Dissipative stationary plasmas: Kinetic modeling, Bennett's pinch and generalizations

Michael K.-H. Kiessling
Institut für Theoretische Physik IV, Ruhr-Universität Bochum, 44780 Bochum, Germany

Joel L. Lebowitz
Departments of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903

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The structure of the self-consistent electromagnetic fields, **E**, **B**, and one-particle distribution functions, f_s, of a stationary dissipative unbounded S-species plasma, which satisfy a system of Maxwell–Vlasov–Fokker–Planck–Boltzmann equations with velocity-independent (effective) dissipation coefficients and applied constant electric field $\mathscr{E}$, is studied. It is proven that when the plasma is invariant along the direction of $\mathscr{E}$, then (a) the f_s are given by S drifting Maxwell–Boltzmann distributions, with densities satisfying self-consistent Poisson–Boltzmann equations of the type first considered by Bennett for $S=2$; (b) all smooth radial current profiles with normalized particle densities satisfy a generalized Bennett relation; (c) Bennett's current profile is the only fully conformally invariant one; (d) there exist other radial solutions with profiles different from Bennett's.

I. INTRODUCTION

Starting with Bennett's seminal work[1] the one-particle distribution functions of stationary current-carrying multicomponent plasmas are frequently postulated to be of drifting Maxwell–Boltzmann type, e.g., Refs. 2–5. While such distribution functions satisfy the time-independent Vlasov equations, see, e.g., Ref. 2, they are not the only ones to do so and Vlasov theory, being nondissipative, does not discriminate between this and infinitely many other choices. Heuristic equilibrium statistical mechanical arguments[6,7] do select the drifting Maxwell–Boltzmann distribution from other solutions, but the significance of this is unclear since the real plasma is not in thermal equilibrium, its current being a dissipative process in response to an external electric field.

In this paper we consider a system of S stationary dissipative Maxwell–Vlasov–Fokker–Planck–Boltzmann equations with very simple dissipation terms in an externally imposed constant electric field $\mathscr{E}\|\mathbf{e}_z$. Similar dissipation terms are found in Refs. 8 and 9. Although somewhat simplistic, these equations are a convenient starting point for the mesoscopic modeling of spatially inhomogeneous current-carrying dissipative plasmas. We present the model equations in Sec. II. In Sec. III we give the proof that their z-invariant solutions are of the drifting Maxwell–Boltzmann form, with species-dependent mean velocity $\mathbf{c}_s$ proportional to $\mathscr{E}$, and with Boltzmann factors given self-consistently by the solutions of a coupled pair of nonlinear Poisson–Boltzmann equations. At this point the problem is equivalent to the S-isothermal Bennett problem with its postulated mean drift speeds. The rest of the paper is devoted to the study of pinched normalizable solutions of the Poisson–Boltzmann equations in all space.

We derive in Sec. IV *a priori* conditions on the parameters for the existence of regular radial solutions with finite numbers of particles of species s per unit length, N_s. In particular, we show that all such solutions have to satisfy certain sum rules, i.e., (36), from which we extract the "Bennett relation" (generalized to S species),

$$\frac{c^{-2}I^2-Q^2}{2}=\sum_{s=1}^{S} N_s k_{\mathrm{B}} T_s, \tag{1}$$

where Q is the total charge per unit length along the invariant direction and I the total current; T_s is the temperature for species s: c is the vacuum speed of light, and k_{B} is Boltzmann's constant. As pointed out by the referee, (1) corresponds to a radial virial identity; see Appendix A. Our derivation is based directly on the Poisson–Boltzmann equations. We then turn, in Sec. V, to Bennett's pinch solution which was derived by him under the hypothesis that the ratio of the positive and negative species densities is constant in space. This required extra conditions on the parameters. These conditions are here shown to be precisely those under which the Poisson–Boltzmann equations are invariant under the full Euclidean conformal group. We further show that when such invariance holds, then for any $S>1$ all normalizable radial profiles are given by Bennett's pinch. Bennett's pinch thus follows from considerably weaker requirements than Bennett's pointwise condition; cf. Refs. 1 and 2. In Sec. VI we address the general radial symmetric case. After mapping the problem to an equivalent Hamiltonian particle problem (see also Appendix B) in two dimensions, for $S=2$ solutions are shown to exist. Their general structure is different from Bennett's solution, with positive and negative species having different asymptotic rate of decay. Concluding remarks are found in Sec. VII.

II. THE MODEL EQUATIONS

We consider a classical, nonrelativistic S-component plasma having charges q_s and masses m_s. The time evolution of the one-particle distribution functions $f_s(\mathbf{x},\mathbf{v},t)$ are given by

$$\partial_t f_s + \mathbf{v}\cdot\partial_{\mathbf{x}} f_s + \mathbf{a}_s\cdot\partial_{\mathbf{v}} f_s = \partial_t f_s|_{\text{diss}} \tag{2}$$

with $\mathbf{v}\in\mathrm{R}^3$, $\mathbf{x}\in\mathrm{R}^3$, $t\in\mathrm{R}^+$; $s=1,\ldots,S$. The acceleration $\mathbf{a}_s$ is given by

$$m_s\mathbf{a}_s(\mathbf{x},t) = q_s[\mathbf{E}(\mathbf{x},t) + c^{-1}\mathbf{v}\times\mathbf{B}(\mathbf{x},t)]. \tag{3}$$

The electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$ satisfy the Maxwell equations with charge and current densities, ρ and $\mathbf{j}$, given self-consistently by velocity moments of the f_s,

$$\nabla\cdot\mathbf{E} = 4\pi\rho(\mathbf{x},t) = 4\pi\sum_s q_s\int_{\mathrm{R}^3} f_s(\mathbf{v},\mathbf{x},t)d^3v, \tag{4}$$

$$\begin{aligned}\nabla\times\mathbf{B} - c^{-1}\partial_t\mathbf{E} &= 4\pi c^{-1}\mathbf{j}(\mathbf{x},t)\\ &= 4\pi c^{-1}\sum_s q_s\int_{\mathrm{R}^3}\mathbf{v}f_s(\mathbf{v},\mathbf{x},t)d^3v,\end{aligned} \tag{5}$$

together with the conditions $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E}+c^{-1}\partial_t\mathbf{B}=0$. An externally produced spatially homogeneous time-independent magnetic induction $\mathscr{B}=(\mathscr{B}\cdot\mathbf{e}_z)\mathbf{e}_z$ and electric field $\mathscr{E}=|\mathscr{E}|\mathbf{e}_z$ are imposed.

