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Existence and Positivity of Solutions of a Fourth-Order Nonlinear PDE Describing Interface Fluctuations

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Abstract

We study the partial differential equation

$$w_t = -w_{xxxx} + \left(\frac{w_x^2}{w}\right)_{xx},$$

which arose originally as a scaling limit in the study of interface fluctuations in a certain spin system. In that application x lies in $\mathbb{R}$, but here we study primarily the periodic case $x \in S^1$. We establish existence, uniqueness, and regularity of solutions, locally in time, for positive initial data in $H^1(S^1)$, and prove the existence of several families of Lyapunov functions for the evolution. From the latter we establish a sharp connection between existence globally in time and positivity preservation: if $[0, T^*)$ is a maximal half open interval of existence for a positive solution of the equation, with $T^* < \infty$, then $\lim_{t\to T^*} w(t,\cdot)$ exists in $C^1(S^1)$ but vanishes at some point. We show further that if $T^* > (1+\sqrt{3})/16\pi^2\sqrt{3}$ then $T^* = \infty$ and $\lim_{t\to\infty} w(t,\cdot)$ exists and is constant. We discuss also some explicit solutions and propose a generalization to higher dimensions.

1. Introduction

In this paper we discuss the evolution of a function $w(t,x)$ under the partial differential equation

$$w_t = -w_{xxxx} + \left(\frac{w_x^2}{w}\right)_{xx}, \tag{1.1}$$

for $t > 0$, subject to an initial condition

$$w(0,x) = g(x)\,. \tag{1.2}$$

We take $x \in X$, where X is either the real line $\mathbb{R}$ or the circle S^1 parameterized by a variable x satisfying $0 \leqq x \leqq 1$. Equation (1.1) may also be written as

$$w_t = -\left(w\,(\log w)_{xx}\right)_{xx}. \tag{1.3}$$

We will occasionally find it convenient to introduce a new dependent variable by setting $w = y^2$; $y = y(t,x)$ then satisfies the equation

$$y_t = -y_{xxxx} + \frac{y_{xx}^2}{y}. \tag{1.4}$$

Equation (1.1) arose (see [2], [3]) in the study, via a "collective variable approximation," of a family of random variables $\{M_n \mid n = 1, 2, \ldots\}$ associated with a certain spin system. Each M_n takes integer values and has probability mass function W_n, that is, $W_n(m) = \Pr\{M_n = m\}$, $m \in \mathbb{Z}$. The functions W_n are determined by a recursion relation (together with the initial condition $W_n(m) = \delta_{mn}$) which we give for completeness in the Appendix but whose details will not concern us here. If for large n we approximate W by a scaled continuous function, $W_n(m) \simeq w(\varepsilon^4 n, \varepsilon m)$, then up to a constant time rescaling the partial differential equation (1.1) is the formal $\varepsilon \to 0$ limit of the recursion relation; see [2].

The recursion for the functions W_n manifestly preserves two characteristic properties of probability mass functions: positivity (either strict positivity or non-negativity) and normalization ($\sum_{m\in\mathbb{Z}} W_n(m) = 1$). Clearly (1.1) also preserves normalization: $\int_X w\,dx$ is constant (assuming, if $X = \mathbb{R}$, that suitable boundary conditions are imposed). We have not been able to show that (1.1) is positivity preserving, but believe on several grounds that this is probably so. First, it is the scaling limit of a positivity preserving discrete recursion. Second, the form (1.4) of the equation is suggestive: at a small quadratic minimum of y the nonlinear term in (1.1) will be large and positive. Finally, we have solved (1.1) numerically, with $X = S^1$, for several positive initial conditions, and verified positivity preservation in each case. We observe from the numerical solutions, however, that the evolution does not preserve order ($g_1 > g_2$ does not imply $w_1(t,x) > w_2(t,x)$ for all t, x) and does not yield monotonic behavior of the minimum: a quartic minimum initially decreases, while a higher order minimum bifurcates into two smaller minima separated by a relative maximum. In Figure 1 we plot some solutions illustrating this behavior.

