

# Electron velocity distribution in a weakly ionized plasma with an external electric field

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The velocity distribution  $f(\mathbf{v})$  of the electron component of a weakly ionized plasma is investigated in a spatially homogeneous external electric field  $\mathbf{E}$ . Both static and time-dependent  $\mathbf{E}$  are considered. The time evolution of  $f$  is described by a Boltzmann equation in which the ions and neutral particles are assumed to have a Maxwellian distribution with *a priori* specified temperatures while the electron–electron interactions are given by a Landau-type collision integral. The (approximate) solution scheme used to solve this equation for a stationary  $f$  (in a constant field) is found to have nonunique solutions for certain ranges of  $E$ , in agreement with that found in earlier investigations using a different method of solution. These results are interpreted to correspond to hysteresis effects when the field is changing very slowly: with the true stable solution undergoing a very sharp changeover, possibly a discontinuous transition, at a certain critical  $E$ . This can be understood intuitively as a transition in the stationary state of the electrons from a low-energy regime dominated by strong coupling to the ions to a high-energy regime dominated by electron–electron and electron–neutral collisions.

## I. INTRODUCTION

We consider a weakly ionized gas in the presence of an externally imposed spatially homogeneous electric field  $\mathbf{E}$  which may be either constant or vary in time. The density of the gas, the degree of ionization, and the strength of the field are assumed to be such that (i) the interactions between the electrons and of the electrons with the ions and neutrals can be described by Boltzmann-type elastic collision integrals, and (ii) collisions between the electrons and the heavy components of the plasma, ions and neutrals, are adequately described by assuming the latter to be in a Maxwellian distribution with *a priori* given temperatures. Under these assumptions the time evolution of the spatially homogeneous electron velocity distribution function  $f(\mathbf{v}, t)$  will satisfy the Boltzmann equation<sup>1–3</sup>

$$\frac{\partial f}{\partial t} - \left(\frac{e}{m}\right) \mathbf{E} \cdot \nabla f = (L_i + L_n)f + Q(f, f). \quad (1)$$

The right side of (1) consists of a linear part corresponding to elastic collisions between the electrons and the heavy components (ions and neutrals) and a quadratic Landau-type collision integral for  $e$ – $e$  interactions.

The stationary solutions of Eq. (1) for a time-independent field  $\mathbf{E}$  were investigated by one of us (AR) in Ref. 4 and we refer the reader there for detailed descriptions of the different collision terms. In the present work we use a different method for the stationary case and also study the distribution function  $f(\mathbf{v}, t)$  for both “slowly” and “rapidly” varying fields  $\mathbf{E}(t)$ . To do this we begin, as in Refs. 2 and 4, by expanding  $f(\mathbf{v}, t)$  in Legendre polynomials and keeping only the first two terms:

$$f(\mathbf{v}, t) = \Phi(v, t) + \cos(\theta)a(v, t) + \dots, \quad (2)$$

where  $v = |\mathbf{v}|$  and  $\theta$  is the angle between  $\mathbf{E}$  and  $\mathbf{v}$ . The first term in (2) is the symmetrical part of the distribution function while the second one determines the electron flux

along the field direction. We assume further that  $a(v, t)$  is small in comparison with  $\Phi$  in the most important domain of  $v$  and neglect the quadratic terms of  $a(v, t)$  in the kinetic equation (1). This leads to a set of coupled equations, linear in  $a$  and quadratic in  $\Phi$ , whose solution for given external parameters we seek.

Equation (1) differs from the frequently studied linear kinetic equation for an essentially zero density “electron swarm”<sup>2</sup> by the omission of inelastic  $e$ – $n$  collisions and the inclusion of elastic  $e$ – $i$  and  $e$ – $e$  collisions. It also differs from the strongly coupled plasma case in that we neglect collective plasma interactions. To limit the number of parameters we have to consider in judging the domain of applicability of (1) we assume equality of the ion and neutral masses and temperatures, setting them equal to  $M$  and  $T_i$ , respectively: we shall denote by  $T$  an “effective” electron temperature even though  $f$  is not Maxwellian. We also assume that the electron–neutral cross section  $S$  and the corresponding mean-free path  $l = (SN)^{-1}$  are independent of  $v$ . Here  $N$  is the neutral particle density. The ion and electron density are assumed equal to  $N_e$ . We can now state requirements on these parameters.

To begin with, let us consider the case of small electric fields so that  $T \approx T_i$ . We need the condition<sup>1,5</sup>  $r_D > N_e^{-1/3}$ , where  $r_D = (kT/4\pi e^2 N_e)^{1/2}$  is the Debye radius. This guarantees quasineutrality and therefore the cutoff of the long-range Coulomb forces. On the other hand, when the electron density is sufficiently small, one can consider the  $e$ – $n$  collisions only and the electron swarm approach is valid. This requires that the rate of energy transfer from electrons to neutrals exceeds greatly the rate of mutual energy exchange in  $e$ – $e$  interaction or the energy transfer between the electrons and the ions. To see when this is true we note that the mean number of collisions of an electron with

TABLE I. Upper bound of the relative electron density  $N_e/N$  when the interaction between electrons in a plasma can be neglected.