The dissipation term will be chosen as

$$\partial_t f_s|_{\text{diss}} = \partial_{\mathbf{v}}\cdot(\lambda_s m_s^{-1}\partial_{\mathbf{v}} f_s + \gamma_s\mathbf{v}f_s) + C_s(f_s,f_s). \tag{6}$$

Here, the λ_s and γ_s are velocity-independent, spatially constant coefficients and $C_s(f_s,f_s)$ represents a Boltzmann-type collision term, for instance a Lenard–Balescu–Guernsey operator[10] which conserves particle, energy, and momentum density, gives rise to a local H theorem,[10] and vanishes only if f_s is a local thermal equilibrium distribution.

Clearly, (6), which couples the different f_s only implicitly through the λ_s and γ_s, is an oversimplification: it can be thought of as modeling in some average way deviations from Vlasov behavior. The parameters λ_s and γ_s can (in some situations) be determined *a posteriori* by measuring a few global characteristics of a stationary plasma in a driving electric field (see below). Other transport phenomena may then be inferred.

The right-hand side of (6) vanishes if $f_s(\mathbf{x},\mathbf{v})=n_s(\mathbf{x})\hat{F}_s(\mathbf{v})$, with

$$\hat{F}_s(\mathbf{v}) = (\beta_s m_s/2\pi)^{3/2}\exp(-\beta_s\tfrac{1}{2}m_s v^2) \tag{7}$$

a Maxwellian velocity distribution function with $\beta_s^{-1}=k_{\mathrm{B}}T_s\equiv\lambda_s/\gamma_s$, $v=|\mathbf{v}|$, and $n_s(\mathbf{x})$ a local s-species density. If $n_s(\mathbf{x})=\bar{n}_s$ constants, then (7) describes a homogeneous (i.e., non-normalizable) stationary state which will be approached in the absence of an external electric field from any initial state as $t\to\infty$; see Ref. 11. We shall not consider such non-normalizable cases in the present work.

III. TRANSLATION INVARIANT STATIONARY DISTRIBUTIONS

Setting

$$\mathbf{E} = -\nabla\phi + E_z\mathbf{e}_z, \tag{8}$$

$$\mathbf{B} = \nabla A\times\mathbf{e}_z + B_z\mathbf{e}_z, \tag{9}$$

we look for stationary normalizable solutions of (2) with ϕ, E_z, and A, B_z independent of z, whence $\nabla\cdot\mathbf{B}=0$. Maxwell's equations, stationarity, and our sign convention for $\mathscr{E}$ then imply

$$E_z = |\mathscr{E}|, \tag{10}$$

$$-\Delta\phi = 4\pi\rho, \tag{11}$$

$$-\Delta A = 4\pi c^{-1}\mathbf{e}_z\cdot\mathbf{j}, \tag{12}$$

$$\nabla B_z = 4\pi c^{-1}\mathbf{e}_z\times\mathbf{j}. \tag{13}$$

For simplicity we continue to use $\mathbf{x}$ now for the spatial variables in $z^\perp$ direction. At $|\mathbf{x}|$ infinity, the particle densities are required to decay to zero so that they are R^2 normalizable, and the magnetic and electric fields approach their leading multipole values with no sources at infinity. We can thus solve (11) and (12) formally to obtain

$$\phi-\phi_0 = -2\ln r*\rho;\quad c(A-A_0) = -2\ln r*j, \tag{14}$$

where $*$ denotes the standard convolution, $r=|\mathbf{x}|$, $j=\mathbf{j}\cdot\mathbf{e}_z$, and ϕ_0, A_0 are arbitrary gauge constants which are not determined by (11) and (12).

The z-independent stationary form of (2) and (6) can now be cast into the form

$$\begin{aligned}m_s\partial_t f_s = 0 = &-m_s\partial_{\mathbf{x}}\cdot(\mathbf{v}f_s) - q_s\partial_{\mathbf{v}}\cdot\{[-\nabla(\phi-u_sA)\\ &+c^{-1}(\mathbf{v}-\mathbf{c}_s)\times\mathbf{B}]f_s\} + \lambda_s\partial_{\mathbf{v}}\cdot\{\hat{F}_s(\mathbf{v}-\mathbf{c}_s)\\ &\times\partial_{\mathbf{v}}[f_s/\hat{F}_s(\mathbf{v}-\mathbf{c}_s)]\} + m_sC_s(f_s,f_s),\end{aligned} \tag{15}$$

where

$$\mathbf{c}_s = (m_s\gamma_s)^{-1}q_s\mathscr{E}\equiv cu_s\mathbf{e}_z. \tag{16}$$

A straightforward calculation verifies that the drifting Maxwell–Boltzmann distribution

$$\begin{aligned}F_s(\mathbf{x},\mathbf{v}) = N_s(\beta_s m_s/2\pi)^{3/2}Z_s^{-1}\exp(-\beta_s\{\tfrac{1}{2}m_s(\mathbf{v}-\mathbf{c}_s)^2\\ +q_s[\phi(\mathbf{x})-u_sA(\mathbf{x})]\}),\end{aligned} \tag{17}$$

where

$$Z_s[\phi,A] = \int_{\mathrm{R}^2}\exp\{-\beta_s q_s[\phi(\mathbf{x})-u_sA(\mathbf{x})]\}d^2x \tag{18}$$

is a normalized solution of (15) in which there are N_s particles of species s per unit length.