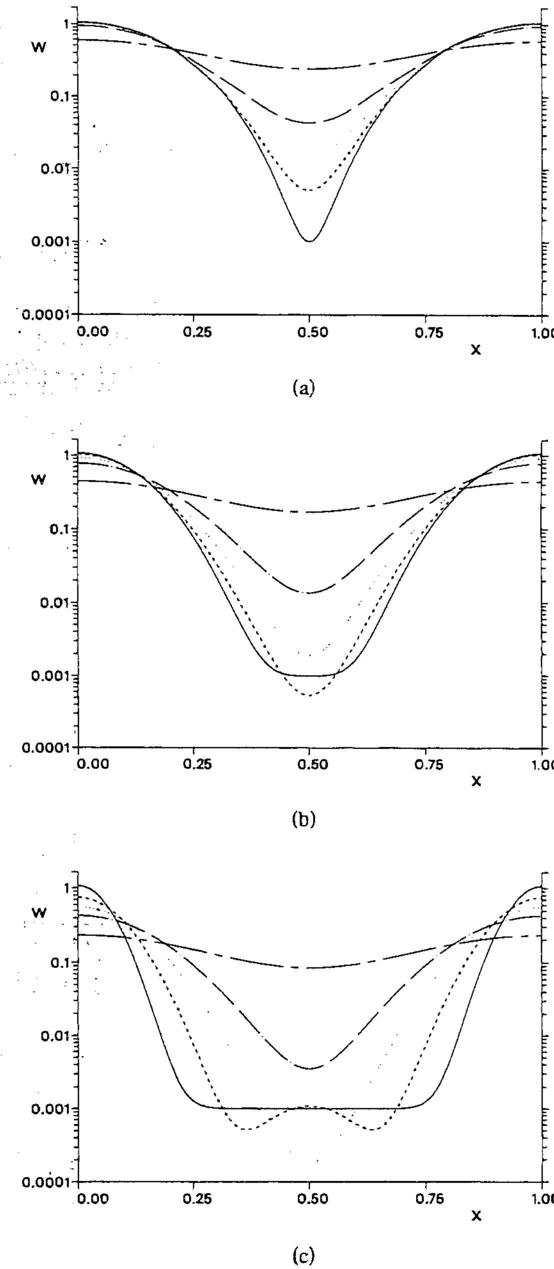

Figure 1. Solutions of equation (1.1) for three different initial conditions. In each case, $w(t)$ is plotted for $t = 0.0$ (solid), $t = 8.0 \times 10^{-6}$ (dashed), $t = 3.2 \times 10^{-5}$ (dotted), $t = 1.0 \times 10^{-4}$ (dash-dot), and $t = 7.2 \times 10^{-4}$ (dash-dash). Here $w(0,x) = \left(\varepsilon^{1/2} + [(1+\cos 2\pi t)/2]^m\right)^2$, with $\varepsilon = 0.001$ and $m = 1, 2$, and 8 in Figures 1 (a), (b), and (c), respectively.

Although the scaling limit discussed above leads to (1.1) with $X = \mathbb{R}$, we have established rigorous control of the behavior of solutions only for $X = S^1$. In this case we prove, in Section 4, existence (locally in time) and uniqueness for solutions of (1.1) in the space $H^1(S^1)$, with positive initial conditions; we establish also the regularity of solutions and determine the asymptotic behavior as $t \to \infty$ when the solution exists for all time. Two questions remain unanswered: whether in general solutions must exist globally in time, and whether the evolution is positivity-preserving. We show in Section 5 that these questions are related by proving that the vanishing of the solution is the only possible impediment to global existence: if $[0, T^*)$ is the maximal interval of existence of some positive solution w of the initial value problem, and $T^* < \infty$, then $w(T^*) \equiv \lim_{t \nearrow T^*} w(t)$ exists and is in fact continuously differentiable, but vanishes at some point of S^1.

In Section 2 we discuss briefly a few explicit solutions of (1.1). Control of $\lim_{t \nearrow T^*} w(t)$ is obtained from certain Lyapunov functions for the evolution, which we discuss in Section 3. Finally, Section 6 describes some extensions of our results to a higher dimensional version of (1.1).

Our problem bears some resemblance to one arising in a recent study (see [1]) of droplet breakup in a Hele-Shaw cell. There the thickness h of a neck between two masses of fluid is described by the fourth-order equation $h_t = -(hh_{xxx})_x$, and the questions of global existence and positivity preservation are again closely related.

2. Explicit Solutions

We can readily identify some explicit solutions of (1.1). It should be noted that additional solutions can be derived from these through the invariance of the equation under the transformations

$$\begin{aligned} &w \to Aw\,, \qquad A \neq 0\,;\\ &x \to x - x_0\,;\\ &t \to t - t_0\,;\\ &x \to ax\,, \quad t \to a^4 t\,, \quad a \neq 0\,. \end{aligned}$$

The Gaussian solution

$$w(t,x) = t^{-1/4}\exp(-x^2/4t^{1/2})\,, \qquad x \in \mathbb{R}\,,$$

is the relevant solution for the original application to the spin system: the random variables M_n in that case are approximately Gaussian, with variance growing as $n^{1/2}$. A related solution is

$$w(t,x) = t^{-1/4}(x^2/t^{1/2})\exp(-x^2/4t^{1/2})\,.$$

It was pointed out to us by M. Kruskal, [9], that general scaling solutions of the form $w(t,x) = t^{\alpha}\Psi(xt^{-1/4})$ may be found by requiring $\Psi(\xi)$ to satisfy the ordinary differential equation

$$\Psi^{(4)} - \left(\frac{\Psi'^2}{\Psi}\right)'' - \frac{\xi}{4}\Psi' + \alpha\Psi = 0\,.$$

Stationary solutions include 1, $\sin^2 x$, $\sinh^2 x$, $\cosh^2 x$, e^x, and x^2. A. Rokhlenko, [11], has noted that there are traveling solutions of the form $w(t,x) = u(x+t)$, where u is the square of an Airy function.

3. Lyapunov Functionals

In this section we discuss the existence of certain functionals $\Phi(f)$, where f is a function defined on X, such that when w is a solution of (1.1), $\Phi(w(t,\cdot))$ varies monotonically in time or is constant. We will refer to such functionals as *Lyapunov functionals* and *conserved quantities*, respectively. It will be clear from our proofs that this monotonicity can be verified whenever (1.1) holds in a sufficiently strong sense to justify various formal manipulations and, in the case $X = \mathbb{R}$, w and its derivatives vanish sufficiently rapidly at infinity to justify neglect of boundary terms in integration by parts. To make a precise statement, however, we will restrict ourselves to the case $X = S^1$. We begin by establishing some notation to be used here and in the next section.

