35V

C NAVIER–STOKES LIMIT OF THE BOLTZMANN EQUATION*

R.ESPOSITO[1], J.L.LEBOWITZ[2], R.MARRA[3]

[1]Dipartimento di Matematica, Università di Roma Tor Vergata.
[2] Mathematics and Physics Departments, Rutgers University.
[3]Dipartimento di Fisica, Università di Roma Tor Vergata.

ABSTRACT. The possibility of deriving a compressible Navier-Stokes behavior by a scaling limit starting from the Boltzmann equation is discussed. The scale invariance turns out to be the main tool for the matter. We consider a Boltzmann flow in a channel and introduce a suitable scaling limit for this system. We find the limit equations under this scaling describing the hydrodynamical dissipative behavior of the compressible gas in the channel. We prove an existence and uniqueness theorem for these equations. Finally we study a stationary case for the previous system under the effect of an external force.

1. INTRODUCTION.

The problem of deriving the Euler (E) and Navier-Stokes (NS) hydrodynamical equations from the Hamiltonian equations of motion of atoms is one of the main open problems of non-equilibrium Statistical Mechanics. The Euler behavior is rather well understood from the conceptual point and with the addition of suitable randomness can actually be proven in a rigorous way [1)], (see also [2)] for a review on the subject). The idea is that the space-time scale separation between the microscopic (particle) and the macroscopic (hydrodynamical) descriptions is responsible of the Euler behavior in the sense that the locally conserved microscopic fields, averaged over macroscopic domains (of size ϵ^{-1}) for times of order ϵ^{-1} converge, in the limit $\epsilon \to 0$, to hydrodynamical fields $\psi \equiv (\rho, u, T)$, ρ being the density, u the velocity field and T the local temperature, satisfying the Euler equations

$$\partial_t \psi = E(\psi) \tag{1.1}$$

In probabilistic terms (1.1) can be interpreted as a law of large numbers.

The situation is quite different for the Navier-Stokes equation. This equation, which describes the behavior of a real fluid in the presence of the viscosity, does not have an immediate interpretation in terms of scale separation. This is clear from the fact that the NS equation does not have a space-time scale invariance property like the Euler

* Research supported in part by NSF Grant DM 89-18903 .

equation. If we restrict our attention to the Euler time scale [3),4)] the viscous terms in the NS equation can be understood as a first order correction to the equation (1.1):

$$\partial_t \psi = E(\psi) + \epsilon F(\psi) \tag{1.2}$$

It is not clear that such first order corrections are uniquely determined, c.f. discussion for a hard rod system in Ref.[5)]. In the context of the Boltzmann equation, they can be computed using the Hilbert method or the Chapmann-Enskog procedure, producing very different equations (linear and nonlinear NS equation respectively).

The point is that different F's in (1.2) cannot be distinguished on the Euler time scale, because the corresponding solutions tend to coincide to lowest order in the limit $\epsilon \to 0$. On the other hand the viscosity produces significant effects on very long times, i.e. times of order ϵ^{-1} with respect the Euler time scale (hence of order ϵ^{-2} with respect to the microscopic scale), so that the first order corrections $F(\psi)$ become important for these times and give rise to distinctive behavior. Not surprisingly there are many difficulties to finding corrections to (1.1) which would be valid for such long times: if the Euler solution is unstable, a perturbation of order ϵ could produce explosive behaviors at such long times. Even assuming enough stability properties to control the explosion of the corrections, they could however be sensible to the initial conditions.

Our purpose is to investigate situations in which it is really possible to distingush between (1.1) and (1.2) in the limit $\epsilon \to 0$. There are essentially two possibilities: either to look at boundary effects in a stationary situation, and in the last Section of this paper we shall discuss this possibility in a very simple example; or to analyze the long time behavior of microscopic systems. Because of the absence of scaling invariance in the compressible Navier-Stokes equation one cannot hope to get such long time behavior as a scaling limit in general, but in the situations discussed in Sections 2 and 3 it is possible to find a limit regime, which is on the other hand the same as predicted by the NS equation with viscosity of order ϵ on times of order ϵ^{-1}.