To show that (17) is the unique solution to (15), assume there exists another normalizable solution, f_s, of (15). The vanishing of the time derivative of its relative entropy (w.r.t. F_s) gives after some partial integrations (with vanishing boundary integrals)

$$\begin{aligned}0 &= -\partial_t\int_{\mathrm{R}^2}\int_{\mathrm{R}^3} f_s\ln\left(\frac{f_s}{F_s}\right)d^2x\,d^3v\\ &= k_{\mathrm{B}}T_s\gamma_s\int_{\mathrm{R}^2}\int_{\mathrm{R}^3} f_s^{-1}F_s^2\left|\partial_{\mathbf{v}}\left(\frac{f_s}{F_s}\right)\right|^2 d^2x\,d^3v\\ &\quad-\int_{\mathrm{R}^2}\int_{\mathrm{R}^3} C_s(f_s,f_s)[\ln(F_s^{-1}f_s)+1]d^2x\,d^3v,\end{aligned} \tag{19}$$

where we used the fact that F_s annihilates the following three operators in (15) separately; the convective (Vlasov) derivative; the Fokker–Planck operator; the Boltzmann-type operator C_s. Since C_s gives rise to a local H theorem, the right-hand side of (19) is manifestly non-negative. It vanishes only if $f_s = h_s(\mathbf{x}) F_s$. But f_s solves (15), whence $\mathbf{v} \cdot \partial_{\mathbf{x}} h_s = 0$, implying $h_s = \text{const}$. Therefore (17) is the only solution of (15) with the given normalization.

The problem of determining stationary, z-invariant solutions of (2)–(6) thus reduces to solving (11)–(13) with their right-hand side given by the zeroth and first velocity moments of (17). Then (13) becomes $\nabla B_z = 0$, whence

$$B_z = \mathscr{B} \cdot \mathbf{e}_z \tag{20}$$

and (11) and (12) turn into the coupled pair of nonlinear integrodifferential equations

$$-\Delta \phi = 4\pi \sum_s N_s q_s Z_s^{-1} \exp[-\beta_s q_s(\phi - u_s A)], \tag{21}$$

$$-\Delta A = 4\pi \sum_s N_s q_s u_s Z_s^{-1} \exp[-\beta_s q_s(\phi - u_s A)] \tag{22}$$

with asymptotic source free conditions (14). Any solution pair (A, ϕ) of (21) and (22) for parameters N_s, β_s, and u_s given by (16), generates a stationary, translation invariant solution of Eqs. (2)–(6), and *vice versa*.

We will call (21) and (22) Bennett's equations. Bennett[1] originally derived a "semirelativistic" analog of (21) and (22), restricted to two species. He considered collisionless streams of charged particles with specified relativistic drift speeds, without thermal dispersion of the velocity component along the stream but with nonrelativistic species-dependent typical thermal motions across the stream. Expanding a normalized form of his formulas [Ref. 1, Eqs. (8) and (9)] to lowest significant order in $(v/c)^2$ gives our (21) and (22), except for the difference in the meaning of the temperatures [i.e., two-dimensional (2-D) versus three-dimensional (3-D) thermal motions]. In fact, relativistic modifications to (21) and (22) show up only in the parameters, whereas the mathematical structure of (21) and (22) obtains in Bennett's semirelativistic treatment and also in a fully relativistic treatment, which couples the isothermal distributions of Jüttner [see Ref. 12, p. 46, Eq. (24)] self-consistently to Maxwell's equations.

IV. SOLVABILITY CONDITIONS

Bennett's equations (21) and (22) are highly nonlinear and not readily susceptible to a general solution. Here we derive some *a priori* conditions on the parameters which have to be satisfied for a classical radial solution with normalized particle densities to exist. In particular, we prove that such solutions exist only if the generalized Bennett relation (1) holds. We mention without proof that it can be shown[13] that (1) must be satisfied under mild integrability conditions for *all* classical solutions, without demanding radial symmetry.

We center a ball B_r of radius r at the origin and define

$$M_s(r) = Z_s^{-1} \int_{B_r} \exp[-\beta_s q_s(\phi - u_s A)] d^2x \tag{23}$$

so that

$$\lim_{r\to\infty} M_s(r) = 1. \tag{24}$$

Let a prime denote derivative w.r.t. r. By Newton's theorem,

$$\phi' = -2 \sum_s N_s q_s r^{-1} M_s(r), \tag{25}$$

$$A' = -2 \sum_s N_s q_s u_s r^{-1} M_s(r). \tag{26}$$

By (24).

$$Q^{-1} \lim_{r\to\infty} r\phi'(r) = -2 = cI^{-1} \lim_{r\to\infty} rA'(r), \tag{27}$$

where

$$Q = \sum_s N_s q_s; \quad I = c \sum_s N_s q_s u_s. \tag{28}$$

By integration,

$$Q^{-1} \lim_{r\to\infty} \frac{\phi(r)}{\ln r} = -2 = cI^{-1} \lim_{r\to\infty} \frac{A(r)}{\ln r} \tag{29}$$

as leading-order behavior of normalizable solutions at infinity. Using (29) in the right-hand side of (21) and (22), we infer the asymptotic decrease of the particle density n_s of species s,

$$n_s(r) \sim r^{-2\beta_s q_s(u_s c^{-1} I - Q)}. \tag{30}$$

Integrability now requires that

$$\beta_s q_s(u_s c^{-1} I - Q) > 1 \tag{31}$$

holds for each s.

To derive the generalized Bennett identity for radial solutions, we compute

$$M_s'(r) = 2\pi r \exp\{-\beta_s q_s[\phi(r) - u_s A(r)]\}/Z_s \tag{32}$$

and

$$M_s''(r) = -\beta_s q_s[\phi'(r) - u_s A'(r)] M_s'(r) + M_s'(r)/r; \tag{33}$$

then eliminate ϕ' and A' in (33) with the aid of (25) and (26). Partial integration of $rM_s''(r)$ w.r.t. r now gives

$$rM_s'(r) = 2M_s(r) + 2 \sum_t N_t q_t q_s \beta_s (1 - u_s u_t) \times \int_0^r M_t(\xi) M_s'(\xi) d\xi. \tag{34}$$

By (30)

$$\lim_{r\to\infty} rM_s'(r) = 0. \tag{35}$$

Taking $r \to \infty$ in (34) therefore yields, for each s

$$1=\sum_t N_t q_t q_s \beta_s (u_s u_t - 1)\langle M_t(r)\rangle_s \tag{36}$$

with the definition

$$\langle F\rangle_s \equiv \lim_{r\to\infty}\int_0^r F(\xi) M_s'(\xi)\,d\xi. \tag{37}$$

We multiply (36) by $N_s k_B T_s$ and sum over s, then add the same formula with indices s,t exchanged, yielding

$$2\sum_s N_s k_B T_s = \sum_{s,t} N_s N_t q_s q_t (u_s u_t - 1) \times[\langle M_t(r)\rangle_s - \langle M_s(r)\rangle_t]. \tag{38}$$

By a partial integration and using (24)

$$\langle M_t(r)\rangle_s + \langle M_s(r)\rangle_t = 1. \tag{39}$$

By (39) and (28) we see that (38) is identical to the generalized Bennett relation (1), which therefore must hold for all radial solutions of (21) and (22).