	$T=300^\circ\text{C}$	0.1 eV	1 eV	5 eV
Ar	$2.0 \times 10^{-10}$	$3.0 \times 10^{-9}$	$1.5 \times 10^{-8}$	$8 \times 10^{-6}$
Ne	$1.2 \times 10^{-11}$	$1.8 \times 10^{-10}$	$3.6 \times 10^{-8}$	$2 \times 10^{-6}$
He	$5.0 \times 10^{-10}$	$8.0 \times 10^{-9}$	$8.0 \times 10^{-7}$	$10^{-5}$
H	$6.0 \times 10^{-9}$	$10^{-7}$	$10^{-5}$	$8 \times 10^{-5}$

neutral particles in a time interval  $\Delta t$  is  $v\Delta t/l$  and the fraction of the energy,  $\epsilon = mv^2/2$ , which is lost in such collisions equals  $2mv\Delta t/Ml$ . The frequency of “ $e$ - $e$  collisions” on the other hand is  $n/v^3$  where<sup>1</sup>

$$n = 4\pi e^4 L N_e / m^2, \quad (3)$$

and  $L = \ln(r_D/r_S)$  is the Coulomb logarithm:  $r_S = e^2/mv^2$  is the radius of strong interactions for charged particles, when their potential energy is of the same order as the kinetic energy. This depends, of course, on the electron energy, but this dependence is weak and we, as is usual,<sup>1,3</sup> consider  $L$  as a constant,  $L \approx 10$ . Since the relative change of energy in an  $e$ - $e$  collision is of order 1, the boundary between the plasma and swarm approaches is given by

$$2mv^4/Mln \approx 1. \quad (4)$$

The linear electron swarm approach thus requires that the left side of (4) exceeds unity. Replacing  $mv^2$  by  $kT$ , this inequality can be written in terms of the fractional electron density,

$$N_e/N \lesssim 0.1 (kT)^2 (Sm/Me^4). \quad (5)$$

Using (5) we obtain Table I for the upper limit of  $N_e/N$  in gas plasmas for different electron energies when one may neglect  $e$ - $i$  and  $e$ - $e$  interactions. If  $N_e/N$  exceeds these values, we must use some form of (1) or a hydrodynamic approach. A physical situation where we expect (1) to give a good description of the behavior of  $f(\mathbf{v}, t)$  corresponds to a noble gas, say helium, plasma in a glowing or high-frequency discharge at a temperature below a few electron-volts (see Table II) as long as  $N_e/N \gtrsim 10^{-6}$ .

The assumption that  $a(v)$  is small and the neglect of higher-order terms in the expansion (2) requires some restriction on the electric field intensity.<sup>2</sup> It was found in Ref. 4 that we must have

$$|\mathbf{E}| < E_{\max} = e(N_e/l)^{1/2}. \quad (6)$$

TABLE II. Values of the maximum electric field intensity (V/cm) when the method of this paper is valid. Numbers are given for different gases under usual conditions.

	$P(\text{Torr})$	Ar	Ne	He	H
Low pressure discharge	$10^{-2}$	$3 \times 10^{-2}$	$10^{-2}$	$1.2 \times 10^{-2}$	
Glowing discharge	1	$10^{-1}$	$3 \times 10^{-2}$	$4 \times 10^{-2}$	$5 \times 10^{-2}$
HF discharge	10	3	1	1.2	1.5
Ionosphere, layer E	$10^{-4}$				$1-3 \times 10^{-6}$
Ionosphere, layer F	$10^{-7}-10^{-4}$				$10^{-7}-10^{-5}$

Table II gives the values of  $E_{\max}$  for typical conditions in different plasmas, where we took plasma parameters from Ref. 1 and a mean electron energy of a few electron-volts. The upper bound for the electric field (6) can probably be relaxed, but this needs an exploration of the convergence of the series (2) for this nonlinear problem. In any case our approximations, which are commonly made in the literature,<sup>2,3,6</sup> appear to be reasonable in the situations we have in mind. In particular we believe that our main result about the existence of hysteresis effects (or phase transition) in the stationary distribution does not depend on these approximations.

### A. Mathematical analysis

Accepting the above approximations, substituting (2) into (1), and using the explicit forms of the collision terms given in Ref. 4, we obtain, after some manipulations also described in Ref. 4, the following set of coupled equations for  $\Phi(v, t)$  and  $a(v, t)$ :

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^v u^2 \Phi du - \frac{eE}{3m} v^2 a(v, t) \\ = \frac{m}{M} \left( \frac{v^4}{l} + n\alpha(v) \right) \left[ \Phi + \left( \frac{kT_i}{mv} \right) \frac{\partial \Phi}{\partial v} \right] \\ + \frac{4\pi m^3}{N_e} nA[\Phi(v, t); \Phi(v, t)], \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial a}{\partial t} - \frac{eE}{m} \frac{\partial \Phi}{\partial v} = - \frac{[v^4/l + n\alpha(v)] a(v, t)}{v^3} \\ + \left( \frac{4\pi m^3}{N_e} \right) nB[\Phi(v, t); a(v, t)]. \end{aligned} \quad (8)$$

where  $\alpha$ ,  $A$ , and  $B$  are defined below.