For $1 \leqq p \leqq \infty$ and r a non-negative integer we let $W^{r,p} \equiv W^{r,p}(S^1)$ denote the space of functions on S^1 with finite norm

$$\|f\|_{W^{r,p}} = \begin{cases} \left[\displaystyle\int_0^1 \sum_{j=0}^{r} |f^{(j)}(x)|^p\,dx\right]^{1/p}, & \text{if } p < \infty\,,\\ \displaystyle\sup_{x\in S^1, 0\leqq j\leqq r} |f^{(j)}(x)|\,, & \text{if } p = \infty\,. \end{cases}$$

In particular, we write $L^p \equiv W^{0,p}$ and $H^r \equiv W^{r,2}$, and let C^r denote the subspace of $W^{r,\infty}$ consisting of r-times continuously differentiable functions. Let Y^r denote either C^r or $W^{r,p}$ for some fixed p. For I an interval of real numbers we let $C^m(I;Y^r)$ denote the set of functions $u : I \to Y^r$ with continuous (strong) derivatives of order m; if I is closed then $C^m(I;Y^r)$ is a Banach space with norm

$$\|u\|_{C^m(I;Y^r)} = \sup_{t\in I, 0\leqq k\leqq m} \left\|u^{(k)}(t)\right\|_{Y^r}\,.$$

We may omit the superscript m when $m = 0$. We let Y^r_+ denote the set of elements of Y^r which are (pointwise) strictly positive and $C^m_+(I;Y^r)$ the elements of $C^m(I;Y^r)$ with images contained in Y^r_+. Finally, if I is an open interval of real numbers, a *classical solution* of (1.1) in Y^r is an element of $w \in C^1_+(I;Y^r) \cap C(I;Y^{r+4})$ satisfying (1.1).

PROPOSITION 3.1. *If Y^r is as above and w is a (positive) classical solution of (1.1) in Y^0, then*

(a) $\int_{S^1} w(t,x)^\beta\, dx$ *is nondecreasing in time if* $0 < \beta < 1$, *is constant if* $\beta = 1$, *and is nonincreasing if* $1 < \beta < 3/2$; *moreover,*

(b) $\int_{S^1} \log(w(t,x))\, dx$ *and* $-\int_{S^1} w \log(w(t,x))\, dx$ *are nondecreasing in time.*

If $w(t,x)$ *is also a classical solution of* (1.1) *in* Y^1, *then*

(c) $\int_{S^1} \left[w_x(t,x)^2/w(t,x)\right]^\alpha dx$ *is nonincreasing in time if* $1 \leqq \alpha \leqq 3/2$.

Proof: Each functional $\Phi(w(t))$ referred to in Proposition 3.1 is of the form $\Phi(w(t)) = \int_{S^1} \phi(w(t,x), w_x(t,x))\, dx$, and the strong differentiability condition, together with one integration by parts in case (c), guarantees that $\Phi(w(t))$ is differentiable and satisfies

$$\frac{d}{dt}\Phi(w(t)) = \int_{S^1} \frac{\delta\phi}{\delta w}(w, w_x) w_t\, dx\,, \tag{3.1}$$

where $\delta\phi/\delta w = \partial\phi/\partial w - (\partial\phi/\partial w_x)_x$. Applying this to $\phi(w) = w^\beta$, using (1.1), and integrating twice by parts yields

$$\begin{aligned}\frac{d}{dt}\int_{S^1} w^\beta\, dx &= -\beta \int_{S^1} (w^{\beta-1})_{xx}\left(w_{xx} - \frac{w_x^2}{w}\right) dx \\ &= -\beta(\beta-1)\int_{S^1} w^{\beta-2}\left(w_{xx}^2 + (\beta-3)\frac{w_{xx}w_x^2}{w} - (\beta-2)\frac{w_x^4}{w^2}\right) dx\,.\end{aligned} \tag{3.2}$$

If to (3.2) we add $\beta(\beta-1)$ times the equation

$$0 = \frac{\beta-3}{9}\int_{S^1} (w_x^3 w^{\beta-3})_x\, dx = \frac{\beta-3}{9}\int_{S^1} w^{\beta-2}\left(3\frac{w_{xx}w_x^2}{w} + (\beta-3)\frac{w_x^4}{w^2}\right) dx\,,$$

we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{S^1} w^\beta\, dx \\ &= -\beta(\beta-1)\int_{S^1} w^{\beta-2}\left(w_{xx}^2 + \frac{2}{3}(\beta-3)\frac{w_{xx}w_x^2}{w} - \frac{1}{9}(\beta^2+3\beta-9)\frac{w_x^4}{w^2}\right) dx\,;\end{aligned}$$

the integrand is a quadratic form in the variables w_{xx} and w_x^2/w which is positive semi-definite for $0 \leqq \beta \leqq 3/2$. Part (a) of Proposition 3.1 follows at once.

