_{\varepsilon,\sigma}(\varrho_N) = \int_{\Lambda^N} \left(\frac{\rho_N}{\eta^{(N)}} \ln \frac{\rho_N}{\eta^{(N)}}\right) \eta^{(N)}\,\mathrm{d}^N\tau \geqslant 0, \tag{4.4}$$

with equality holding if and only if $\rho_N = \eta^{(N)}$ almost everywhere. This completes the construction of the variational principle (4.2).

Now recall that $M_{\varepsilon,\sigma} \subset P(\Lambda)$ is the set of maximizers for $\mathbf{R}_{\varepsilon,\sigma}(\varrho)$. Let $\varrho_{\varepsilon,\sigma} \in M_{\varepsilon,\sigma}$. By 4.2, we have $\ln\left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right] \geqslant \mathbf{S}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}^{\otimes N})$. By Corollary 3.2, $\varrho_{\varepsilon,\sigma}$ has density in $L_1^{1,+} \cap L^\infty$. Therefore, a simple computation for $\mathbf{S}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}^{\otimes N})$ now gives, for any finite $N \in \mathbb{N}$, fixed $\varepsilon \in \mathbb{R}$ and $\sigma > 0$, the estimate

$$\ln\left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right] \geqslant N\mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) + \sum_{k=0}^{3}\sum_{l=0}^{2} a_{k,l} N^{-k}(N-1)^{-l}, \tag{4.5}$$

with all $a_{k,l}$ independent of N. In particular, the $a_{k,l}$ exist because $\rho_{\varepsilon,\sigma} \in L^\infty(\Lambda)$. Dividing (4.5) by N and taking $N \to \infty$ proves Proposition 4.1. □

The counterpart of Proposition 4.1 is the following:

PROPOSITION 4.2. *For $\varepsilon \in \mathbb{R}$ and $\sigma > 0$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \ln\left[|\Lambda|^{-N} Z(N,\varepsilon,\sigma)\right] \leqslant \max_{\rho \in L_1^{1,+}(\Lambda)} \mathbf{R}_{\varepsilon,\sigma}(\varrho). \tag{4.6}$$

We prepare the proof of Proposition 4.2 by the following Lemma. We continue to write $\mu^{(N)}$ for $\mu^{(N,\varepsilon,\sigma)}$, and we set $\eta_n^{(N)} = \mathrm{d}\mu_n^{(N)}/\mathrm{d}^n\tau$, where $\mathrm{d}\mu_n^{(N)} = \mu_n^{(N)}(\mathrm{d}^n\tau) = \mu^{(N)}(\mathrm{d}^n\tau \otimes \Lambda^{N-n})$.

LEMMA 4.3. *For every $\varepsilon \in \mathbb{R}$, $\sigma > 0$, $n \in \mathbb{N}$ and $p \in [1,\infty)$ there is a $C(n,p,\varepsilon,\sigma) < \infty$ such that $\|\eta_n^{(N)}\|_{L^p(\Lambda^n)} < C$.*

Proof. Clearly, $\eta_n^{(N)} \leqslant |\Lambda|^{N-n} Z^{-1}$, while

$$Z^{-1}(N,\varepsilon,\sigma) \leqslant |\Lambda|^{-N} \exp[-N\mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) + \mathrm{O}(1)]$$

by 4.5. In case that $\mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) = 0$, which occurs iff $\rho_{\varepsilon,\sigma} = |\Lambda|^{-1}$ and $\|\,|\Lambda|^{-1}\|_{H^{-1}}^2 = 2\varepsilon$, we have $\eta_n^{(N)} \leqslant C|\Lambda|^{-n}$, and the proof is complete in this case. Here and in the following, C denotes a generic constant, independent of N, that may change from line to line. Let, therefore, ε and σ be such that $\rho_{\varepsilon,\sigma} \neq |\Lambda|^{-1}$, in which case $\mathbf{R}_{\varepsilon,\sigma}(\varrho_{\varepsilon,\sigma}) = -h < 0$ strictly. We then obtain the nonuniform L^∞

bound $\eta_n^{(N)} \leqslant C|\Lambda|^{-n}\exp(hN)$. To prove that for $p < \infty$ we have $\eta_n^{(N)} \in L^p(\Lambda^n)$ uniformly in N, it suffices to show that $\|\eta_n^{(N)}\|_{L^p(\Lambda^n)} < C$ for $N > N_0$. The following constructions are valid if N is big enough. Furthermore, we assume $p > 1$ (the case $p = 1$ is trivial).