Physically acceptable $|u_s|<1$ and $T_s>0$ require a combination of positive and negative species to make the right-hand side of (38) positive. We thus recover the well-known fact that the attractive magnetic interactions cannot compensate the repulsive Coulomb interactions in a plasma with only positive (negative) species. For $q_s q_{s'}<0$ for at least one pair (s,s') there is a solution manifold of (38).

V. BENNETT'S SOLUTION

A solution with normalized densities in all space was constructed by Bennett[1] for $S=2$. Bennett looked for radial solutions with a constant ratio $n_1(r)/n_2(r)$ of particle densities throughout space. This is readily generalized to S species. It implies a constant ratio of current and charge densities, and of magnetic and electric potentials (after an appropriate choice of the irrelevant constants ϕ_0 and A_0). In our nonrelativistic setup, his ansatz reads

$$cI^{-1}A(\mathbf{x}) = \chi(\mathbf{x}) = Q^{-1}\phi(\mathbf{x}) \tag{40}$$

which with (21) and (22) gives the conditions

$$\beta_s q_s (u_s c^{-1} I - Q) = 2, \tag{41}$$

$s=1,\ldots S$, on the parameters. If they are met, (40) maps both (21) and (22) into

$$-\Delta\chi = 4\pi\frac{\exp(2\chi)}{\int_{\mathbb{R}^2}\exp(2\chi)d^2x} \tag{42}$$

which is known as Liouville's[14] equation (see Bandle's book[15]). It is structurally invariant under the Euclidean conformal group of translations, rotations, dilations, and inversions at the unit circle: it is also gauge-invariant under $\chi\mapsto\chi+\text{const}$. The one-parameter family of radial solutions,

$$\chi_k(r) = -\log(1+k^2r^2) \tag{43}$$

with k^{-1} an arbitrary scale length, is known as Bennett's pinch. The corresponding current density j and charge density ρ are given by

$$I^{-1}j(r) = \pi^{-1}k^2(1+k^2r^2)^{-2} = Q^{-1}\rho(r). \tag{44}$$

Like Eq. (42), its solution (44), written as a probability measure $Q^{-1}\rho(r)d^2x$, is form invariant under rotations, dilations $r\to\lambda r$; $k\to\lambda^{-1}k$, and inversions $r\to 1/r$; $k\to 1/k$. By translating the origin we see that (44) is in fact invariant under the full Euclidean conformal group.

Bennett (Ref. 1, p. 893) left open whether other solutions of (21) and (22) exist. For the Liouville equation (42), Chen and Li recently showed[16] that the spatial translations of Bennett's pinch (43) yield all normalizable smooth solutions; see also Ref. 13.

We note, however, that (41) and (42) were obtained by Bennett as a consequence of the special ansatz (40). We now prove that for arbitrary $S>1$, (41) are precisely the conditions under which the system (21) and (22) is fully conformally invariant and further that this invariance and $|u_s|<1$ implies (40). That is, for $|u_s|<1$, our partial differential equation (PDE) system (21) and (22) is equivalent to (42) in the conformal invariant case. By Chen and Li's result this implies that Bennett's pinch yields the general physical, normalizable conformal solution to (21) and (22).

We notice that the system (21) and (22) is already invariant under rotations, dilations, and translations. Invariance under the full Euclidean conformal group requires that (21) and (22) is also form invariant under a Kelvin transformation $\mathbf{x}\mapsto\mathbf{x}/|\mathbf{x}|^2$ and

$$\phi(\mathbf{x})\mapsto\phi(\mathbf{x}/|\mathbf{x}|^2) - Q\ln|\mathbf{x}|^2, \tag{45}$$

$$A(\mathbf{x})\mapsto A(\mathbf{x}/|\mathbf{x}|^2) - c^{-1}I\ln|\mathbf{x}|^2. \tag{46}$$

This can only hold if (41) is satisfied, whence (41) is a set of necessary, and also sufficient conditions that (21) and (22) be invariant under the Euclidean conformal group. Because of (38) only $S-1$ conditions of the type (41) have to be postulated for the parameters to obtain full conformal invariance.

Next, consider $\Psi = I\phi - cQA$, which satisfies

$$-\Delta\Psi = 4\pi\sum_s N_s q_s (I - u_s cQ) \times\frac{\exp[-(\beta_s q_s\Psi + 2cA)/I]}{\int_{\mathbb{R}^2}\exp[-(\beta_s q_s\Psi + 2cA)/I]d^2x} \tag{47}$$

with

$$\Psi(\mathbf{x})\to 0;\quad \text{as } |\mathbf{x}|\to\infty, \tag{48}$$

which obtains from (21) and (22) after eliminating the products $\beta_s q_s u_s$ using (41). By (27) and (48), $\nabla\Psi$ is square integrable, whence given any smooth A for which $\int_{\mathbb{R}^2}\exp(-2cA/I)d^2x<\infty$, a solution Ψ of (47) and (48) is a stationary point of the functional

$$\int_{\mathbb{R}^2}|\nabla\Psi|^2\,d^2x + 8\pi\sum_s N_s k_B T_s I(I - u_s cQ) \times\ln\int_{\mathbb{R}^2}\exp\left(-\frac{(\beta_s q_s\Psi + 2cA)}{I}\right)d^2x. \tag{49}$$

Because of (1) and $|u_s|<1$ we have $|I|>|u_s cQ|$, thus

$$I(I - u_s cQ) > 0 \tag{50}$$

which implies that (49) is *convex*. Therefore, given such A, a solution Ψ of (47) and (48) is unique. Using this and (41) we now verify that $\Psi\equiv 0$ is the only solution of (47) and (48) for all such A. Since $\Psi\equiv 0$ is equivalent to (40), up to a gauge constant, we are back to Liouville's equation. Notice that in the above we did not require radial symmetry.