We shall consider as our starting point a Boltzmann fluid, i.e. one described by the Boltzmann equation, instead of a particle system. This is a real simplification of the problem because the transition from the microscopic scale to the kinetic scale is a non trivial step in the above program. Although not fully understood, we take it as accomplished, because the Boltzmann fluid retains the difficulties pointed out above, related to the instabilities of the macroscopic equations. Moreover, since the Euler limit of the Boltzmann equation is controlled, one can hope that in this context one has to face only the technical difficulties specific of the Navier-Stokes problem. This turns out to be true at least in the case of the Incompressible Navier-Stokes (INS) equation, that can be obtained from the Boltzmann equation via a suitable scaling limit. In fact the

INS equation is invariant under a space-time scaling of parabolic type, together with the scaling of the mean velocity.

The situation is much more complicated in the compressible case and the main subject of this work is the presentation of some special systems described by equations which have scaling invariance properties, but at the same time exhibit compressible behaviors. We conclude this section with a short overview of the results on the hydrodynamical limits for the Boltzmann equation.

The Knundsen number Kn is the ratio between the mean free path and the characteristic macroscopic length, hence it is proportional to the previously introduced parameter ϵ. It is well known that the limit behavior for $Kn \to 0$, under the hyperbolic scaling ($x = \epsilon^{-1}x', t = \epsilon^{-1}t'$) corresponds to the compressible Euler regime; this result goes back to Hilbert. From the rigorous point of view, Caflisch [6] proved the following:
Let ρ, u, T be a smooth solution of the Euler equations for $t \in [0, t_0]$; then there exists a solution $f^\epsilon(x, v, t)$ of the rescaled Boltzmann equation such that, for $t \leq t_0$, we have in some suitable norm

$$||f^\epsilon - M(\rho, u, T)|| < c\epsilon$$

for ϵ small enough, with $M(\rho, u, T)$ the Maxwellian with parameters ρ, u, T.

To get sensible viscous effects one has to take the Reynolds number Re finite while taking the limit $Kn \to 0$. Since the Mach number $Ma = U/c$ (where U is a typical velocity and c is the sound velocity) is related to Kn and Re by the relation $Ma \sim Re \times Kn$ we have to consider $Ma \sim \epsilon$. This corresponds to the incompressible regime. To have a non trivial behavior, of course, the time has to be very long and therefore one has to consider the parabolic space-time scaling (($x = \epsilon^{-1}x', t = \epsilon^{-2}t'$).
De Masi *et al.* [7] proved that, if $u(x, t)$ is a smooth enough solution of the incompressible Navier-Stokes equation on a torus for $t \in [0, t_0]$, one can construct a solution f^ϵ to the rescaled (parabolically) Boltzmann equation, with special initial conditions, such that, for $t \in [0, t_0]$, in a suitable norm

$$||f^\epsilon - M(\rho, \epsilon u, T)|| < c\epsilon^2$$

where ρ and T are suitably chosen positive constants. Moreover they proved that, on times that are long on the Euler scale, but short on the Navier-Stokes scale the solution of the rescaled Boltzmann equation behaves in the limit as the solution of the incompressible Euler equation.

Partial results in this direction have been obtained in situations in which much less regularity is available (see Refs. [8],[9]). More recently the restrictions on the initial conditions for the Boltzmann equation assumed in Ref.[7] have been removed in Refs [10],[11].

2. The Compressible regime.

As discussed above, in order to get limiting behaviors on the Navier-Stokes scale we have to look for scale invariant situations, and this is in general incompatible with finite Mach numbers. In the following we shall discuss an example in which the Mach number and the Reynolds number can both stay finite thanks to special symmetry properties.

We consider the motion of the fluid in a 2-dimensional infinitely long periodic channel. In this case there is a stationary Euler behavior, so that we can hope to get long time behavior without facing the explosion of the Euler modes. Let $x \in \mathbb{R}$ be the direction of the channel and let us choose periodic boundary conditions on the other direction, $y \in S_1$, S_1 being the unit circle. We assume that the density ρ_o, temperature T_0 and velocity field U_0 depending only on y with U_0 parallel to the x direction $(U_0(y) = (u_0(y), 0))$ is a stationary solution to the compressible Euler equations for an ideal fluid

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho U) = 0 \\
&\rho(\partial_t U + U \cdot \nabla U) + \nabla(\rho T) = 0 \\
&\rho(\partial_t T + U \cdot \nabla T) + \rho T \nabla \cdot U = 0.
\end{aligned} \tag{2.1}$$

Taking now ρ_0, T_0, U_0 as initial condition for the Navier-Stokes equation the behavior is no more stationary. The effect of the viscosity, together with the non-homogeneity of the initial velocity, gives rise to a gradient of temperature and, as a consequence, to a flow in the y direction. Our goal is to recover this behavior in a suitable scaling limit starting from the Boltzmann equation.