Pick a domain $\tilde{\Lambda} \subset \Lambda$ such that $|\Lambda\backslash\tilde{\Lambda}| = \mathrm{e}^{-phN}|\Lambda|$, with $\mathrm{dist}(\partial\Lambda, \partial\tilde{\Lambda}) > b\,\mathrm{e}^{-phN}$ for some fixed positive b. The L^∞ bound for $\eta_n^{(N)}$ and summation give

$$\begin{aligned}\int_{\Lambda^n} &(\eta_n^{(N)})^p\,\mathrm{d}^n\tau\\ &= \sum_{k=0}^{n}\binom{n}{k}\int_{(\Lambda\backslash\tilde{\Lambda})^k}\int_{\tilde{\Lambda}^{n-k}}(\eta_n^{(N)})^p\,\mathrm{d}^n\tau\\ &\leqslant C^p|\Lambda|^{(1-p)n}\,\mathrm{e}^{phN}(1-[1-\mathrm{e}^{-phN}]^n) + \int_{\tilde{\Lambda}^n}(\eta_n^{(N)})^p\,\mathrm{d}^n\tau. \end{aligned} \tag{4.7}$$

Since $\mathrm{e}^{phN}(1-[1-\mathrm{e}^{-phN}]^n) = n + \mathrm{O}(\mathrm{e}^{-phN})$, it suffices to show that $\eta_n^{(N)} \in L^p(\tilde{\Lambda}^n)$ uniformly for N big enough.

By the permutation symmetry of $H^{(N)}$, we can consider $\eta_n^{(N)}$ as function of the first n coordinates. Proceeding as in (4.7), we obtain

$$\begin{aligned}\eta_n^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_n) \leqslant{}& \frac{1}{Z}\int_{\tilde{\Lambda}^{N-n}}\exp\Big[-N\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N)}\Big)^2\Big]\prod_{j=n+1}^{N}\mathrm{d}\tau_j + \\ &+C|\Lambda|^{-n}\,\mathrm{e}^{hN}(1-[1-\mathrm{e}^{-phN}]^{N-n}),\end{aligned} \tag{4.8}$$

with $\lim_{N\to\infty}\mathrm{e}^{hN}(1-[1-\mathrm{e}^{-phN}]^{N-n}) = 0$, for $p > 1$. To control the integral over $\tilde{\Lambda}^{N-n}$ in (4.8), we write $H^{(N)} = H^{(n)} + H^{(N-n)} + W^{(N,n)}$, where it is understood that

$$H^{(n)} = H^{(n)}(\mathbf{r}_1,\ldots,\mathbf{r}_n) \quad\text{and}\quad H^{(N-n)} = H^{(N-n)}(\mathbf{r}_{n+1},\ldots,\mathbf{r}_N).$$

Clearly, $W^{(N,n)} = \sum_{i=1}^{n}\sum_{j=n+1}^{N}G(\mathbf{r}_i,\mathbf{r}_j)$. Using now $(W^{(N,n)} + H^{(n)})^2 \geqslant 0$, next $G(\mathbf{r}_i,\mathbf{r}_j) > 0$ on $\tilde{\Lambda}\times\tilde{\Lambda}$, and $c_* \geqslant F(\mathbf{r}_j) \geqslant -\varepsilon^*N + c^*$ on $\tilde{\Lambda}$, where $\varepsilon^* = ph/4\pi$, and where c_* and c^* are N-independent constants, it is easily seen that

$$\begin{aligned}-N&\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N)}\Big)^2\\ &\leqslant -N\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N-n)}\Big)^2 + a_1 + \\ &\quad + a_2\frac{1}{N}\Big[W^{(N,n)} + \hat{H}^{(n)}\Big] + a_3\frac{1}{N^2}\hat{H}^{(N-n)},\end{aligned} \tag{4.9}$$