VI. OTHER SOLUTIONS

We consider now two species ($s=\pm$) in the general case when (41) is not satisfied. This implies lack of inversion symmetry of (21) and (22), but invariance under the subgroups of dilations, translations, and rotations, as well as gauges still prevails. Looking for solutions which are radially symmetric around the origin leaves the dilation and gauge groups. We set $Q=0$. The case $Q\neq 0$ can be obtained from it by a boost[17] along $\mathbf{e}_z$, which leaves perpendicular scales and the mathematical structure of (21) and (22) unaffected and changes the particle densities only in their overall amplitude, preserving $c^2\rho^2-j^2$. We show the equivalence of (21) and (22) to a two-degrees-of-freedom Hamiltonian model and, using this, prove existence and find the structure of solutions in this case.

We transform to new dependent and independent variables, setting $\tau=\ln r$, and

$$x(\tau)=\phi(e^\tau)+\bar{u}\tau I/c, \tag{51}$$

$$y(\tau)=A(e^\tau)+\tau I/c, \tag{52}$$

where

$$\bar{u}=\frac{\Sigma_s(N_sT_s)^{-1}u_s}{\Sigma_s(N_sT_s)^{-1}}. \tag{53}$$

It suffices to consider only the case $\bar{u}<0$, the other being analogous. The full conformal case (41) obtains for $\bar{u}=0$.

We now choose the gauge such that

$$\int_{-\infty}^{\infty}\exp[-\beta_sq_s(x-u_sy)]d\tau\equiv 2 \tag{54}$$

for $s=\pm$. Using this Eqs. (21) and (22) assume the form

$$\ddot{x}(\tau)=-\sum_s N_sq_s\exp[-\beta_sq_s(x-u_sy)], \tag{55}$$

$$\ddot{y}(\tau)=-\sum_s N_sq_su_s\exp[-\beta_sq_s(x-u_sy)], \tag{56}$$

and dilation symmetry of (21) and (22) now means invariance of (55) and (56) under translations $\tau\rightarrow\tau+\text{const}$. The finiteness of $\phi(0)$ and $A(0)$ and the asymptotic behavior (29) yield the asymptotic conditions

$$\lim_{|\tau|\rightarrow\infty}\tau^{-1}x(\tau)/\bar{u}=I/c=-\lim_{|\tau|\rightarrow\infty}|\tau|^{-1}y(\tau). \tag{57}$$

Upon reinterpreting (x,y) as a position vector and τ as a time variable, our problem becomes identical to that of a classical point particle which moves in a time-independent nonconservative force field. Despite the nonconservative character of the force field, we can construct an autonomous two-degrees-of-freedom Hamiltonian,

$$H=\tfrac{1}{2}(p_x^2-p_y^2)-U(x,y), \tag{58}$$

where

$$U(x,y)=\sum_s N_sk_BT_s\exp[-\beta_sq_s(x-u_sy)]. \tag{59}$$

Equations (55) and (56) are equivalent to Hamilton's equations of motion for this H. Hamiltonians of this type with nonpositive definite kinetic energy term have a number of curious properties and have been considered recently by Pfirsch[18] in a very different plasma-physical context.

A. Existence of solutions

We will establish existence of solutions by proving that among those trajectories in (x,y) space which have the correct asymptotic incoming slope $w_{\text{in}}=1/\bar{u}$ in the quadrant $\{(x,y):x>0,y<0\}$, one can find one with the correct asymptotic outgoing slope $w_{\text{out}}=-1/\bar{u}$ in the quadrant $\{(x,y):x<0,y<0\}$ if the incoming data for $\tau\rightarrow-\infty$ fulfill (57). This suffices because H is a conserved integral of motion and $U(x,y)$ decays asymptotically to zero along those trajectories, whence

$$H=\frac{I^2}{2c^2}(\bar{u}^2-1)=\frac{1}{2}(p_x^2-p_y^2)|_{\text{in}}=\frac{1}{2}(p_x^2-p_y^2)|_{\text{out}} \tag{60}$$

along such a trajectory. Together with the correct asymptotic slopes this implies the correct outgoing momenta when the correct incoming ones are given. Our procedure consists of first locating a corridor for the $\tau\rightarrow-\infty$ asymptotic trajectory which has the following property: the $\tau\rightarrow+\infty$ slopes of the two extremal trajectories which have their $\tau\rightarrow-\infty$ asymptotes situated at one of the boundaries of the corridor sandwich the required future slope. The second step is a continuity argument that establishes the existence of the right outgoing slope for some data inside the corridor.

We compute the lines of force $\{(x^F,y^F)\}$ in parameter representation

$$\begin{aligned}x^F(\zeta;\zeta_0|x_0)=x_0&-\frac{k_BT_+}{q_+}\left(1+\frac{N_+T_+}{N_-T_-}\right)^{-1}\frac{\zeta-\zeta_0}{1+u_-\bar{u}}\\&-\left[1+\frac{k_BT_+}{q_+}\left(1+\frac{N_+T_+}{N_-T_-}\right)^{-1}\frac{1-u_+\bar{u}}{(1+u_-\bar{u})^2}\right]\\&\times\ln\frac{1-u_+\bar{u}+(1+u_-\bar{u})e^{\zeta}}{1-u_+\bar{u}+(1+u_-\bar{u})e^{\zeta_0}},\end{aligned} \tag{61}$$

$$y^F(\zeta;\zeta_0|x_0)=(\beta_+q_+u_+-\beta_-q_-u_-)^{-1}\zeta+x^F(\zeta)/\bar{u}. \tag{62}$$

For fixed ζ_0, the lines of force are labeled by $x_0=x^F(\zeta=\zeta_0)$. Some are shown in Fig. 1. All lines are asymptotically straight for $\zeta\rightarrow\pm\infty$. There is exactly one entirely straight line of force. We call it the critical line, C. It is parallel to a family of straight lines, $\{(x_v,y_v)\}$, on each of which

$$\frac{\ddot{y}}{\ddot{x}}=\frac{dy^F}{dx^F}=\frac{1}{v}=\text{const}. \tag{63}$$