The terms on the right sides of (7) and (8) coming from  $e$ - $i$  collisions are proportional to

$$\alpha(v) = \sqrt{\frac{2}{\pi}} \left( \frac{M}{kT_i} \right)^{3/2} \int_0^v \exp\left( -\frac{Mu^2}{2kT_i} \right) u^2 du, \quad (9)$$

$e$ - $n$  collisions are represented by the terms proportional to  $l^{-1}$ , and the nonlinear functionals  $A$  and  $B$  describe the contributions from  $e$ - $e$  interactions. They are

$$A[\Phi; \Phi] = \Phi(v, t) \int_0^v u^2 \Phi(u, t) du + \frac{1}{3v} \frac{\partial \Phi}{\partial v} \left( \int_0^v u^4 \Phi(u, t) du + v^3 \int_v^\infty u \Phi(u, t) du \right), \quad (10)$$

$$B[\Phi; a] = 2\Phi a + v^{-3}(va' - a) \left[ \int_0^v \left(1 - \frac{u^2}{3v^2}\right) u^2 \Phi du + \frac{2v}{3} \int_v^\infty u \Phi du \right] + \frac{a''}{3v^3} \left( \int_0^v u^4 \Phi du + v^3 \int_v^\infty u \Phi du \right) \\ + \frac{\Phi'}{3} \left[ \int_0^v \left(\frac{u^3}{v^3} - \frac{3u^5}{5v^5}\right) a du + \frac{2}{5} \int_v^\infty a du \right] + \frac{\Phi''}{5} \left( v^{-4} \int_0^v u^5 a du + v \int_v^\infty a du \right). \quad (11)$$

The primes in (11) denote differentiation with respect to  $v$ , and we use the normalization

$$\int \Phi(\mathbf{v}, t) d^3v = \frac{N_e}{m^3}. \quad (12)$$

The function  $\alpha(v)$  in (9) behaves like  $v^3$  when  $v \rightarrow 0$ , but is close to unity almost everywhere on the scale of electron velocities.

The functionals (10) and (11) satisfy the identities

$$\int_0^\infty A[\Phi; \Phi] v dv = 0, \quad \int_0^\infty B[\Phi; a] v^3 dv = 0,$$

following from the nature of the collision integral.<sup>3,7</sup> Multiplying now (7) by  $v$  and integrating over all  $v$  leads to the energy balance equation,

$$\frac{d}{dt} \bar{\epsilon}(t) = - \left( \frac{4\pi m^4}{N_e} \right) \left[ \frac{eE}{3m} \int_0^\infty v^3 a(v, t) dv + \frac{m}{M} \int_0^\infty \left( \frac{v^4}{l} + n\alpha(v) \right) \left[ \Phi + \left( \frac{kT_i}{mv} \right) \frac{\partial \Phi}{\partial v} \right] v dv \right], \quad (13)$$

where

$$\bar{\epsilon}(t) = \frac{m^3}{N_e} \int \frac{mv^2}{2} \Phi(v, t) d^3v.$$

The first term on the right side of (13) is just  $\mathbf{E} \cdot \mathbf{J}$  where  $\mathbf{J}$  is the current and we have used the fact that  $e-e$  collisions conserve energy (and momentum). Equation (13) will play an important role in our analysis.

## II. STATIONARY DISTRIBUTION

We consider first the case where  $\mathbf{E}$  is constant in time and try to find the stationary solutions of (7) and (8) by setting the time derivatives there equal to zero. This is the problem considered in Ref. 4 but unlike what was done there we shall not assume *a priori* that  $\Phi(v)$  is a Maxwellian. An inspection of the time-independent form of (7) and (8) shows that we can use (7) to obtain  $a(v)$  explicitly in terms of  $\Phi$ . Substituting this  $a(v)$  into (8) yields an equation for  $\Phi$  alone. Unfortunately the resulting equation is of cubic order and is very sensitive to small errors in  $\Phi$ . We note further that while we assumed that  $a(v)$  is small, only the linear term on the right side of (7) is multiplied by the small parameter  $m/M$ . The nonlinear term, on the other hand, can be large unless  $\Phi(v)$  is close to a Maxwellian with some temperature  $T$ . To obtain more information about  $\Phi$  without doing any numerical analysis we begin by

making the rough approximation of throwing away the nonlinear terms in (7) and (8). While this is rather inconsistent since the  $e-e$  collision term is of the same order as the  $e-i$  term, we nevertheless note that the  $e-e$  term does not change the electron energy directly and thus this rough approximation does not affect (13). Accepting it for a moment and substituting  $a(v)$  from (7) into (8) we obtain a linear equation for  $\Phi(v)$  whose solution is

$$\tilde{\Phi}(v) = C \exp[-\tilde{U}(v)], \quad (14)$$

with

$$\tilde{U}(v) = \int_0^v \left( \frac{kT_i}{mu} + \frac{cu^5}{(nl+u^4)^2} \right)^{-1} du, \quad (15)$$

and

$$c = (M/3m)(eEl/m)^2.$$

This can be put in the dimensionless form

$$\tilde{U}(q) = \int_0^q \left[ 1 + \left( \frac{y}{q_1} \right)^3 / (1 + 4R_p y^2)^2 \right]^{-1} dy,$$

where

$$q = \frac{mv^2}{2kT_i}, \quad R_i = \frac{(kT_i)^2}{4\pi e^4 L N_e l}, \\ q_1 = \left( \frac{3m}{2M} \frac{\pi e^4 L N_e}{R_i E^2 l} \right)^{1/3}.$$

We plot in Fig. 1  $\tilde{\Phi}$  vs  $q$  for different values of  $R_p$ , taking  $M/2m = 36\,000$  (argon), and  $E = e(N_e/l)^{1/2}$  the maximum field in (6). There are four regions: In region I  $\tilde{\Phi}(v)$  is close to a Maxwellian distribution  $M(v) = \text{const} \times \exp[-mv^2/2kT_i]$  with the ion temperature  $T_i$ . This continues until  $mv^2/2$  reaches the magnitude  $[kT_i(m/3M)(4\pi e^3 N_e L/E^2)]^{1/3}$ . After this (region II)  $\tilde{\Phi}(v)$  is almost  $v$  independent up to  $v = v_1$ , where

$$mv_1^2/2 = \epsilon_1 = (\pi e^4 N_e L l)^{1/2}. \quad (16)$$

In region III  $\tilde{\Phi}(v)$  has the form of the Druyvesteyn distribution<sup>6,8</sup>

$$\Phi_D = \text{const} \times \exp[-(mv^2/2\epsilon_2)^2], \quad \epsilon_2 = \sqrt{M/3meEl}. \quad (17)$$

Finally, in region IV, corresponding to very large velocities,  $\tilde{\Phi}$  again turns into a Maxwellian with temperature  $T_i$ .