The width d of the channel is assumed of order ϵ^{-1} in microscopic units. The Reynolds number Re, roughly speaking, is the ratio between the dissipative term and the transport term in the Navier-Stokes equation; since in our case the space dependence is only in the y direction, the magnitude of the transport term is proportional to the y-velocity , U_y. Hence the Reynolds number for the problem is: $Re \sim |U_y| d/\nu$. Therefore, to keep it finite we are forced to consider U_y of order ϵ. On the other hand the Mach number $Ma \sim U_x/c$ is finite, and the flow is compressible.

Since on the Euler scale the solution is stationary, we look at long times and consider the rescaled (parabolically) Boltzmann equation which, due to the symmetry in the x direction, can be written:

$$\partial_t f^\epsilon + \epsilon^{-1} v_y \cdot \partial_y f^\epsilon = \epsilon^{-2} Q(f^\epsilon, f^\epsilon). \tag{2.2}$$

with the initial condition

$$f^\epsilon(y,v,0) = M(\rho_0, T_0, U_0) \tag{2.3}$$

The solution is constructed as a series in ϵ via a Hilbert-like expansion similar to the one introduced in Ref.[7]. Of course we do not expect that it converges, but only that it is asymptotic to the solution. The argument below is formal: we shall determine consistently all the terms in the expansion, but to prove the asymptotiticity we need to control the remainder term after a suitable truncation of the series, and this step is not yet complete.

Since we expect that the solution to the lowest order coincides with the local equilibrium, i.e. with the local Maxwellian $M_\epsilon = M(\rho, T, U_\epsilon)$ with parameters $\rho(y,t), T(y,t)$, $U_\epsilon = (u(y,t), \epsilon w(y,t))$ evolving according to the previous considerations, we assume the following form for f^ϵ:

$$f^\epsilon = M_\epsilon + \sum_{n=1}^{\infty} \epsilon^n f_n \tag{2.4}$$

Expanding M_ϵ in a Taylor series in ϵ, $M_0 \equiv M_\epsilon|_{\epsilon=0} = M(\rho(y,t), T(y,t), (u(y,t), 0))$ we have

$$M_\epsilon = M_0 + \sum_{n=1}^{\infty} \epsilon^n \phi_n \tag{2.5}$$

We also set $g_n = f_n + \phi_n$. Since $Q(M,M) = 0$ for any Maxwellian M, substituting (2.4) and (2.5) in the Boltzmann equation (2.2) and equating terms of the same order in ϵ we obtain the following set of equations

$$\epsilon^{-1} : v_y \partial_y M_0 = 2Q(M_0, f_1) \tag{2.6}$$

$$\epsilon^{0} \quad : \partial_t M_0 + v_y \partial_y g_1 = 2Q(M_0, f_2) + Q(2\phi_1 + f_1, f_1) \tag{2.7}$$

and for $k \geq 1$

$$\epsilon^{k} \quad : \partial_t g_k + v_y \partial_y g_{k+1} = 2Q(M_0, f_{k+2}) + \sum_{\substack{n,m\geq 1 \\ n+m=k+2, n\neq m}} Q(g_n, f_m)$$

$$+ \sum_{n\geq 2} Q(f_n + 2\phi_n, f_n)\delta_{2n,k+2} \tag{2.8}$$

We need some notation: let $\mathcal{H}$ be the Hilbert space of measurable functions of the velocity endowed with the scalar product

$$(f,g) = \int (M_0(v))^{-1} f(v) g(v) dv$$

Let $\xi = \{\xi_\alpha\}_{\alpha=0,\dots,3} = \{M_0(v), \tilde{v}_x M_0(v), v_y M_0(v), (\tilde{v}^2 - 2T)M_0(v)\}$ with $\tilde{v} = (\tilde{v}_x, v_y)$ $= (v_x - u, v_y)$ be the set of collision invariants and P the projection operator $Ph = \sum_{\alpha=0}^{3} (\xi_\alpha, h)\, \xi_\alpha$. The operator Q has the following property:

$$(\xi_\alpha, Q(h,g)) = 0; \quad PQ(h,g) = 0 \tag{2.9}$$

for any h and g.
Hence (2.6) implies:

$$P(v_y \partial_y M_0) = 0 \tag{2.10}$$

$$(1-P)(v_y \partial_y M_0) = 2Q(M_0, f_1) \tag{2.11}$$

(2.10) implies

$$\partial_y(\rho T) = 0 \tag{2.12}$$

Therefore the compatibility condition at the order ϵ^{-1} leads to the constancy of the pressure to the lowest order.
The first correction to the Maxwellian, f_1 is determined by (2.11) as follows. Put $\mathcal{L}f \equiv 2Q(M_0, f)$; the restriction of $\mathcal{L}$ to $(1-P)\mathcal{H}$ (that we still denote by $\mathcal{L}$) has a well defined inverse, so that f_1 is determined up to terms in the orthogonal to $(1-P)\mathcal{H}$ and can be written

$$f_1 = \mathcal{L}^{-1}(1-P)(\tilde{v}_y \partial_y M_0) + \sum_{\alpha=0}^{3} f_1^{(\alpha)} \xi_\alpha \tag{2.13}$$

By the properties of $\mathcal{L}$ (see [12])), there are two positive functions γ and ω depending only on T and $\tilde{v}^2$ such that the following expression holds for f_1:

$$\begin{aligned} f_1 = & -\gamma(\tilde{v}^2) M_0 v_y \tilde{v}_x \partial_y u - \omega(\tilde{v}^2) v_y \frac{\tilde{v}^2 - 4T}{2T^2} M_0 \partial_y T \\ & + \rho^{(1)} + u^{(1)} \tilde{v}_x + w^{(1)} \tilde{v}_y + T^{(1)} M_0 \frac{\tilde{v}^2 - 2T}{T^2} \end{aligned} \tag{2.14}$$

The functions $(f_1^{(\alpha)}) \equiv (\rho^{(1)}, u^{(1)}, w^{(1)}, T^{(1)})$ are still undetermined. We assume that $w^{(1)} = 0$ because it can always be absorbed in the Maxwellian M_ϵ changing the definition of w.

Next step is to substitute f_1 from (2.14) and ϕ_1 from (2.5) in (2.7) to determine f_2. The compatibility condition

$$P(\partial_t M_0 + v_y \partial_y g_1) = 0 \tag{2.15}$$

gives rise to the following equations:

$$\partial_t \rho + \partial_y(\rho\ w) = 0 \tag{2.16}$$

$$\partial_t(\rho\ u) + \partial_y(\rho\ u\ w) - \partial_y(\nu \partial_y u) = 0 \tag{2.17}$$

$$\partial_t(\rho(\frac{1}{2}\ u^2 + T) + \partial_y[\rho\ w(2T + \frac{1}{2}u^2) - \nu u\ \partial_y u] - \partial_y(K\ \partial_y T) = 0 \tag{2.18}$$

where the transport coefficients $\nu = \int dv \gamma(\tilde{v}^2) M_0 v_y \tilde{v}_x$ and $K = \int dv v_y \omega(\tilde{v}^2) M_0 \frac{\tilde{v}^2 - 4T}{2T^2}$ are functions of the temperature only. Moreover

$$\partial_y (T\rho^{(1)} + \rho T^{(1)}) = 0 \tag{2.19}$$

that ensures the constancy of the pressure up to the first order. The condition (2.19) is known as the Boussinesq relation.

Notice that $u^{(1)}$ and one of the quantities $T^{(1)}$ and $\rho^{(1)}$ are still free and can be used to satisfy the compatibility condition to the next order. Once eqs. (2.16)–(2.19) are satisfied, f_2 can be determined as f_1 before, up to terms in the orthogonal to $(1-P)\mathcal{H}$ by means of (2.7). The procedure can be iterated to determine all the terms f_{k+2} for $k \geq 1$ using (2.8), because at each step there are 4 new undetermined functions available to satisfy the compatibility conditions. These are linear conditions that give rise to linear partial differential equations whose solvability is easy to prove. Hence the only requirement necessary for the construction of the expansion is the solvability of the system (2.16)–(2.18) under the condition (2.12).