They are given by

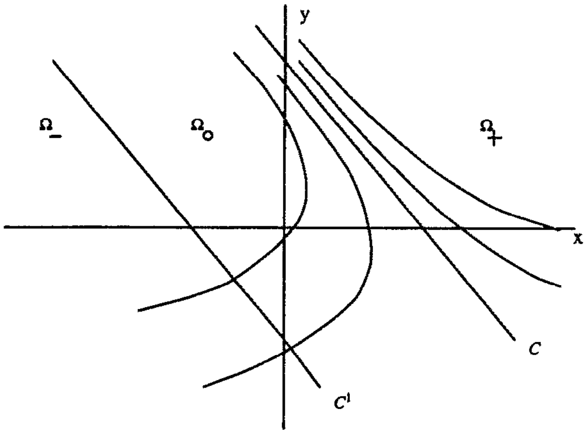

FIG. 1. Lines of force, critical (C) and cocritical (C') lines, and Ω regions for the equivalent particle model in (x,y) space. The flow of the force field is "downward" in Ω_+ and from Ω_0 into Ω_-.

$$(\beta_+q_+-\beta_-q_-)(x_v-\bar{u}y_v)=\ln\frac{vu_+-1}{vu_--1} \tag{64}$$

so that their slope is $1/\bar{u}$. In particular, the incoming data (57) thus point antiparallel to $(\ddot{x},\ddot{y})$ along C, which is obtained for $v=\bar{u}$. Similarly, at $v=-\bar{u}$ there exists a second, cocritical line C', on which $(\ddot{x},\ddot{y})$ points parallel to the outgoing data (57). The lines C and C' divide (x,y) space into the three disjoint subspaces, Ω_-, Ω_0, Ω_+, shown in Fig. 1.

We infer from Fig. 1 that a particle launched on C stays on C and gets reflected downward. If a particle is launched on C' it gets deflected deeper into Ω_-, but clearly will aim too high to the left in the asymptotic future. These outgoing scenarios sandwich the required data. Clearly, if the launch is made in Ω_+ or Ω_- close to C or C', respectively, the asymptotic future behavior will be farther away from the required (57). The launch has thus to take place in the corridor Ω_0, a transition region with a more complicated structure.

We now define the "launching parameter" b as the distance between C and the asymptote to the incoming trajectory (see Fig. 2), and $\hat{b}$ as the distance between C and C'. To prove existence of trajectories with the correct outgoing slopes, we have to show that the outgoing inverse slope $1/w(b)$ of a trajectory varies continuously from $1/w>-\bar{u}$ to $1/w<-\bar{u}$ when b is decreased from $b>\hat{b}$ to some value $b<\hat{b}$. Since clearly $1/w(\hat{b})>-\bar{u}$, and $1/w(b=0)=\bar{u}<0<-\bar{u}$, one may be tempted to conclude the proof with this observation. However, the launching condition on the critical line itself is singular and we cannot yet conclude that $\lim_{b\searrow 0}1/w(b)=1/w(0)$, which is precisely what we have to show.

To make the following discussion more transparent, we rotate and then translate the coordinates (x,y) to a new coordinate system (X,Y), such that the Y direction is parallel to the critical line of force, $b=0$ defines $X=0$, and $b>0$ means $X<0$. The transformation is

$$\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}-\begin{pmatrix} K \\ 0 \end{pmatrix} \tag{65}$$

with

$$\cos\theta=\frac{1}{\sqrt{1+\bar{u}^2}};\quad \sin\theta=\frac{-\bar{u}}{\sqrt{1+\bar{u}^2}} \tag{66}$$

and

$$K=\frac{\cos\theta}{\beta_+q_+-\beta_-q_-}\ln\frac{1-\bar{u}u_+}{1-\bar{u}u_-}. \tag{67}$$

The equations of motion become

$$\ddot{X}=-F(Y)[(1-\bar{u}u_+)G_+(X)-(1-\bar{u}u_-)G_-(X)], \tag{68}$$

$$\ddot{Y}=-F(Y)[(\bar{u}+u_+)G_+(X)-(\bar{u}+u_-)G_-(X)], \tag{69}$$

with

$$F(Y)=-N_+q_+\cos\theta\exp[-\beta_+q_+\cos\theta(\bar{u}-u_+)Y], \tag{70}$$

$$G_\pm(X)=\exp[-\beta_\pm q_\pm\cos\theta(1+\bar{u}u_\pm)(X+K)]. \tag{71}$$

The asymptotic data imply now vertical upward motion for $\tau\to-\infty$, with $X(-\infty)\leq 0$, and positive slope $\tan(2\theta)$ for $\tau\to\infty$, situated in the lower left quadrant.

We have $(\ddot{X})(X,Y)<0$ for $X<0$, and $\ddot{X}\sim X$ for $X\nearrow 0$, for any given Y. We also have $(\ddot{Y})(X,Y)<0$ in a left neighborhood of $X=0$, and $(\ddot{Y})(X,Y)$ may or may not become positive for sufficiently negative X. For given $X<0$, both accelerations increase exponentially in strength as $Y\nearrow+\infty$, and decrease exponentially fast to zero as $Y\searrow-\infty$.

We now show that for any given $Y_0<0$ and small $X_0<0$ we can find an $\epsilon>0$ so that there is a τ_0 such that $Y(\tau_0)<Y_0$ and $X(\tau_0)>X_0$ for the reflected trajectory, by choosing $X(-\infty)>-\epsilon$. This implies that we can make the outgoing trajectory arbitrarily steep with arbitrarily negative $Y\leq -1$ arbitrarily close to the line $X=0$ at some finite time τ_0. Since the acceleration terms decrease exponentially with Y for $Y\to-\infty$ with the X motion giving only a small correction, the motion after time τ_0 can only contribute an exponentially small correction to the outgoing slope. This then proves the uniform approach of the outgoing slope to that of the "critical trajectory $X\equiv 0$" when we let $X(-\infty)\nearrow 0$.