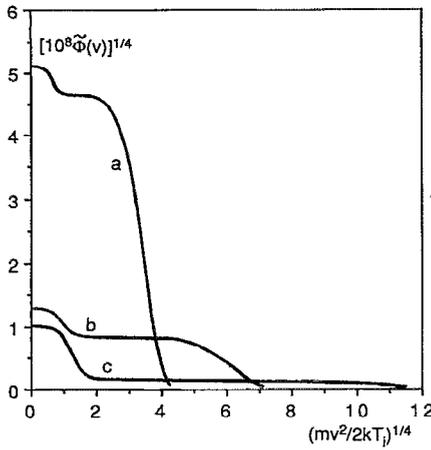


FIG. 1. Plots of the fourth root of  $\tilde{\Phi}(v)$ , the symmetric component of the normalized electron distribution function without  $e-e$  interactions vs  $q = mv^2/2kT_i$ . The parameters  $R_i^{-1/2} = \epsilon_1/kT_i$  and  $q_1$  are (a) 6 and 0.3, (b) 60 and 1.4, (c) 600 and 6.4, respectively. See discussion following Eq. (15) for precise definitions.

The magnitude of  $\tilde{\Phi}(v)$  in region IV is, however, so small that we may consider the distribution as given by  $\Phi_D$  for all  $v > v_1$ .

To understand the form (15) we note that the slow electrons, which gain little energy from the field, are strongly coupled to the ions and therefore have a distribution close to a Maxwellian with the ion temperature. This strong coupling can be seen from (7), where the term responsible for  $e-i$  interaction is proportional to  $\alpha(v) \simeq 1$ , while the  $e-n$  interaction term there behaves like  $v^4$ . Regions III and IV are controlled by  $e-n$  collisions and  $\tilde{\Phi} \sim \Phi_D$ , the usual form obtained from electron swarm theory. Region II represents a transition from ion to neutral-dominated collisions. Note that for  $T_i = 300$  K, and a typical electron-atom collision cross section of order  $10^{-16}$  cm<sup>2</sup>,  $R_i^{-1/2} \simeq 600 \sqrt{N_e/N}$ , so curves (a), (b), and (c) correspond to values  $10^{-4}$ ,  $10^{-2}$ , and 1 for  $N_e/N$ , respectively.

What happens after we turn on the  $e-e$  interaction? Each  $e-e$  collision is very effective in energy exchange and, when the electron density is not very low, they try to bring the electron distribution function close to a Maxwellian with an effective electron temperature  $T$  considerably higher than the ion temperature. This produces a dynamic cooperative effect: there is now a coupling between the slow electron population, with reduced energy transfer from them to the ions, and the faster electrons, which absorb energy from the external field more effectively. This produces a new electron distribution, one that changes from a Maxwellian into a Druyvesteynian form (17) for large electron velocities when the collisions of electrons with the neutral particles become so frequent that it dominates the mutual energy exchange. Using (4) we find that this change occurs at energies

$$\epsilon(v) = \epsilon_1 \sqrt{M/2m}. \quad (18)$$

We now make the basic approximation that  $\Phi(v)$  is very close to a Maxwellian with an *unknown* temperature

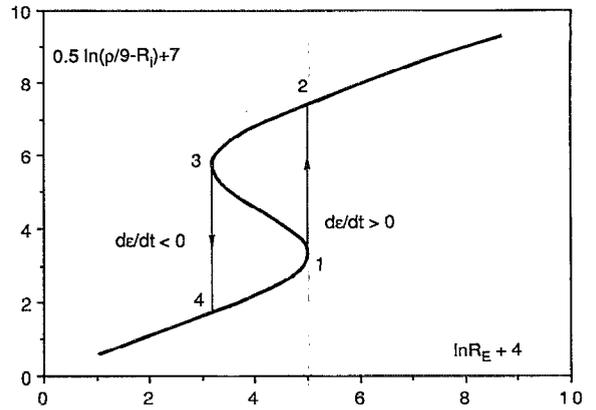


FIG. 2. The dependence of the mean electron energy squared (minus that of the ions) on the electric field intensity when  $R_i = 5 \times 10^{-4}$ .

for energies smaller than (18) and then makes a smooth transition to the Druyvesteyn form. We do this by means of a simple interpolation formula

$$U(v) = \frac{\eta \epsilon}{kT} + \frac{(1-\eta)\epsilon^2}{\epsilon_2^2}, \quad \eta = \left[ 1 + \left( \frac{\sqrt{2m} \epsilon}{M \epsilon_1} \right)^2 \right]^{-1}. \quad (19)$$

We now substitute (19) into (14) and use the resulting  $\Phi(v)$  in (8). We then solve numerically the linear [in  $a(v)$ ] equation (8) for different values of  $T$ . The resulting family  $a(v)$  is then used in the time-independent form of (13) to find a correct self-consistent value of  $T$  in (19). A brief description of the solution technique is given in the Appendix. Note that  $T$  here is not proportional to the mean electron energy. It is only when the field is very small that  $\bar{\epsilon} \sim \frac{3}{2} kT$ .