The monotonicity of $\int_{S^1} \log w\, dx$ follows from the remark that the time derivative of this quantity is obtained from (3.2) by omitting the factor β and then setting $\beta = 0$; from this point on, the argument is a special case of the one above. Monotonicity of $\int_{S^1} w \log w\, dx$ is obtained easily by using the form (1.3) of w_t in (3.2).

To verify (c) it is most convenient to work in terms of the variable $y = w^{1/2}$; since w is strictly positive, y will be a strong solution of (1.4) in Y^1. Note that $w_x^2/w = y_x^2$, so that from (1.4) and the analogue of (3.1),

$$\begin{aligned}&\frac{d}{dt}\int_{S^1} (y_x^2)^\alpha\, dx \\ &\quad= -2\alpha\int_{S^1} ((y_x^2)^{\alpha-1}y_x)_x\left(-y_{xxxx} + \frac{y_{xx}^2}{y}\right) dx \\ &\quad= -2\alpha\int_{S^1}\left[((y_x^2)^{\alpha-1}y_x)_{xx}\, y_{xxx} - (y_x^2)^{\alpha-1}y_x\left(\frac{y_{xx}^2}{y}\right)_x\right] dx \\ &\quad= -2\alpha\int_{S^1}(y_x^2)^{\alpha-1}\left[(2\alpha-1)y_{xxx}^2 - 2y_{xxx}\frac{y_{xx}y_x}{y} + \left(\frac{y_{xx}y_x}{y}\right)^2\right] dx \\ &\qquad - 4\alpha(2\alpha-1)(\alpha-1)\int_{S^1}(y_x^2)^{\alpha-2}y_x y_{xx}^2 y_{xxx}\, dx\,.\end{aligned}$$

The integrand of the first term is, for $\alpha \geqq 1$, a positive semi-definite quadratic form in the variables y_{xxx} and $y_{xx}y_x/y$; the second term, after one more integration by parts, becomes

$$\frac{4}{3}\alpha(2\alpha-1)(\alpha-1)(2\alpha-3)\int_{S^1}(y_x^2)^{\alpha-2}y_{xx}^4\, dx\,,$$

which is non-positive for $1 \leqq \alpha \leqq 3/2$.

Remark 3.2. The origin of the of the evolution equation (1.1), discussed in the Introduction, and the exact conservation of $\int_X w\, dx$, make it natural to view w as a probability density function. In this case two of the Lyapunov functionals of Proposition 3.1 are familiar: $-\int_X w \log w\, dx$ is the *entropy* and $\int_X (w_x^2/w)\, dx$ the *Fisher information* of the probability distribution.

4. The Evolution on S^1

We now turn to the existence theory for the initial value problem (1.1)–(1.2) posed in $X = S^1$. As discussed in Remark 4.1 below, we will assume throughout this section that the initial value g in (1.2) lies in H_+^1. Equations (1.1)–(1.2) are formally equivalent to the integral equation

$$w(t) = e^{-At}g + \int_0^t e^{-A(t-s)}F(w(s))_{xx}\, ds\,, \tag{4.1}$$

where $A = \partial_x^4$ and $F(f) = f_x^2/f$, which may be written as

$$w(t) = e^{-At}g + \partial_x^2\int_0^t e^{-A(t-s)}F(w(s))\, ds\,. \tag{4.2}$$

By a *mild* solution of the initial value problem (1.1)–(1.2) on an interval $[0,T]$ we shall mean an element of $C_+([0,T];H^1)$ which satisfies (4.2); we write $T^* = T^*(g)$ when $[0,T^*)$ is the maximal half-open interval such that a mild solution exists on every closed subinterval $[0,T] \subset [0,T^*)$.

Remark 4.1. e^{-At} is an analytic semigroup on L^2 and (1.1) may be written as

$$\frac{dw}{dt} + Aw = \tilde{F}(w) ,$$

where $\tilde{F}$ maps H^3_+ to H^0. Since H^3 is the domain of $A^{3/4}$, standard theory (see [5] and [10]) assures us that for initial data $g \in H^3_+$ the integral equation (4.1) has a unique solution on some interval $[0,T]$, which is in fact a classical solution in H^0. In our discussion in Section 5 of the relation between global existence and strict positivity of the solution, however, we will need to know the existence of solutions for initial data with only one derivative — specifically, for data in C^1. In this section we take $g \in H^1$ since this slight additional generality does not complicate the proofs. It has been pointed out to us by T. Kato, [8], that solutions in fact exist for $g \in W^{1,1}$, but we will not write down this generalization. Our method is closely related to that of [5] and [10] and also to that of [6] and [4]; see also [7].

The next theorem and its corollary summarize our results.

THEOREM 4.2. *Suppose that $g \in H^1_+$. Then the following hold.*

(a) LOCAL EXISTENCE AND UNIQUENESS OF A MILD SOLUTION: *For some $T > 0$ there exists a unique $w \in C_+([0,T];H^1)$ satisfying* (4.2).