We notice that we can find a small positive $\delta>-X_0$ such that for all $X\in(X_0,0)$ there exist some positive constants $C_1(\delta)$ and $C_2(\delta)$ such that

$$\ddot{X}\geq C_1F(Y)X;\quad \ddot{Y}\leq C_2F(Y). \tag{72}$$

We define $X_*(\tau)$ and $Y_*(\tau)$ by

$$\ddot{X}_*=C_1F(Y_*)X_*;\quad \ddot{Y}_*=C_2F(Y_*) \tag{73}$$

so that for $\tau\leq\tau_*$, we have $0>X(\tau)\geq X_*(\tau)$ and $Y(\tau)\leq Y_*(\tau)$ once we make sure this holds asymptotically for $\tau\to-\infty$, and τ_* is the largest τ for which $X_*(\tau)>X_0$.

Both differential equations can be solved explicitly. Let

$$\omega=q_+\cos\theta\sqrt{N_+\beta_+(u_+-\bar{u})C_2/2} \tag{74}$$

and $z(\tau)=\tanh[\omega\tau]$, then

$$Y_*(\tau)=[\beta_+q_+(u_+-\bar{u})\cos\theta]^{-1}\ln(1-z^2[\tau]), \tag{75}$$

$$X_*(\tau)=-\epsilon P_\nu(-z|\tau|), \tag{76}$$

where

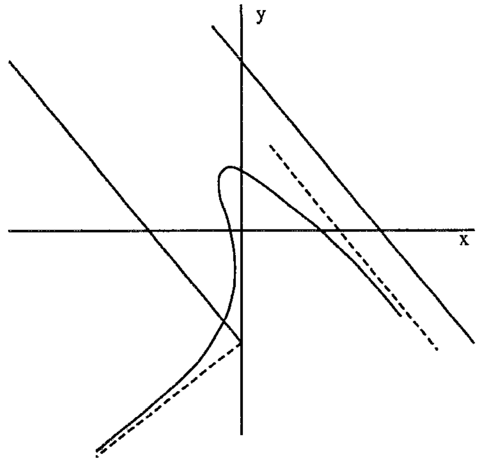

FIG. 2. Structure of the general trajectory of a solution in (x,y) space, together with its asymptotes (dashed lines) and the critical and cocritical lines. The "particle" comes in from lower right.

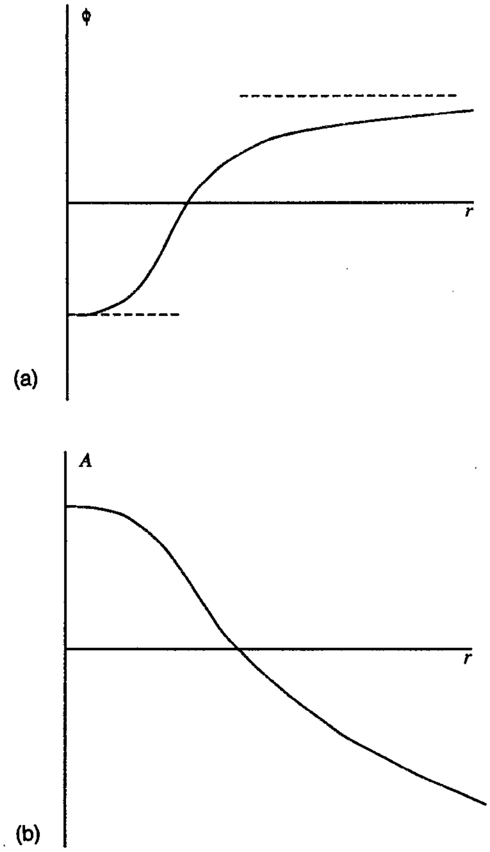

FIG. 3. (a) Electric potential $\phi(r)$ constructed from the trajectory of Fig. 2. (b) Magnetic potential $A(r)$ constructed from the trajectory of Fig. 2.

$$P_\nu(-z)=K_\nu\int_0^\infty \frac{(\sinh \xi)^{2\nu+1}}{(\cosh \xi-z)^{\nu+1}}\,d\xi \tag{77}$$

is a Legendre function[19] of the first kind of order ν, with

$$\nu=-\frac{1}{2}\left[1-\sqrt{1-8C_1/C_2\beta_+q_+(u_+-\bar{u})\cos\theta}\right] \tag{78}$$

and K_ν a normalization constant. Since $P_\nu(-z)$ is strictly positive for real $z<1$ and diverges $\sim\ln(1-z)$ for $z\nearrow 1$, we see that we can always choose an $\epsilon\sim|X_0|\tau_0^{-1}$ so that $Y_*(\tau_0)=Y_0$ and $X_*(\tau_0)>X_0$ for arbitrarily large negative Y_0 and arbitrarily small negative X_0, with τ_0 the larger of the two roots solving (75) with these data. Clearly, since this implies that the acceleration terms can be made exponentially small, a simple Picard iteration argument now shows that the motion for $\tau>\tau_0$ can pick up only an exponentially small change in slope, which *a forteriori* is true for the real trajectory as well. This proves the uniform approach of the asymptotic outgoing slope of the trajectory to $+\infty$ as $X(-\infty)\rightarrow 0$. Therefore in (x,y) space the future inverse slope converges uniformly to $1/\bar{u}$ as $b\rightarrow 0$. By continuity there now exists a $b_0<\hat{b}$ such that $w_{\text{out}}(b_0)$ matches the outgoing data (57). This concludes the proof.

B. Spatial structure of solutions

It remains to determine the qualitative structure of the solutions. Having shown that the asymptotic motion is linear with the right slopes from the shape of the lines of force we now readily infer the general structure of the trajectories in (x,y) space, (see Fig. 2).

In the fully conformal situation C would be vertical and the trajectory of the solution identical to some lower portion of C.

The trajectory in Fig. 2 gives the general shape of a possible solution of (55) and (56). As a function of τ, $y(\tau)$ first increases monotonically from $-\infty$ reaches a maximum and then decreases again monotonically to $-\infty$: $x(\tau)$ decreases from $+\infty$ in the infinite past to $-\infty$ in the infinite future either monotonically or with at most one local minimum and one maximum in between. Taking the known asymptotic behavior into account, this now translates into the shapes $\phi(r)$ and $A(r)$ shown in Fig. 3. This completes the discussion of the qualitative radial profiles for the two-species case. A quantitative result can be obtained straightforwardly in the infinitesimal neighborhood of Bennett's solution. In fact, an infinitesimally nearby solution has the mathematical structure (74)–(78) in X,Y space, with appropriate constants replacing C_1, C_2, and ϵ.