### A. Results and discussions

Our procedure of solving (7) and (8) differs considerably from the one used in Ref. 4 and the asymptotics of  $\Phi(v)$  and  $a(v)$  are also different. Nevertheless, the new solution produces results similar to the ones found in Ref. 4. In particular it shows in Fig. 2 almost the same dependence of the mean electron energy and current on the electric field intensity (see Figs. 3 and 4) as in Ref. 4, including the fact that the mean electron energy is nonmonotone if the ion temperature is low. This can be seen clearly in Fig. 2 which shows the existence of three different mean energy states  $\bar{\epsilon}(E)$  of the electrons for some range of intensities of the external electric field. The non-single-valued character of  $\bar{\epsilon}(E)$  which presumably corresponds to a sharp, possibly discontinuous, increase in the mean energy at a certain value of  $E$  is surprising.

We can understand this behavior physically by noting the roles of the different terms in Eqs. (7) and (8). In particular the  $i-e$  interaction is dominant at low velocities (the ratio of it to the  $e-e$  interaction and  $e-n$  interaction is, as already noted, proportional to  $v^{-4}$ ). In the absence of  $e-e$  collisions these slow electrons strongly interact with the ions and transfer to them the energy obtained from the field; see Fig. 1 where the distribution is Maxwellian with

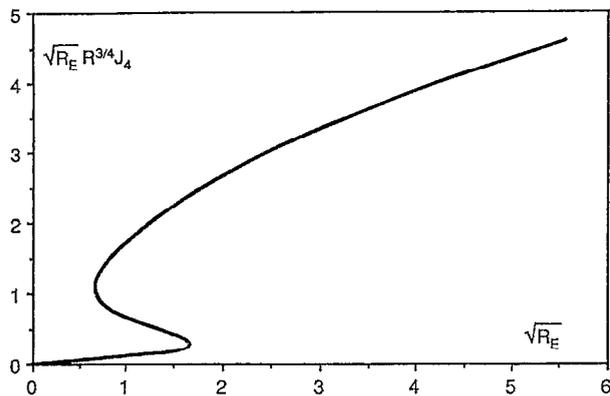


FIG. 3. The current-voltage characteristics of the plasma in dimensionless units:  $R_i = 5 \times 10^{-4}$ ,  $\gamma = 1/36\,000$ .

temperature  $T_i$  in region I. After turning on the  $e-e$  interaction this distribution changes since due to the mutual energy exchange between the electrons the number of slow electrons goes down together with the energy flow to the ions. This means that there are two regimes possible: (i) the weak electric field regime, when the mean electron energy is small and the electrons strongly interact with ions, and (ii) the strong field regime, where the electron energy is high and to some extent decoupled from that of the ions. The transition between these two regimes can then be quite sharp. This interpretation is confirmed by our studies of simple model systems in which the same behavior is found.

In the next section we construct  $\bar{\epsilon}(E)$  in a form which can be realized physically, if Eqs. (7) and (8) adequately describe the real system. This leads to a hysteresis type behavior of  $\bar{\epsilon}(E)$  as the field is changed.

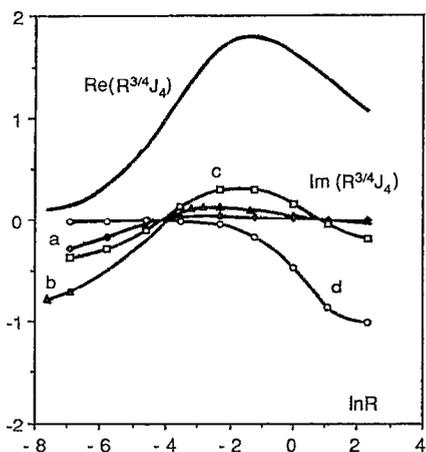


FIG. 4. The real and imaginary parts of the electron conductivity as functions of the mean electron energy in a harmonically oscillating weak electric field for four different frequencies: (a)  $\theta = 0.1$ , (b)  $\theta = 1$ , (c)  $\theta = 10$ , (d)  $\theta = 100$  [ $\theta = \omega\tau_1$ , see (20)]. The real part of the conductivity is almost the same for all these cases. The values of the imaginary components were multiplied by 100, 30, 30, and 10 for curves (a), (b), (c), and (d), respectively.

### III. TIME-DEPENDENT FIELD

We return now to the time-dependent equations (7) and (8) assuming that  $E(t)$  is given. Let us consider first the case when  $E(t)$  changes sufficiently slowly that we can neglect the time derivative  $\partial a/\partial t$  in (8) compared to the other terms there. This will require, at the minimum, that the characteristic time of the field change,  $\tau$ , e.g., its period, is large compared to  $\tau_1$ , where  $\tau_1$  is the maximum value of the reciprocal of the term  $(v/l + n/v^3)$  which multiplies  $a(v,t)$  on the right side of (8); we have set  $\alpha(v) = 1$  there in accordance with the discussion after (12).

A straightforward computation yields

$$\tau_1 \approx (l^3 m^2 / 4\pi e^4 L N_e)^{1/4} \approx l/v_1, \quad (20)$$

where  $mv_1^2 = 2\epsilon_1$  defined in (16). Here  $\tau_1$  can be thought of as the time scale on which  $a(v,t)$  relaxes to a given  $E$  and  $T$ . It can be related to effective or mean times between collisions of electrons with neutrals and ions (see Sec. I and Ref. 1)  $\tau_1 \sim (\tau_{en}^3 \tau_{ei})^{1/4}$ . Note that  $\tau_1$  is independent of the mean electron energy if  $l = \text{const}$  and  $N_e$  is fixed.