where $\hat{H}^{(n)}$ and $\hat{H}^{(N-n)}$ are obtained from $H^{(n)}$ and $H^{(N-n)}$ by deleting the image interactions $\sum F$, and where $a_1(N) = n\sigma^{-2}\varepsilon^*(|\varepsilon|+\varepsilon^*)+O(N^{-1})$, $a_2(N) = \sigma^{-2}(\varepsilon+\varepsilon^*)+O(N^{-1})$, and $a_3(N) = n\sigma^{-2}\varepsilon^* + O(N^{-1})$, independent of position. Clearly, there exist N-independent constants c_i such that $0 < c_i - a_i \ll 1; i = 1, 2, 3$, if N is big enough. Inserting (4.9) into the integral in (4.8), and then applying Hölder's inequality with $\alpha = 4\pi N/nc_2$, $\beta = 8\pi N^2/(N-n)c_3$, and $\gamma^{-1} = 1 - \alpha^{-1} - \beta^{-1}$, we get

$$\eta_n^{(N)} \leqslant e^{c_1} e^{c_2 N^{-1}\hat{H}^{(n)}} Z^{-1}\tilde{Z}^{1/\gamma}\|e^{a_3 N^{-2}\hat{H}^{(N-n)}}\|_{L^\beta}\|e^{a_2 N^{-1}W^{(N,n)}}\|_{L^\alpha}, \tag{4.10}$$

with

$$\tilde{Z}(N,\varepsilon,\sigma) = \int_{\tilde{\Lambda}^{N-n}} \exp\Big[-\gamma N\frac{1}{2\sigma^2}\Big(\varepsilon - \frac{1}{N^2}H^{(N-n)}\Big)^2\Big]\,d^{N-n}\tau. \tag{4.11}$$

Clearly, $\exp(c_2 N^{-1}\hat{H}^{(n)}) \in L^p(\tilde{\Lambda}^n)$ uniformly for N big enough. Turning to Z, we have the following estimate. Let A denote the difference between the left side of (4.9) and γ times the first term on its right side. Let $\langle\cdots\rangle$ denote expected value w.r.t.

$$|\tilde{\Lambda}|^{-n}\,d^n\tau \otimes \tilde{Z}^{-1}\exp[-\gamma N(2\sigma^2)^{-1}(\varepsilon - N^{-2}H^{(N-n)})^2]\,d^{N-n}\tau \in P(\tilde{\Lambda}^N).$$

Then $Z \geqslant \exp(\langle A\rangle)\tilde{Z}$, as a consequence of $\tilde{\Lambda} \subset \Lambda$, $\exp(\cdots) \geqslant 0$, and Jensen's inequality applied w.r.t. $\langle\cdots\rangle$. Hence, in (4.10), $Z^{-1}\tilde{Z}^{1/\gamma} \leqslant e^{-\langle A\rangle}\tilde{Z}^{-1/\alpha-1/\beta}$. By applying now Jensen's inequality w.r.t. $|\tilde{\Lambda}|^{-(N-n)}\,d^{N-n}\tau$ to the integral (4.11), and noticing that $\alpha \sim N$ and $\beta \sim N$, we find that $\tilde{Z}^{-1/\alpha-1/\beta} < C$. Moreover, the fact that the map $\sigma^{-2} \mapsto N^{-1}\ln[|\tilde{\Lambda}|^{-(N-n)}\tilde{Z}(N,\varepsilon,\sigma)]$ is nonpositive, decreasing, convex, and bounded below by C/σ^2 implies that $\langle(\varepsilon - N^{-2}H^{(N-n)})^2\rangle < C$; cf. the strategy in the proof of Lemma 3 in [16]. This estimate and the obvious boundedness of the remaining terms in $\langle A\rangle$ show that $e^{-\langle A\rangle} < C$. Finally, the two terms in (4.10) given by the L^α and the L^β norms, respectively, are bounded above independently of N, which was proven in [16, 15]. This concludes the proof. □

Proof of Prposition 4.2. By Lemma 4.3, each sequence $\{\mu_n^{(N)}\}_{N=1}^\infty$ on Λ^n lives in a weakly $L^p(\Lambda^n)$ compact subset of $P(\Lambda^n)$, any $p \in [1,\infty)$. By Tychonov's theorem, the sequence $\{\mu^{(N)}\}_{N=1}^\infty$, considered on Ω, is restricted to the corresponding weakly $L^p(\Omega)$ compact subset of $P^{ex}(\Omega)$, in the product topology. Existence of limit points now follows by the theorem of Bolzano and Weierstrass.