Left for future studies is the case $S\geqslant 3$. It is readily seen that the transformation to the equivalent point particle analog is possible for arbitrary S, but the dimensionality of the model is always 2, whereas the number of exponentials in the acceleration is S, so that the mechanical analog will in general be nonautonomous. This means a significant complication.

VII. CONCLUDING REMARKS

In this paper we have investigated a simple kinetic model of an unbounded dissipative stationary S component plasma in a constant external driving electric field $\mathscr{E}$. Our model shows that a dissipative treatment of a plasma will severely limit the number of possible stationary distribution functions as compared to dissipation free Vlasov theory; cf. Refs. 20 and 21 for related observations concerning dissipative versus ideal magnetohydrodynamics, and Refs. 22–26 for *ad hoc* macroscopic variational approaches that select certain profiles from others. We did not, however, consider the time evolution for given initial data of the distributions and the fields. This means in particular that we do not know if or how a final stationary state with particle numbers N_s is achieved in the course of time. We also do not know how the final N_s are chosen. Since we are in an unbounded domain, the final N_s need not be the same as their initial values at $t=0$. For example, if the external electric field is too small to satisfy (1), we would expect to get, as $t\rightarrow\infty$, only the trivial solution in which all N_s and therefore also I and Q go to zero. Less dramatically, the plasma may "shed" only some "excess" particles to "infinity." We hope to investigate this further later. Meanwhile we believe that our results point to the conclusion that *typically* a stationary current-carrying S-isothermal plasma will be non-neutral. Indeed, given that one ignites a neutral gas, the applied electric induction is unlikely to drive the individual species to exactly the "right" mean speeds for which neutrality of the stationary state would be observed in the laboratory frame. Non-neutral plasma beams are of course well known,[2] but many theoretical beam profiles result from some special ansatz which allows for explicit solutions like in Bennett's original work.

As potential application, our model indicates that measurement of a few global quantities like current and charge, and the asymptotic decay of the density profiles, may reveal species individual information on effective transport coefficients via equations like (30). A further investigation of this aspect in more realistic effective dissipation models seems worth pursuing.

We finally notice that we have not investigated the stability of our solutions against perturbations which are invariant along $\mathcal{E}$; cf. Refs. 6, 7, and 27. It is also conceivable that some of our solutions are unstable to a macroscopic kink or sausage instability[28,29] or a velocity space instability of Buneman type.[2] The dissipation changes the problem considerably as compared to the Vlasov setting discussed in Ref. 2 for instance. We hope to address these questions as well as plasmas in confined geometry in future work.

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APPENDIX A: VIRIAL IDENTITY

The generalized Bennett relation (1) is a special case of a virial identity. Assuming cylindrical symmetry, taking the first velocity moment of (15) and summing over species yields, after a few partial integrations, the momentum balance equation

$$-\partial_r p+\rho E_r+c^{-1}j_z B_\varphi=0 \tag{A1}$$

with (r,φ,z) denoting cylinder coordinates and $\partial_r p$ the radial pressure gradient, cf. Ref. 2. Multiplying (A1) by r^2, expressing ρ and j_c through E_r and B_φ using (4) and (5), then integrating over dr gives formally, after a partial integration,

$$16\pi\int_0^\infty p(r)r\,dr=\lim_{r\rightarrow\infty}(r^2B_\varphi^2-r^2E_r^2) \tag{A2}$$

assuming that

$$\lim_{r\rightarrow\infty} r^2p(r)=0. \tag{A3}$$

To prove (A3) and to evaluate the right-hand side of (A2) one has to control ϕ and A asymptotically by using the Poisson–Boltzmann equations, as in Sec. IV. Noting furthermore that here $p=\Sigma_s k_{\rm B}T_s n_s$, one arrives at (1).

Our derivation of (1) is done directly from the Poisson–Boltzmann equations, thus avoiding the detour via the first velocity moment of the stationary kinetic equation. However, the physically interesting connection with the more familiar notion of a virial identity is worth noting, and we are grateful to the anonymous referee for drawing our attention to the identity (A2).

APPENDIX B: HAMILTON PRINCIPLE

Since it may be useful for the discussion of certain aspects of radial solutions to (21) and (22) in the setting of (55)–(57) we show that they satisfy a variational principle. By Legendre transform from H (58) we obtain the Lagrangian

$$L=\frac{1}{2}\left[(\dot x-\dot x_\infty)^2+(\dot y_\infty^2-\dot y^2)\right]+U(x,y) \tag{B1}$$

defined on the space of $\mathbb{R}^2$-valued functions $\tau\mapsto\{x(\tau),y(\tau)\}$ for which both $\ddot x$ and $\ddot y$ are in $\mathcal{S}(\mathbb{R})$, the Schwartz space of C^∞ functions with rapid decrease at infinity, for which $(\dot x,\dot y)$ satisfies the asymptotic conditions (57), and for which $\ddot y\leq 0$ (the latter because $I>0$ as a consequence of $\mathbf{e}_z\cdot\mathcal{E}>0$). From (55)–(57) it immediately follows that, for a solution the asymptotic momenta are approached exponentially fast. Then the action integral

$$\mathcal{A}[x,y]=\int_{-\infty}^{\infty}L\{x(\tau),y(\tau)\}d\tau \tag{B2}$$

exists and is non-negative on the specified function space. The equations of motion (55)–(57), i.e., including the asymptotic conditions are obtained from the Hamiltonian principle $\delta\mathcal{A}=0$ where the variation is with respect to functions $\{x(\tau),y(\tau)\}$ in the above function space.

The action functional $\mathcal{A}$ is non-negative on the constructed function space, but a solution of our PDE system does not correspond to a global minimum of $\mathcal{A}$. Without further constraints, one can construct a sequence of functions along which $\mathcal{A}\searrow 0$, but clearly 0 is not taken for any function in our function space. If the contraints (54) are taken into account, one can show existence of a global minimum on the space of continuously differentiable functions, but the minimizer is not twice continuously differentiable and hence does not solve our PDE system. Solutions instead correspond to critical points of $\mathcal{A}$, which may be either a relative minimum or a saddle point; maxima do not exist. However, the global minimum may provide a useful lower estimate for $\mathscr{A}$.

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