When  $\tau \gg \tau_1$  we have the same equation for  $a(v,t)$  as in the stationary problem. We shall assume further that in this adiabatic mode  $\Phi(v,t)$  also has its stationary form (19), but the parameters  $E$  and  $T$  are functions of time,  $E(t)$  being explicitly given while  $T(t)$  and thus also the mean energy  $\bar{\epsilon}(t)$  having to be found via the self-consistent method described in Sec. II for constant  $E$ , but using now the time-dependent form of (13). This yields an ordinary differential equation which, after some manipulations to make it dimensionless, is given in the Appendix [Eq. (A7)].

An analysis of this equation, given in the Appendix, shows that the characteristic relaxation time of the energy to the instantaneous value of the field is given by  $\tau_r \approx (M/4m)\tau_{en}$ . When the mean-free path of an electron is  $2 \times 10^{-2}$  cm, which corresponds to a partial pressure of the neutral component of order 1 Torr, and the mean electron energy is 1–10 eV, then  $\tau_r \sim (1-3) \times 10^{-6}$  sec in an argon plasma. Here  $\tau_1$  depends on the charged particle density like  $N_e^{-1/4}$  [see (20)] and, roughly speaking, is about  $m/M$  times  $\tau_r$ . It means that  $\tau_1 \sim 10^{-9} - 10^{-10}$  sec for the considered conditions.

To see what happens to  $\bar{\epsilon}(t)$  as the field is changed slowly we note that the curve in Fig. 2 represents states in which the electrons are in equilibrium with the field. The region to the left (right) of the curve corresponds to negative (positive) values of  $d\bar{\epsilon}/dt$ . It is easy to see that the part of the curve 1D3 corresponds to an unstable equilibrium. Thus starting from weak fields the mean electron energy will increase as the field intensity increases up to point 1 in Fig. 2. At this point any increase of  $E$  puts the system into the region with  $d\bar{\epsilon}/dt > 0$  and the system, using its own time scale, “jumps” to point 2 in the figure. For larger fields it again increases smoothly. Similar considerations show an analogous behavior when  $E(t)$  is decreasing; a smooth decrease of the electron energy is possible up to point 3 and after point 4, but the energy must jump down between points 3 and 4. Curve  $\bar{\epsilon}(E)$  thus has the

hysteresis-type behavior, and the section corresponding to 1D3 is not available for the electron ensemble in this process. This is true even when the field changes so slowly that  $\tau > \tau_r$ .

Let us consider now briefly the case of rapidly changing  $E(t)$ , i.e.,  $\tau \ll \tau_r$ . We can now assume that the mean electron energy  $\bar{\epsilon}$  does not change during a time of order  $\tau$  and also that the symmetrical term of the distribution function  $\Phi(v)$  has only very small corrections which change much faster than the relaxation time  $\tau_r$ . The main part of  $\Phi(v)$  will again be given by a smooth function of the time having the form (19). This  $\Phi$  will be determined (approximately) by averaging Eq. (7) over a time period which is large compared to  $\tau$ . This will change  $E(t)a(v,t)$  there to some  $\bar{E}(t)a(v,t)$ . We can then use this  $\Phi$  in (8).

To see how this works let us consider the case of a periodic field with a high frequency  $\omega$ :

$$E = E_0 e^{i\omega t}, \quad \omega \gg (\tau_r)^{-1}. \quad (21)$$

We can also write the condition on  $\omega$  in terms of the plasma frequency  $\omega_p = (4\pi e^2 N_e / m)^{1/2}$ ,  $\omega / \omega_p \gg (4m/M)(l_e/l)^{3/4}$ , where  $l_e = N_e^{-1/3}$  is the mean distance between electrons in plasma.

The solution of (8) can then be represented in the form

$$a(v,t) = b(v) e^{i\omega t}. \quad (22)$$

The substitution of (21) and (22) into (8) gives the equation

$$\frac{eE_0}{m} \frac{d\phi}{dv} = \left( \frac{v}{l} + \frac{n\alpha(v)}{v^3} + i\omega \right) b(v) - \frac{4\pi m^3}{N_e} n B[\Phi(v); b(v)]. \quad (23)$$

Equation (23) can be solved in a manner similar to the stationary version of (8), but the calculation requires much more computer time. The joint solution of (23) and (13) allows us to obtain some plasma transport properties at high frequencies both in the stationary and nonstationary states.

From the expression for the electron current density

$$\mathbf{J} = -em^3 \int f(\mathbf{v},t) \mathbf{v} d^3v \equiv \sigma \mathbf{E} \quad (24)$$

we easily obtain the electron conductivity as a function of  $\omega$ :

$$\sigma(\omega) = - \left( \frac{4\pi em^3}{3E_0} \right) \int_0^\infty b(v) v^3 dv. \quad (25)$$

Note that  $b(v)$ , the solution of (23), is now a complex function as well as  $\sigma(\omega)$ . We solved Eq. (23) to linear order in the alternating field but did not assume the mean electron energy to be equal to  $kT_e$ . The results of this calculation are given in Fig. 4.

## IV. SUMMARY

We have investigated the electron velocity distribution of some weakly ionized plasmas in a physically interesting domain where the time evolution of  $f(\mathbf{v},t)$  is governed by the spatially homogeneous nonlinear Boltzmann equation (1). A simple approximate method for solving (1) for the case of a moderately strong time-independent electric field was presented. The primary ingredient in the scheme is the ansatz (19) for the spherical part of the velocity distribution which interpolates between a Maxwellian and a Druyvesteyn form. The temperature of the Maxwellian was then obtained self-consistently from Eq. (13) describing the energy flow in the system.