Since the expansion is probably not convergent in general, we try to truncate the series and to control the remainder. The strategy is the same as in Refs[6),7)] : we consider a special truncation of the series for f^ϵ:

$$f^\epsilon = M_\epsilon + \sum_{k=1}^{8} \epsilon^k f_k + \epsilon^4 f_R \tag{2.20}$$

M_ϵ and the functions f_k for $k = 1, \ldots, 8$ are determined by the equations (2.6), (2.7) and (2.8) for $k = 1, \ldots, 6$. f_R, which is defined by (2.20), then has to satisfy the following equation in order that f_ϵ be a solution of the Boltzmann equation:

$$\begin{aligned} &\partial_t f_R + \epsilon^{-1} v_y \partial_y f_R = \\ &\epsilon^{-2} 2Q(f_R, M_\epsilon) + \epsilon^{-1} 2Q(f_1, f_R) + \epsilon^2 Q(f_R, f_R) + 2Q(\sum_{k=2}^{8} f_k, f_R) + \epsilon^3 A \end{aligned} \tag{2.21}$$

where A depends on the $f_k, k = 2, \ldots, 8$ and is given by

$$\begin{aligned} A = &- \partial_t (g_7 + \epsilon g_8) v_y \partial_y (f_8 + \sum_{i=9}^{\infty} \phi_i) + \\ &2Q(f_8, \sum_{i=2}^{\infty} \phi_i) + \sum_{k \geq 9} \epsilon^{k-9} \sum_{\substack{n,m \geq 1 \\ n+m=k+2}} 2Q(f_n, f_m) \end{aligned} \tag{2.22}$$

Notice that the factor ϵ^3 in front of A in (2.21) is crucial to the control of f_R and this is the reason for writing (2.20) with the factor ϵ^4 in front of f_R instead of ϵ^9. Of course the exponents we have chosen are somewhat arbitrary, and this is just a convenient choice. Nevertheless, we are not yet able to prove that f_R is finite in some norm, because we have not enough control on the hydrodynamical part of f_R. The difficulty is present also at the hydrodynamical level (assuming compressible Navier-Stokes equation as a starting point, see below), and it seems that in this case one should be able to control sound waves propagating with larger and larger velocity when ϵ goes to 0, but with a net effect on the system that should be negligible in the limit.

3. Properties of the macroscopic equations.

Equations (2.12), (2.16)–(2.18) describe the behavior of the Boltzmann fluid in the channel on the hydrodynamical scale in the limit $\epsilon \to 0$. They are scale invariant under the transformation

$$\begin{aligned} &y' = \epsilon^{-1} y, \quad t' = \epsilon^{-2} t, \quad w'(y',t') = \epsilon w(y,t), \\ &u'(y',t') = u(y,t), \quad \rho'(y',t') = \rho(y,t), \quad T'(y',t') = T(y,t) \end{aligned} \tag{3.1}$$

Note that the component u of the velocity in the x direction stays finite so that we have a true compressible flow in the limit $\epsilon \to 0$. The system (2.12), (2.16)–(2.18) can also be obtained by starting with the full compressible Navier-stokes equation for the flow in the channel:

$$\begin{aligned} &\partial_t \rho + \partial_y (\rho w) = 0 \\ &\rho(\partial_t u + w \partial_y u) - \partial_y(\nu \partial_y u) = 0 \\ &\rho(\partial_t w + w \partial_y w) + \partial_y P - \partial_y(\nu \partial_y w) = 0 \\ &\rho(\partial_t T + w \partial_y T + T \partial_y w) - \nu[(\partial_y w)^2 + (\partial_y u)^2] - \partial_y(K \partial_y T) = 0 \end{aligned} \tag{3.2}$$

and rescaling the variables according to (3.1). The formal limit as ϵ goes to zero of (3.2) is just the system we are discussing. In fact, an equivalent form of the scaling (3.1) is to keep the variable y fixed, scale ν in $\epsilon\nu$ and look at times $\epsilon^{-1} t$. Therefore it can also be interpreted as the long time behavior in the vanishing viscosity limit of a compressible Navier-Stokes fluid in a channel. The proof of convergence is at the moment absent also for this problem, and we believe that the bad control of the high speed sound waves is the main difficulty also in this case.
In this respect it is worth mentioning that Klainerman and Majda [13] considered the incompressible limit of the Navier-Stokes equation. The fast sound modes and their interference with temperature variations prevented in that case a proof of a result that is available for the Boltzmann equation. In fact, they assumed that the temperature

variations are small and proved the analogous of the results in Refs[10),11)] only in that case. Hence the problem of the convergence of the equations (3.2) to the equations (2.12), (2.16)–(2.18) is perhaps more difficult than the convergence of the Boltzmann equation to it.