(b) REGULARITY OF MILD SOLUTIONS: *If $w \in C_+([0,T];H^1)$ satisfies* (4.2) *then $w \in C((0,T];H^r)$ for all r; moreover, w is a classical solution of* (1.1) *in H^r for any r.*

COROLLARY 4.3. LYAPUNOV FUNCTIONALS: *If $g \in H^1_+$ and $w \in C_+([0,T];H^1)$ is a mild solution of the initial value problem* (1.1)–(1.2), *then the* (*monotonicity*) *conclusions of Proposition* 3.1 *hold on the interval* $(0,T^*(g))$.

We begin our discussion with a series of lemmas, closely related to standard tools in the theory of analytic semigroups. In stating them we introduce the usual fractional powers A^ρ of the operator A. (For a general definition of A^ρ when A is the generator of an analytic semigroup; see, e.g., [10], here A^ρ is given by multiplication of the k-th Fourier coefficient by $(2\pi|k|)^{4\rho}$.)

LEMMA 4.4. *Suppose that $1 \leqq p \leqq q \leqq \infty$, that j is a non-negative integer, and that $\rho \geqq 0$. Then for $t > 0$,*

$$\|A^\rho \partial_x^j e^{-At} f\|_{L^q} \leqq C t^{-(4\rho+j+1/p-1/q)/4} \|f\|_{L^p} . \tag{4.3}$$

Moreover, if $4\rho + j + 1/p - 1/q < 4$ and $u \in C([0,T];L^p)$, then

$$\|A^\rho \partial_x^j \int_0^t e^{-A(t-s)}u(s)\,ds\|_{L^q} \leqq C\|u\|_{C([0,T];L^p)} t^{1-(4\rho+j+1/p-1/q)/4} . \tag{4.4}$$

Here C is a constant depending on ρ, j, p, and q.

Proof: Equation (4.4) is an immediate consequence of (4.3). To prove the latter we first suppose that $\rho = 0$. The operator e^{-At} has kernel $G(x-y;t)$ with

$$G(\xi;t) = \sum_{k\in\mathbb{Z}} t^{-1/4} H\left(\frac{\xi+k}{t^{1/4}}\right),$$

where

$$H(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-p^4} e^{ip\xi}\,dp .$$

Thus $\|\partial_\xi^j G(\xi;t)\|_{L^r} = t^{-(j+1-1/r)/4}\|H\|_{L^r(\mathbb{R})}$, and (4.3) follows from Young's inequality by taking $1/r + 1/p = 1 + 1/q$. The case $p = q$, $j = 0$ of (4.3) is a consequence (see [10]) of the fact that e^{-At} is an analytic semigroup on L^p, and the general result then follows by writing $A^\rho \partial_x^j e^{-At} = A^\rho e^{-At/2} \partial_x^j e^{-At/2}$.

In stating the next lemmas we will use the following notation. For Y any Banach space of functions on S^1 and $0 < \rho < 1$ we let $C^\rho(I;Y)$ denote the elements of $C(I;Y)$ which are Hölder continuous of order ρ on I, and $C^{m+\rho}(I;Y)$ the elements u of $C^m(I;Y)$ such that $u^{(m)} \in C^\rho(I;Y)$.

LEMMA 4.5. *Suppose that $u \in C([0,T];L^p)$ for some p, $1 \leqq p \leqq \infty$, and that $z(t) = \partial_x^2 \int_0^t e^{-A(t-s)}u(s)\,ds$. Then $z \in C^\rho([0,T];W^{1,q})$, with Hölder constant depending only on ρ, p, q, and $\|u\|_{C([0,T];L^p)}$, whenever ρ and q satisfy $p \leqq q \leqq \infty$ and $4\rho + 1/p - 1/q < 1$.*

Proof: This is a standard argument from analytic semigroup theory; we recall the estimate (e.g., [10])

$$\|(e^{-At} - I)f\|_{L^q} \leqq Ct^\rho \|A^\rho f\|_{L^q} , \tag{4.5}$$

valid when $0 < \rho \leqq 1$ and $f \in L^q$ is in the domain of A^ρ. Now for $t, \tau > 0$ and $t + \tau \leqq T$ we write

$$\begin{aligned} z(t+\tau) - z(t) &= (e^{-A\tau} - 1)\partial_x^2 \int_0^t e^{-A(t-s)}u(s)\,ds \\ &\quad + \partial_x^2 \int_t^{t+\tau} e^{-A(t+\tau-s)}u(s)\,ds \\ &\equiv z_1 + z_2 . \end{aligned}$$

From (4.5) and (4.4),

$$\|z_{1,x}(t)\|_{L^q} \leqq C\tau^\rho \left\| A^\rho \partial_x^3 \int_0^t e^{-A(t-s)} u(s)\, ds \right\|_{L^q}$$
$$\leqq C\tau^\rho T^{1-(4\rho+1/p-1/q)/4} \|u\|_{C([0,T];L^p)} ,$$

and from (4.4),

$$\|z_{2,x}(t)\|_{L^q} \leqq C\tau^{(1-1/p+1/q)/4} \|u\|_{C([0,T];L^p)} .$$

LEMMA 4.6. *Suppose that $u \in C^\rho([0,T];L^p)$ for some ρ, p satisfying $0 < \rho < 1$ and $1 \leqq p \leqq \infty$, and that $z(t) = \int_0^t e^{-A(t-s)} u(s)\, ds$. Then $z \in C^{1+\rho}((0,T];L^p)$, $\partial_x^4 z \in C^\rho((0,T];L^p)$, and $z' = -\partial_x^4 z + u$.*

Proof: This is a slight generalization of Lemma 2.14 of [4]; the proof in this form is easily obtained from, for example, Theorems 4.2.4, 4.3.4, and 4.3.5 of [10].