Let now $\mu^{(N_k)} \xrightarrow{L^p} \mu^{(\varepsilon,\sigma)}$, for a subsequence $\{N_k\}_{k=1}^\infty$, then $\mu_n^{(N_k)} \xrightarrow{L^p} \mu_n^{(\varepsilon,\sigma)}$ for any n. Pick any n and let $N_k > n$. Then $N = [N_k/n]n + n_0$, where $[N_k/n]$ is the integer part of N_k/n, and $n_0 < n$. Since G and F have only logarithmic singularities, Lemma 4.3 and a direct computation give

$$\frac{1}{N_k}\mathbf{P}_{\varepsilon,\sigma}\left(\mu^{(N_k)}\right) = \frac{1}{N_k}\left[\frac{N_k}{n}\right]\mathbf{P}_{\varepsilon,\sigma}\left(\mu_n^{(N_k)}\right) + O\left(\frac{1}{n}\right). \tag{4.12}$$

We next let $N_k \to \infty$ in (4.12). Clearly, $N_k^{-1}[N_k/n] \to n^{-1}$ as $N_k \to \infty$. This and weak L^p convergence give

$$\lim_{N_k\to\infty} \frac{1}{N_k}\mathbf{P}_{\varepsilon,\sigma}\left(\mu^{(N_k)}\right) = \frac{1}{n}\mathbf{P}_{\varepsilon,\sigma}\left(\mu_n^{(\varepsilon,\sigma)}\right) + \mathrm{O}\left(\frac{1}{n}\right) \tag{4.13}$$

for all n. In particular, the subsequent limit $n \to \infty$ in (4.13), recalling (3.11), gives

$$\lim_{N_k\to\infty} \frac{1}{N_k}\mathbf{P}_{\varepsilon,\sigma}\left(\mu^{(N_k)}\right) = \mathbf{p}_{\varepsilon,\sigma}\left(\mu^{(\varepsilon,\sigma)}\right). \tag{4.14}$$

On the other hand, for the entropy functional we have

$$\limsup_{N_k\to\infty} \frac{1}{N_k}\mathbf{S}\left(\mu^{(N_k)}\right) \leqslant \mathbf{s}\left(\mu^{(\varepsilon,\sigma)}\right). \tag{4.15}$$

This follows from sub-additivity (3.14), nonpositivity, and weak upper semi-continuity [30], see [16, 15]. Combining the estimates (4.15) and (4.14), we get

$$\limsup_{N\to\infty} \frac{1}{N}\mathbf{S}_{\varepsilon,\sigma}\left(\mu^{(N)}\right)$$

$$\leqslant \sup_{\mu^{(\varepsilon,\sigma)}} \mathbf{s}_{\varepsilon,\sigma}\left(\mu^{(\varepsilon,\sigma)}\right) \leqslant \sup_{\mu} \mathbf{s}_{\varepsilon,\sigma}(\mu) = \max_{\varrho} \mathbf{R}_{\varepsilon,\sigma}(\varrho), \tag{4.16}$$

where $\sup_{\mu^{(\varepsilon,\sigma)}}$ means supremum w.r.t. all limit points of $\{\mu^{(N)}\}$, $\sup_\mu$ means supremum w.r.t. all elements in $P_p^{\mathrm{ex}}(\Omega)$, and $\max_\varrho$ means maximum w.r.t. all elements in $P(\Lambda)$. The last step is Corollary 3.4. □