Hydrodynamic Equations

Let us go back to the properties of the equations (2.12), (2.16)-(2.18). It is convenient to rewrite them as follows:

$$\partial_t \rho + \partial_y(\rho w) = 0 \tag{3.3$_a$}$$

$$\rho(\partial_t u + w\partial_y u) - \partial_y(\ \nu \partial_y\ u) = 0 \tag{3.3$_b$}$$

$$\rho(\partial_t T + w\partial_y T + T\partial_y w) - \nu(\partial_y u)^2 - \partial_y(K\ \partial_y\ T) = 0 \tag{3.3$_c$}$$

$$\partial_y(\rho T) = 0 \tag{3.3$_d$}$$

The system (3.3) is a system of four equations in the unknowns ρ, u, w, T. It is not in the form of an initial value problem, because it does not contain the time derivative of w. In fact the initial value of w cannot be prescribed arbitrarily, because it turns out to depend on the initial value of $\partial_y u$.
For sake of simplicity, we assume the ν and K are constant and that $T(y,0) = 1$, $\rho(y,0) = 1$. We also assume that $u(y,0) = u_0(y)$ is in the Sobolev space H_1 on the unit circle. If u_0 is constant the system (3.3) has the trivial solution $u(y,t) = u_0$, $T(y,t) = 1$, $\rho(y,t) = 1$ and $w(y,t) = 0$ for any $t \geq 0$. In general we can prove [14)] the following theorem

Theorem 3.1.

1) For any given $u_0 \in H_1[0,1]$ there is a $t_0 > 0$ s.t. for $t \leq t_0$ there is a classical solution to (3.3) with the given initial data, and it is unique.

2) There exists $\delta > 0$ s.t., if $u_0 \in H_1$ and $|u_0|_1 < \delta$ then there exists an unique classical solution of the system (3.3) globally in time.

Notice that this means that w is determined also at time $t = 0$.

We just sketch the proof. It consists of two steps. First we transform the problem into an initial value problem using Lagrangian coordinates, that allows us to eliminate the variable w in the system (3.3). Then we prove existence and uniqueness theorem for the new system. We denote by Y the lagrangian variable corresponding to y, via the transformation

$$y = \Phi(Y,t) \tag{3.4}$$

defined by the ordinary differential equation

$$\frac{d}{dt}\Phi = w(\Phi, t); \quad \Phi(Y, 0) = Y$$

With the notation

$$u(Y,t) = u(\Phi(Y,t),t); \quad T(Y,t) = T(\Phi(Y,t),t); \quad P = \rho T$$

the system (3.3) becomes

$$\partial_t u = \nu P \partial_Y (T^{-1} \partial_Y u) \tag{3.5}$$

$$\partial_t T = \frac{K}{2T} P \partial_Y (\partial_Y T) + \frac{\nu}{2} T^{-1} P (\partial_Y u)^2 + \frac{T}{2P}\dot{P} \tag{3.6}$$

$$\frac{d}{dt} P = \nu P \int_0^1 dY' (T(Y'))^{-1} (\partial_{Y'} u)^2 \tag{3.7}$$

with initial conditions $u(Y,0) = u_0(Y)$, $T(Y,0) = 1$, $P(Y,0) = 1$.
Note that P depends only on time because of $(3.3)_d$.
It is convenient to introduce the variable S defined by

$$S = \frac{T}{\sqrt{P}}$$

which is related to the entropy s by the relation $s = 2 \log S$. In terms of it, the equations (3.5)–(3.7) become

$$\partial_t u = \nu \sigma \partial_Y ((S^{-1} \partial_Y u) \tag{3.8}$$

$$\partial_t S = \frac{K\sigma}{2} \partial_Y ((S^{-1} \partial_Y S) + \frac{\nu}{2S} (\partial_Y u)^2 \tag{3.9}$$

where $\sigma = \int_0^1 dY S(Y)$. The initial conditions become

$$u(Y,0) = u_0(Y); \quad S(Y,0) = \sqrt{T_0}(Y) = 1$$

After solving the system (3.8), (3.9) one can find ρ and w from the relations

$$\rho \partial_Y \Phi = \rho_0 = 1; \quad \partial_t \rho + \rho^2 \partial_Y w = 0$$

The non-linear system (3.8), (3.9) is of the parabolic type. Its solution can be constructed by getting a differential inequality for the function

$$H(t) = \frac{1}{2}(|u(.,t)|_1^2 + |\eta(.,t)|_1^2$$

where $\eta = S - 1$. We prove the following Lemma

Lemma 3.2. *Let $B(t) = |u(.,t)|_2^2 + |\eta(.,t)|_2^2$ Then there are b and c such that*

$$H(T) + b\int_0^T d\tau A(\tau) \leq H(0)\exp c\int_0^T d\tau A(\tau) \tag{3.10}$$

The inequality (3.10) implies that if $H(0)$ is small then $H(T)$ and $\int_0^T d\tau A(\tau)$ are finite for any T. The proof of the part 2) of the theorem is then achieved by an iterative procedure based on the a priori estimate (3.10). The local result is obtained by a similar, but simpler procedure.