In the next lemma we record some one-dimensional Sobolev estimates; constants are given explicitly since they will be used in Section 5 to derive sufficient conditions for global existence. Here, and throughout the remainder of the paper, we adopt the notation $\bar{f} = \int_{S^1} f\, dx$ for any $f \in L^1$.

LEMMA 4.7. *If $f \in W^{1,1}$ then $f \in C^0$; moreover, if $\bar{f} = 0$ then*

$$\|f\|_{C^0} \leqq \frac{1}{2} \|f_x\|_{L^1} , \tag{4.6}$$

$$\|f\|_{C^0} \leqq \frac{1}{2\sqrt{3}} \|f_x\|_{L^2} , \tag{4.7}$$

$$\|f\|_{L^2} \leqq \frac{1}{2\pi} \|f_x\|_{L^2} . \tag{4.8}$$

Proof: The first statement is well known. Suppose then that $\bar{f} = 0$; we may assume that $f(0) = 0$, so that

$$|f(x)| = \frac{1}{2} \left| \int_0^x f_x\, dx + \int_x^1 f_x\, dx \right| \leqq \frac{1}{2} \|f_x\|_{L^1} .$$

Moreover, $\bar{f} = 0$ and $\|f_x\|_{L^2} < \infty$ imply that $f' \in E^\perp$, where $E \subset L^2$ is the subspace generated by 1 and x, so that

$$|f(x)| = \left| \int_0^x f'\, dx \right| \leqq \|P\chi_x\|_{L^2} \|f_x\|_{L^2} ,$$

with P the orthogonal projection onto $E^\perp$ and χ_x the characteristic function of the interval $[0,x]$. The estimate (4.7) follows from $\sup_{x\in S^1} \|P\chi_x\| = 1/2\sqrt{3}$. Finally, (4.8) is obtained immediately from expansion in Fourier series. We remark that the constants in (4.6)–(4.8) are easily seen to be the best possible.

Finally, we give two lemmas on the behavior of the non-linearity in (1.1).

LEMMA 4.8. *Let $S \doteqdot \{ f \in W^{r,p} \mid \bar{f} = a, f(x) \geqq \nu, \|f\|_{W^{r,p}} \leqq M \}$ for some $a, \nu, M > 0$. Then*

(a) If $r = 1$ and $p \geqq 2$ then F is a bounded, Lipschitz mapping from S to $L^{p/2}$;

(b) If $r \geqq 2$ then F is a bounded, Lipschitz mapping from S to $W^{r-1,p}$.

Proof: (a) For $f, h \in S$,

$$\|F(f)\|_{L^{p/2}} = \left\| \frac{f_x^2}{f} \right\|_{L^{p/2}} \leqq \frac{M^2}{\nu} \tag{4.9}$$

and

$$\begin{aligned} \|F(f) - F(h)\|_{L^{p/2}} &= \left\| \frac{(f_x + h_x)(f_x - h_x)}{f} - \frac{(h_x)^2 (f-h)}{fh} \right\|_{L^{p/2}} \\ &\leqq \left(\frac{2M}{\nu} + \frac{M^2}{2\sqrt{3}\nu^2} \right) \|f_x - h_x\|_{L^p} , \end{aligned} \tag{4.10}$$

where we have used estimate (4.7).

(b) It suffices to give bounds for $\|\partial_x^{r-1} F(f)\|_{L^p}$ and $\|\partial_x^{r-1} (F(f) - F(h))\|_{L^p}$, for $f, h \in S$. Now $\partial_x^{r-1} F(f)$ is a sum of terms of the form

$$\frac{\partial_x^{k_1} f \cdots \partial_x^{k_m} f}{f^{m-1}} , \tag{4.11}$$

where $2 \leqq m \leqq r+1$, $1 \leqq k_j \leqq r$, $\sum_j k_j = r+1$. The estimates proceed as in (a), using $\|\partial_x^k f\|_{C^0} \leqq M$ and, from (4.6), $\|\partial_x^k (f-h)\|_{C^0} \leqq (1/2) \|\partial_x^{k+1} (f-h)\|_{L^{p/2}}$, for $k < r$.

LEMMA 4.9. *Suppose that $w \in C_+([0,T];H^1)$. If $w_x \in C^\rho((0,T];L^4)$ then $F(w) \in C^\rho((0,T];L^2)$; if $\partial_x^r w \in C^\rho((0,T];L^2)$ for some $r \geqq 2$, then $\partial_x^{r-1} F(w) \in C^\rho((0,T];L^2)$.*

Proof: This is an immediate consequence of Lemma 4.8.

We can now give the proofs of our main results.