Proof of Theorem 2.1. Part 2.7 of Theorem 2.1 follows directly from Propositions 4.1 and 4.2. Corollary 3.4 then implies that any limit point $\mu^{(\varepsilon,\sigma)}$ is a maximizer of $\mathbf{s}_{\varepsilon,\sigma}(\mu)$. Suppose now $\nu(\mu^{(\varepsilon,\sigma)}|\mathrm{d}\varrho)$ is not concentrated on $M_{\varepsilon,\sigma}$. Then, with the help of Proposition 3.3 and extremal decomposition, we arrive at the contradiction

$$\mathbf{s}_{\varepsilon,\sigma}(\mu^{(\varepsilon,\sigma)})$$

$$= \int_{P(\Lambda)} \nu(\mu^{(\varepsilon,\sigma)}|\mathrm{d}\varrho)\mathbf{R}_{\varepsilon,\sigma}(\varrho) < \mathbf{R}_{\varepsilon,\sigma}(\varrho^{(\varepsilon,\sigma)}) = \mathbf{s}_{\varepsilon,\sigma}(\mu^{(\varepsilon,\sigma)}). \tag{4.17}$$

Therefore, $\nu(\mu^{(\varepsilon,\sigma)}|\mathrm{d}\varrho)$ is concentrated on $M_{\varepsilon,\sigma}$. □

5. Vanishing of the Variance σ^2 for $N = \infty$

After having taken $N \to \infty$, we now turn to the subsequent limit $\sigma \to 0$.

Proof of Theorem 2.2. Let first $0 < \varepsilon < \infty$. Using that $L^{1,+}_{1;\varepsilon} \subset L^{1,+}_{1}$, then using the fact that the penalty term vanishes for $\rho \in L^{1,+}_{1;\varepsilon}$, and finally using Proposition 2.1 of [14], which states that $\max\{\mathbf{S}(\varrho) : \rho \in L^{1,+}_{1;\varepsilon}(\Lambda)\}$ is well defined for $0 < \varepsilon < \infty$, we have

$$s_\sigma(\varepsilon) = \max_{\rho \in L^{1,+}_{1}(\Lambda)} \mathbf{R}_{\varepsilon,\sigma}(\varrho) \geqslant \max_{\rho \in L^{1,+}_{1;\varepsilon}(\Lambda)} \mathbf{R}_{\varepsilon,\sigma}(\varrho) = \max_{\rho \in L^{1,+}_{1;\varepsilon}(\Lambda)} \mathbf{S}(\varrho), \tag{5.1}$$

for all $\sigma > 0$. In particular, since $\max\{\mathbf{S}(\varrho) : \rho \in L^{1,+}_{1;\varepsilon}(\Lambda)\} > -\infty$, we thus have $\inf_{\sigma>0} s_\sigma(\varepsilon) > -\infty$. On the other hand, a direct computation shows that the derivative of the map $\sigma^{-2} \mapsto N^{-1} \ln\left[|\Lambda|^{-N} Z(N, \varepsilon, \sigma)\right]$ is nonpositive, hence the map $\sigma^{-2} \mapsto s_\sigma(\varepsilon)$ is monotonic decreasing. We conclude that the limit (2.9), defining $s(\varepsilon)$, exists for $\varepsilon \in \mathbb{R}^+$.

Now let $\mu^{(\varepsilon)}$ be a limit point of the measure $\mu^{(\varepsilon,\sigma)}$ on Ω. Since $\mu^{(\varepsilon,\sigma)} \in P^{\mathrm{ex}}(\Omega)$, also $\mu^{(\varepsilon)} \in P^{\mathrm{ex}}(\Omega)$, whence a de Finetti–Dynkin type decomposition measure $\nu(\mu^{(\varepsilon)}|\mathrm{d}\varrho)$ exists. Clearly, by (5.1), $\nu(\mu^{(\varepsilon)}|\mathrm{d}\varrho)$ is concentrated on the subset of absolutely continuous ϱ. Let ρ_ε denote any density in the support of $\nu(\mu^{(\varepsilon)}|\mathrm{d}\varrho)$, and let $\rho_{\varepsilon,\sigma} \to \rho_\varepsilon$ as $\sigma \to 0$, weakly in $L^{1,+}_{1}(\Lambda)$. By the nonpositivity of the penalty term, weak upper semi-continuity of entropy, and an obvious variational estimate, we have

$$\limsup_{\sigma\to 0} s_\sigma(\varepsilon) \leqslant \limsup_{\sigma\to 0} \mathbf{S}(\varrho_{\varepsilon,\sigma}) \leqslant \mathbf{S}(\varrho_\varepsilon) \leqslant \max_{\rho \in L^{1,+}_{1;\varepsilon}(\Lambda)} \mathbf{S}(\varrho). \tag{5.2}$$