4. A STATIONARY CASE.

Stationary flow in a channel provides a simple interesting example of the relevance of the Navier-Stokes correction, related to the presence of boundaries. The solution of the Euler equations in this situation is trivial while the NS has a rich structure. The physical system is a fluid in a channel subjected to an external specific force g constant (in space and time) and directed along the x direction (channel direction). A stationary state will be reached when the heat produced by the friction due to the relative motion of the particles in the x direction under the action of the force is balanced by the energy lost through the thermal walls. The hydrodynamical equations for this system, in the compressible case, with the state equation $P = \rho T$, are

$$\partial_y(\nu(T)\partial_y u) + \rho g = 0 \tag{4.1}$$

$$\partial_y(K(T)\partial_y T) + \nu(T)(\partial_y u)^2 = 0 \tag{4.2}$$

$$\partial_y(\rho T) = 0 \tag{4.3}$$

The third equation follows from the assumption that the y component of the velocity field is zero i.e. that the flow is laminar. The boundary conditions are

$$u(0) = u(1) = 0; \qquad T(0) = T(1) = T_0 \tag{4.4}$$

The solution of this system does exist for ν and K independent on T and is stable for small g. It is given, to the first order in g, by a parabolic profile for $u(y)$ and a quartic profile for $T(y)$ (see Fig.1).

Molecular Dynamics simulations [15] on a Hamiltonian system of particles interacting via Lennard-Jones potential in a channel with boundary conditions given below show a macroscopic behavior similar to the one in Fig. 1.

We want to get such a behavior from the Boltzmann equation. The system is bounded by thermal walls (in the y direction), i.e. the particle behavior at the walls is the following: when a particle hits the edges of the channel ($y = 0$ or $y = 1$) it is reflected diffusely with a velocity distributed according to a maxwellian with zero mean velocity and prescribed temperature T_0. In the language of classical kinetic theory this means that the *accomodation coefficient* equals one. This corresponds to the Boltzmann equation with the external force g and thermal conditions on the incoming flow at the boundary:

$$f(0, v_y > 0) = M(\rho_0, T_0)\chi(v_y > 0); \quad f(1, v_x < 0) = M(\rho_1, T_0)\chi(v_x < 0) \tag{4.5}$$

$\chi(v_y \lesseqgtr 0)$ being the indicator function of the set $\{v_y \lesseqgtr 0\}$. The macroscopic behavior is recovered in the limit $Kn \equiv \epsilon$ going to zero. The size of the channel is ϵ^{-1}, hence we scale $y \to \epsilon y$. On the other hand we scale the force by a factor ϵ^2, because in this way it will produce finite changes in the velocity in the x direction on the relevant hydrodynamical scale of times, that is, as before, $t = \epsilon^{-2}\tau$. We look for stationary solutions, hence we have to solve the equation:

$$v_y \frac{\partial f^\epsilon}{\partial y} + \epsilon g \frac{\partial f^\epsilon}{\partial v_x} = \epsilon^{-1} Q(f^e, f^e) \tag{4.6}$$

In [16] we prove that the stationary state of (4.6) with zero mean velocity in the y direction, $w = 0$, behaves in the limit $\epsilon \to 0$ as a Maxwellian with parameters obtained from the solution of the system (4.1)-(4.3). The method follows the lines of the Hilbert-like expansion explained in section 2. The assumed form of the solution of (4.6) is

$$f^\epsilon = M(\rho, T, (u,0)) + \sum_{n=1}^{6} \epsilon^n (f_n + f_{BL}^{(n)}) + \epsilon^3 R^\epsilon \tag{4.6}$$