Proof of Theorem 4.2 (a): Local existence and uniqueness. Define $v(t) = e^{-At}g$ and let $\mu = \inf_{x\in S^1} g(x)$; because e^{-At} is continuous on H^1 we may by (4.6) choose T_1 so that $v(t) \geqq \mu/2$ for $t \in [0,T_1]$. Now for positive $T \leqq T_1$ let R be the set of functions in $C([0,T];H^1)$ satisfying $\overline{u(t)} \equiv \overline{g}$ and $\|u_x - v_x\|_{C([0,T];L^2)} \leqq \alpha$, where $\alpha = \sqrt{3}/2\mu$; from (4.7), $u \in R$ implies that $u(t) \geqq \mu/4$, $t \in [0,T]$. Thus for $u \in R$ we may define $B(u)$ by

$$B(u)(t) = \partial_x^2 \int_0^t e^{-A(t-s)} F(u(s))\, ds\,,$$

and set $K(u) \equiv v + B(u)$. Fixed points of K yield mild solutions of the initial value problem.

We will show that, when T is suitably restricted, K maps R into R and is a contraction in the metric $\rho(u,w) = \|u_x - w_x\|_{L^2}$ on R; thus, K has a unique fixed point on R. Clearly $\overline{K(u)(t)} = \overline{v(t)} = \overline{g}$. We estimate $\|K(u)_x - v_x\|_{L^2}$ and $\|K(u)_x - K(w)_x\|_{L^2}$ from (4.4); applying (4.9) and (4.10) with $p = 2$, $\nu = \mu/4$, and $M = (\|g_x\|_{L^2} + \alpha)$ we see that it suffices to require that

$$\frac{4M^2CT^{1/8}}{\mu} \leqq \alpha \tag{4.12}$$

and

$$\frac{8CM(1 + M/\sqrt{3}\mu)T^{1/8}}{\mu} \leqq 1\,, \tag{4.13}$$

with C a constant from (4.4). Equations (4.12) and (4.13) will be satisfied if T is chosen sufficiently small.

Proof of Theorem 4.2 (b): Regularity of mild solutions. We first prove that $w_x \in C^\rho((0,T];L^q)$ whenever $1 \leqq q < \infty$ and $0 < \rho < 1/4q$. Write

$$w(t) = v(t) + B(w)(t) \equiv e^{-At}g + \partial_x^2 \int_0^t e^{-A(t-s)} F(w(s))\, ds \tag{4.14}$$

as usual. Certainly $v \in C^k((0,T];W^{r,q})$ for any k, r and q. On the other hand, since $w \in C_+([0,T];H^1)$, Lemma 4.8 (a) implies that $F(w) \in C([0,T];L^1)$, and Lemma 4.5 that $B(w) \in C^\rho([0,T];W^{1,q})$.

We next show that $w \in C^\rho((0,T];H^2)$; again it suffices to verify this for $B(w)$. By the result above and Lemma 4.9, $F(w) \in C^\rho((0,T];L^2)$ and hence, since

$$\partial_x^2 B(w)(t) = \partial_x^4 \int_0^t e^{-A(t-s)} F(w(s))\, ds\,,$$

Lemma 4.6 implies that $B(w) \in C^\rho((0,T];H^2)$.

We now verify inductively that $w \in C^\rho((0,T];H^r)$ for $r \geqq 3$. Fix $\varepsilon > 0$ and for $t > \varepsilon$ write

$$\partial_x^{r+1} w = \partial_x^{r+1} e^{-A(t-\varepsilon)} w(\varepsilon) + \partial_x^4 \int_\varepsilon^t e^{-A(t-s)} \partial_x^{r-1} F(w(s))\, ds\,.$$

The induction hypothesis and Lemma 4.9 imply that $\partial_x^{r-1}F(w) \in C^\rho((0,T];L^2)$, so that again Lemma 4.6 implies that $w \in C^\rho((0,T];H^r)$.

Finally, for any $r \geqq 0$ we may write

$$\partial_x^r w = \partial_x^r e^{-A(t-\varepsilon)} w(\varepsilon) + \int_\varepsilon^t e^{-A(t-s)} \partial_x^{r+2} F(w(s))\, ds\,.$$

The last statement of Lemma 4.6 now implies that $\partial_x^r w$ is differentiable in L^2 and satisfies $(\partial_x^r w)' = \partial_x^{r+4} w + \partial_x^{r+2} F(w)$, so that w is a classical solution of (1.1) in H^r.

Proof of Corollary 4.3: Lyapunov functionals. This is an immediate consequence of Proposition 3.1 and part (b) of Theorem 4.2. We remark also that all of the functionals referred to in Proposition 3.1, with the exception of $\int_{S^1} (w_x^2/w)^\alpha\, dx$ for $\alpha > 1$, are defined also on the half-closed interval $[0,T^*)$ and, by continuity, vary monotonically there.

5. Relation Between Positivity Preservation and Global Existence

As indicated in the Introduction, equation (1.1) arose from a positivity preserving discrete scheme. While we have not been able to prove either positivity preservation or global existence in time for the partial differential equation, we can give a sharp connection between the two.