Therefore, (5.2) and (5.1) prove that

$$s(\varepsilon) = \max_{\rho \in L^{1,+}_{1;\varepsilon}(\Lambda)} \mathbf{S}(\varrho). \tag{5.3}$$

Repeating almost verbatim the argument (4.16), with $\mathbf{S}$ replacing $\mathbf{R}_{\varepsilon,\sigma}$, we find that ρ_ε is the density of a measure in M_ε. By Proposition 2.3 of [14], the density of any $\varrho_\varepsilon \in M_\varepsilon$ satisfies the Euler–Lagrange equation (2.11) for (2.10), with β and α real numbers chosen to satisfy the constraints $\rho \in L^{1,+}_{1;\varepsilon}(\Lambda)$. This proves our theorem for $\varepsilon > 0$.

Let now $\varepsilon < 0$ be fixed. Then, since G is the kernel for a positive operator, we have

$$\left(\varepsilon - \frac{1}{2}\int_\Lambda \rho_{\varepsilon,\sigma} G * \rho_{\varepsilon,\sigma}\, \mathrm{d}\tau\right)^2 \geqslant \varepsilon^2. \tag{5.4}$$

With (5.4) and $\mathbf{S}(\varrho) \leqslant 0$, we find $s_\sigma(\varepsilon) \leqslant -\varepsilon^2/2\sigma^2$. Thus,

$$\limsup_{\sigma\to 0} s_\sigma(\varepsilon) = -\infty. \tag{5.5}$$

Similarly, if $\epsilon = 0$, either $\liminf \int_\Lambda \rho_{\varepsilon,\sigma} G * \rho_{\varepsilon,\sigma}\, \mathrm{d}\tau \geqslant 2C > 0$. In that case, repeating the above argument with C^2 replacing ϵ^2, we get (5.5). Or, if $\int_\Lambda \rho_{\varepsilon,\sigma} G * \rho_{\varepsilon,\sigma}\, \mathrm{d}\tau \to 0$, then $\lim_{\sigma\to 0} \mathrm{supp}\,(\rho_{\varepsilon,\sigma}) \cap \Lambda = \emptyset$, whence any limit point of $\rho_{\varepsilon,\sigma}$ concentrates at $\partial\Lambda$. Once again we find (5.5). □

6. Nonequivalence: An Outlook

By standard arguments, the limit of the canonical free energy per vortex pair, established in [15] and [16], coincides with the Legendre–Fenchel transform of the limit of the micro-canonical entropy per vortex, while the reverse transformation gives the convex hull of the entropy per vortex. If the entropy per vortex and its convex hull coincide, we say that (thermodynamic) equivalence of ensembles holds.

Equivalence has been proved for vortices in a disk domain when the Hamiltonian is postulated to be the only conserved quantity, [14]. However, since by rotational symmetry angular momentum is a conserved quantity too, we are entitled to consider a constrained micro-canonical ensemble in which the generalized angular momentum per vortex is also fixed. Nonequivalence of the so-constrained micro-canonical and correspondingly constrained canonical vortex ensembles in a disk is evidenced in the Monte Carlo study of [34]. These authors found an energy region in which the map $\varepsilon \mapsto s(\varepsilon)$ (constrained entropy per vortex) is convex instead of concave. While the entropy maximizing vortex structures in this region have negative vortex-pair specific heat, globally, they do exist stably in the finite volume micro-canonical ensemble with angular momentum constraint. Most remarkably, similar structures are observable in actual experiments, see the discussion in [34]. Interestingly, for the same system in an infinite domain [13, 35] equivalence of the constrained ensembles has been established in the works [15, 36, 37]. A detailed account of more examples of nonequivalence is in preparation and will be published elsewhere.

Finally, it should be recalled that this discussion concerns the high energy tail of the finite N vortex distribution. The low energy tail of neutral vortex systems has been treated rigorously in [38], where the traditional thermodynamic limit for S/N as a function of E/N at fixed density $N/|\Lambda|$ is constructed and the thermodynamic equivalence of ensembles established.

Acknowledgement

This work was supported through NSF Grant # DMS-9623220, and through AFOSR Grant # 92-J-0115.

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