The mean velocity in the y direction vanishes to zero order in ϵ because we take the Maxwellian M to depend on the velocity field $(u,0)$. The terms $f_{BL}^{(n)}$ are boundary layer contributions due to the fact that in general f_n does not satisfy the boundary conditions. The solvability condition at the first order in ϵ gives the hydrodynamical equations (4.1)-(4.2) while (4.3) is determined by the solvability conditions at zero order. The functions f_n are determined as in the time dependent case, suitably modified to take into account the boundary layer terms. On the other hand, the boundary layer terms are functions of $\zeta = \epsilon^{-1} y$ determined by the the equation

$$v_y \partial_\zeta f_{BL}^{(n)} + \epsilon^2 g \partial_{v_x} f_{BL}^{(n)} = \mathcal{L} f_{BL}^{(n)} \tag{4.7}$$

with suitable boundary conditions. Equation (4.7) is a perturbation of the one treated in Ref.[17] and can be controlled in a similar way. The equation for the remainder R (dropping the ϵ dependence) is

$$v_y \partial_y R + \epsilon g \partial_{v_x} R = \epsilon^{-1} 2Q(M,R) + 2Q(\sum_{n=1}^{6} \epsilon^n (f_n + f_{BL}^{(n)}), R) + \epsilon^2 Q(R,R) + \epsilon^2 A \tag{4.8}$$

with A a bounded quantity depending on f_n, $n = 1, \ldots, 6$. We also prescribe the vanishing of the incoming part of R at the boundary. In order to control the remainder we introduce the space $B_{\ell,s}$ of the measurable functions on $[0,1] \times \mathbb{R}^2$ with the norm

$$||h||_{\ell,s} = \sup_{v \in \mathbb{R}^2} (1+v^2)^{\ell/2} |h(.,v)|_s \tag{4.9}$$

The study of the remainder equation is performed along the lines in Refs[6],[7], and a suitable use of the symmetry properties of the problem. Let us remark that in this case we do not have to deal with the problem of infinitely fast sound waves, because of the stationarity of the problem.

The result is stated in the following theorem:

Theorem 4.1. *Given a smooth solution ρ, T, u of (4.1)-(4.3) in $H_s([0,1])$, $s \geq 1$ with the boundary conditions (4.4), for g small enough, there exists a constant C and a solution $f^\epsilon(x,v,t)$ of the rescaled Boltzmann equation (4.6) with the boundary conditions (4.5) such that we have*

$$||f^\epsilon - M(\rho, u, T)||_{3,1} < C\epsilon$$

for ϵ small enough.

Aknowledgments

R.E. and R.M. thank very much the warm hospitality at the Mathematics Departement of the Rutgers University, where part of these results were done. We are greatly indebted with A.De Masi and E.Presutti who helped us to focus the crucial points of this work.

References

[1] Olla,S., Varadhan,R. and Yau, in preparation.
[2] Sphon,H., *The dynamics of systems with many particles*, Springer-Verlag, New york, (1991).
[3] Dobrushin,R.L. and Sokolovskii,F., *Higher order hydrodynamic equations for a system of independent random walks*, Technical Report '90-79 (1990).
[4] Boldrighini,C.,Dobrushin,R.L. and Sukov,Yu.M., *One-dimensional hard-road caricature of hydrodynamics: Navier-Stokes correction*, preprint (1991).
[5] Spohn,H., Ann. Phys.(USA) 141, 353-364 (1982).
[6] Caflisch,R.E., Commun. on Pure and Applied Math. 33, 651-666 (1980).
[7] De Masi,A., Esposito,R. and Lebowitz,J.L., Commun. Pure Appl. Math. 42, 1189-1214 (1989).
[8] Bardos,C., Golse,F. and Levermore,D., J.Stat.Phys. 63, 323-344 (1991).
[9] ———, *Fluid dynamical limits of kinetic equations II. Convergence proof for the Boltzmann equation*, preprint (1990).
[10] Bardos,C. and Ukai,S., *The classical incompressible Navier-Stokes limit of the Boltzmann equation.*, preprint (1991).
[11] Asano,K., in preparation.
[12] Cercignani,C., *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York, (1988).
[13] Klainerman,S. and Majda,A., Commun.Pure and Appl. Math. 34, 481-524 (1981).
[14] Esposito,R., Lebowitz,J.L. and Marra,R., in preparation.
[15] Hannon,L., Lie,G.C. and Clementi,E., Journal Stat.Phys. 51, 965-979 (1988).
[16] Esposito,R., Lebowitz,J.L. and Marra,R., in preparation.
[17] Bardos,C., Caflisch, R.E. and Nicolaenko,B., Commun.Pure and Appl. Math. *XXXIX*, 323-352 (1986).