The method was then generalized to time-dependent fields which vary either very "slowly" or "rapidly" compared to the relevant characteristic times of the system, which we list here:  $\tau_{ee}$ ,  $\tau_{ei}$  and  $\tau_{en}$  are effective mean time intervals between two successive collisions of an electron with electrons, ions, and neutrals, respectively. A time of order  $\tau_{ee}$  is sufficient<sup>1</sup> for the electron "Maxwellization" with their own temperature;  $\tau_1 = (\tau_{en}^3 \tau_{ei})^{1/4}$  is the relaxation time for the nonsymmetrical part of the distribution function  $a(v,t)$ ;  $\tau_r \sim (M/4m)\tau_{en}$  is the relaxation time of  $\Phi(v,t)$ , that is, the time for getting the mean electron energy in equilibrium with the applied field. We always have  $(\tau_r/\tau_1) = (M/4m)(\tau_{en}/\tau_{ei})^{1/4} \gg 1$  even for fast electrons, because of the very large factor  $M/4m$ .

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## APPENDIX: METHOD OF SOLUTION

We outline briefly the technique of solving (8) and the determination of the mean electron energy as a function of the electric field intensity for the stationary problem. We define dimensionless variables and parameters as in Ref. 4,

$$x = v \sqrt{m/kT}, \quad R = R(T) = (kT)^2 / (4\pi e^4 L N_e l),$$

$$R_i = R(T_i), \quad s = 4\pi e^3 L N_e / kTE, \quad (A1)$$

$$R_E = (M/12m)(eEl)^2 / 4\pi e^4 L N_e l,$$

and also the dimensionless functions

$$a(v) = -(x/s)\Phi\phi(x), \quad (A2)$$

$$\Phi(x) = e^{-U(x)}. \quad (A3)$$

The time-independent equation (8) can then be written in the form

$$\begin{aligned} \varphi [I_2(1+Rx^4) - 2x^3\Phi + QY - WZ] + \left(2xYZ - \frac{Q}{x}\right) \frac{d\varphi}{dx} \\ - Z \frac{d^2\varphi}{dx^2} + \frac{j_4Y}{3} - \frac{j_6(W+Z)}{5x^2} + \frac{x^3(2Y/3 - W)F}{5} \\ = x^3YI_2, \end{aligned} \quad (\text{A4})$$

Here we have used the abbreviations

$$\begin{aligned} Y &= \frac{1}{x} \frac{dU}{dx}, \quad W = \left(\frac{dU}{dx}\right)^2 - \frac{d^2U}{dx^2}, \\ Z &= \frac{1}{3}[i_4(x) + x^3f(x)], \quad Q = Z + x^2i_2(x) + x^3f(x), \\ i_n(x) &= \int_0^x \Phi(y)y^n dy, \quad f(x) = \int_x^\infty \Phi(y)y dy, \quad (\text{A5}) \\ j_n(x) &= \int_0^x \Phi(y)\varphi(y)y^n dy, \\ F(x) &= \int_x^\infty \Phi(y)\varphi(y)y dy, \\ I_n &= i_n(\infty), \quad J_n = j_n(\infty), \quad n=1,2,3,\dots \end{aligned}$$

Note that (A4) and (A5) do not use any particular form of  $U(x)$  and that  $T$  does not represent the electron energy in a relatively strong electric field when the distribution does not look like a Maxwellian. The mean electron energy is calculated from (14), (19), and (A3) to be

$$\bar{\epsilon} = kTI_4/2I_2,$$

and we define the dimensionless representative of the mean energy by

$$\rho = (\bar{\epsilon}/\epsilon_1)^2 = (I_4/I_2)^2 R. \quad (\text{A6})$$

Note that  $\rho = 9R$  for the Maxwellian distribution.

Equation (13) can be rewritten now as

$$\begin{aligned} \frac{d\rho}{dy} = \tau I_4 R (\tau I_2^2)^{-1} [4R_E J_4 - I_5 + 4I_3 \sqrt{R_i/R} \\ - R^{-1}(I_1 - \sqrt{R_i/R})], \end{aligned} \quad (\text{A7})$$

where  $y = t/\tau$  and

$$\tau_r = (M/4m)l\sqrt{m/kT}. \quad (\text{A8})$$

For the stationary state we set  $d\rho/dy = 0$  in (A7). This leads to a relation between the parameters introduced earlier,

$$R_E = [I_5 - 4I_3 \sqrt{R_i/R} + R^{-1}(I_1 - \sqrt{R_i/R})]/4J_4. \quad (\text{A9})$$

We want to solve (A4), find  $\varphi(x)$ , and use (A9) to find  $\bar{\epsilon} = \bar{\epsilon}(E)$ . To do this we now introduce our approximation (19),

$$U(x) = 0.5(x^2 + \gamma R^2 x^8/8R_E)(1 + \gamma R x^4)^{-1}, \quad (\text{A10})$$

which corresponds to (19) where we have set  $\gamma = 2m/M$ . The solution was done by the spline-collocation method.<sup>9</sup> The calculation procedure requires an interval  $0 \leq x \leq X$  to be given as well as  $\varphi(x)$  and  $d\varphi/dx$  at the ends of the interval. We must be able to determine the functions  $I_n$  and

$J_n$  in order to solve Eq. (A7) and calculate the physical averages afterwards. Practically satisfactory accuracy can be reached if we assume  $\Phi(x) = 0$  after  $U(x) \simeq 20$ . This gives the upper limit  $x = X$ . The analysis of (A4) shows that

$$\varphi(0) = \frac{d\varphi}{dx}(0) = 0, \quad (\text{A11})$$

$$\varphi(X) \rightarrow X^2 U'(X) / [1 + RX^4 + XU'(X) - I_4 W/3I_2],$$

when  $X$  is large. The calculations are not sensitive to the magnitude of the derivative of  $\varphi(X)$ . Differentiating (A11) or using  $\varphi'(X) \simeq -\varphi(X)/X$  or  $\varphi'(X) = 0$  made almost no difference in the result. Comparing (A11) with the calculation of asymptotics in Ref. 4 we find that the present way is simpler and does not require preliminary numerical experiments.

The computational program calculates the single-value function  $R_E(\rho)$  iteratively. We fixed  $R_i = 5 \times 10^{-4}$  and chose some  $R$ . We then start the calculation by taking some arbitrary  $R_E = \xi_1$  in (A10), solve (A4), find  $I_1, I_3, I_5, \varphi(X)$ , and then calculate  $\varphi(x)$  and  $J_4$ . These values are substituted in (A9) to obtain a new  $R_E$ , say  $\eta_1$ . If  $\eta_1 \neq \xi_1$ , the calculation is repeated with this  $R_E = \xi_2$ , where  $\xi_2 = \eta_1$  and we obtain for  $R_E$  a new value  $\eta_2$  by (A9). If  $\eta_2 \neq \xi_2$ , we use the linear interpolation formula

$$\xi_{k+1} = (\eta_{k-1}\xi_k - \xi_{k-1}\eta_k) / (\xi_k - \xi_{k-1} - \eta_k + \eta_{k-1})$$

for  $R_E = \xi_3$  and carry out a new calculation, etc. It was usually enough to use a few iterations to get the discrepancy in  $R_E$  to less than 1%. We then obtain the function  $R_E(\rho)$  and find that its behavior is insensitive to the exact value of  $\gamma$  with  $R_E \sim \rho$  for  $\rho \gg 100$ .

We took  $\gamma^{-1} = 36\,000$  for this calculation corresponding to an argon plasma. The location of the transition from a Maxwellian to a Druyvesteyn-type distribution

$$mv^2/2\epsilon_1 \simeq \sqrt{M/2m}$$

is only qualitative, we also made the calculation taking this ratio to be three times smaller ( $\gamma^{-1} = 4000$ ). In this case the distribution function is "less" Maxwellian, the temperature  $T$  as well as  $R$  become worse characteristics, and we have more problems satisfying (A9). Still, there are almost no changes in the form of the curve  $R_E(\rho)$  given in Fig. 2; the difference does not exceed 1% when we take  $\gamma = 0, 1/36\,000$ , and  $1/4000$ .

In the case of rapidly varying fields we solve the dimensionless version of (25), which is very similar to (A4) with only one difference:

$$1 + Rx^4 \rightarrow 1 + Rx^4 + i\theta R^{3/4} x^3,$$

where

$$\theta = \omega\tau_1 \simeq \omega/\omega_p (l/l_e)^{3/4}, \quad (\text{A12})$$

and  $\theta \cdot R^{3/4}$  is independent of the neutral particles density. When we consider moderately strong fields  $E$ , we can obtain a self-consistent solution as before by using instead of (A9) a slightly different equation:

$$\overline{R_E J_4} = (1/4) [I_5 - 4I_3 \sqrt{R_i/R} + R^{-1} (I_1 - \sqrt{R_i/R})]. \quad (\text{A13})$$

Here

$$\overline{R_E J_4} = \frac{1}{2} R_E \int_0^\infty x^4 \operatorname{Re}[\Phi(x)] dx$$

is obtained after we separate the real and imaginary components of  $b(v)$  so that Eq. (A13) involves only real variables. This allows us to find  $R$ ,  $T$ , and therefore  $\bar{\epsilon}(E_0)$  for a given field. These parameters are also necessary for the calculation of conductivity by (25), which depends on the electric field intensity in this nonlinear regime. The complex plasma conductivity (25) can be written in these notations in the form

$$\sigma(\omega) = e(N_e l)^{3/4} R^{3/4} J_4 / I_2 [3(4\pi L)^{1/4} \sqrt{m}],$$

where only  $J_4(\omega)$  is complex.

- <sup>1</sup>V. E. Golant, A. P. Zhylinsky, and I. E. Sakharov, *Fundamentals of Plasma Physics* (Wiley, New York, 1980).
- <sup>2</sup>A. V. Phelps and L. C. Pitchford, *Phys. Rev. A* **31**, 2392 (1985).
- <sup>3</sup>E. M. Lifshitz and L. P. Pitaevsky, *Physical Kinetics* (Pergamon, New York, 1981).
- <sup>4</sup>A. Rokhlenko, *Phys. Rev. A* **43**, 4438 (1991).
- <sup>5</sup>F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum, New York, 1984).
- <sup>6</sup>I. P. Shkarofsky, T. W. Johnston, and M. P. Bachynsky, *The Particle Kinetics of the Plasmas* (Addison-Wesley, Reading, MA, 1966).
- <sup>7</sup>R. L. Liboff, *Introduction to the Theory of Kinetic Equations* (Wiley, New York, 1969); R. L. Liboff, *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions* (Prentice-Hall, Englewood Cliffs, NJ, 1990).
- <sup>8</sup>R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience, New York, 1962); *Transport Processes in Plasmas* (North-Holland, Amsterdam, 1988), Vols. I and II.
- <sup>9</sup>Y. S. Zav'yalov, B. I. Kvasov, and L. M. Miroshnichenko, *The Methods of Spline-Functions* (Nauka, Moscow, 1980) (in Russian).