THEOREM 5.1. *Suppose that $g \in H^1_+$ and that $w \in C_+([0,T^*(g));H^1)$ is a mild solution of the initial value problem (1.1)–(1.2) defined on a maximal half-open interval. If $T^*(g) < \infty$, then $\lim_{t\nearrow T^*} w(t)$ exists in C^1, but the limiting function vanishes at at least one point of S^1.*

Proof: Suppose that $T^* \equiv T^*(g) < \infty$. Fix ε with $0 < \varepsilon < T^*$; we claim that $w(t)$ is uniformly Hölder continuous in C^1 on $[\varepsilon, T^*)$. Then for any sequence of times $t_n \nearrow T^*$, $\{w(t_n)\}$ is Cauchy in C^1; this implies the existence of $w(T^*) \equiv \lim_{t\nearrow T^*} w(t)$ in C^1. If $w(T^*)$ were strictly positive then the solution could be extended by Theorem 4.2 (a), so $w(T^*)$ must vanish at at least one point of S^1.

To verify the claim we write

$$w(t) = e^{-A(t-\varepsilon)} w(\varepsilon) + \partial_x^2 \int_\varepsilon^t e^{-A(t-s)} F(w(s))\, ds\,. \tag{5.1}$$

The first term is as regular as we wish. On the other hand, we know by Corollary 4.3 and Proposition 3.1 (c) that for $1 \leqq p \leqq 3/2$, $F(w(t)) \in L^p$ on $[\varepsilon, T^*)$, with $\|F(w(t))\|_{L^p}$ non-increasing. Thus for $\varepsilon < T < T^*$, $\|F(w)\|_{C([\varepsilon,T];L^p)}$ is finite and independent of T. We take $p > 1$ and apply Lemma 4.5 to conclude that the second term is uniformly Hölder continuous in $W^{1,\infty}$ on $[\varepsilon, T^*)$. Since Theorem 4.2 (b) implies that $w \in C([\varepsilon, T^*); C^1)$, w must also be uniformly Hölder continuous in C^1.

The fact that positivity of solutions suffices for global existence enables us to give several sufficient conditions for such existence.

THEOREM 5.2. *Suppose that $g \in H^1_+$. Then $T^*(g) = \infty$ and $\lim_{t\to\infty} \|w(t) - \bar{g}\|_{H^1} = 0$ if any of the following three conditions holds*:

$$\text{(i)} \quad \int_{S^1} \frac{g_x^2}{g}\,dx < 16\,\bar{g}\,, \qquad \text{(ii)} \quad \|g_x\|_{L^2} < 4\,\bar{g}/\sqrt{3}\,, \qquad \text{(iii)} \quad T^*(g) > \frac{(1+\sqrt{3})}{16\pi^2\sqrt{3}}\,.$$

Proof: Let w be a solution of the initial value problem (4.2) on $[0, T^*)$ as above. We introduce the variable $y(t,x) = \sqrt{w(t,x)}$ and define the quantity $\psi(t)$ by

$$\psi^2(t) = \frac{1}{\bar{g}} \int_{S^1} y_x^2(t)\,dx\,.$$

By Proposition 3.1 (c) and the remark in the proof of Corollary 4.3,

$$\psi(t) \leqq \psi(0) = \frac{1}{2\bar{g}^{1/2}} \left[\int_{S^1} \frac{g_x^2}{g}\,dx \right]^{1/2}, \tag{5.2}$$

for all $t < T^*$. Let $y_{\max}(t)$ and $y_{\min}(t)$ denote respectively the maximum and minimum values of y. From the argument leading to (4.6), and the Schwarz inequality,

$$y_{\max}(t) - y_{\min}(t) \leqq \frac{1}{2}\|y_x(t)\|_{L^1} < \frac{\bar{g}^{1/2}}{2}\psi(t) \leqq \frac{\bar{g}^{1/2}}{2}\psi(0)\,; \tag{5.3}$$

since certainly $y_{\max} \geqq \bar{g}^{1/2}$, (5.3) yields $y_{\min}(t) > \bar{g}^{1/2}(1 - \psi(0)/2)$. Now condition (i) of the theorem implies that this lower bound is strictly positive and hence, by Theorem 5.1, that $T^*(g) = \infty$. On the other hand, if condition (ii) holds, then $g > \bar{g} - \|g - \bar{g}\|_{C^0} \geqq \bar{g}/3$ from (4.7) and hence $\int g_x^2/g\,dx \leqq (3/\bar{g})\|g_x\|_{L^2}^2 < 16\bar{g}$, so that global existence follows from condition (i).

We next introduce the quantity

$$\phi(t) \equiv \frac{1}{\bar{g}^{1/2}} \int_{S^1} \frac{y_{xx}^2(t)}{y(t)}\,dx\,,$$

which by Theorem 4.2 (b) is finite for $t > 0$. From (4.7) we have

$$y(t) \leqq \bar{y}(t) + \frac{1}{2\sqrt{3}}\|y_x(t)\|_{L^2} \leqq \bar{g}^{1/2}\left[1 + \frac{\psi(t)}{2\sqrt{3}}\right],$$

and hence, from (4.8),

$$\psi(t)^2 = \frac{1}{\bar{g}}\int_{S^1} y_x^2(t)\,dx \leqq \frac{1}{4\pi^2\bar{g}}\int_{S^1} y_{xx}^2(t)\,dx \leqq \frac{1}{4\pi^2}\phi(t)\left[1 + \frac{\psi(t)}{2\sqrt{3}}\right]. \tag{5.4}$$

Equation (5.4) implies that $\psi(t) \leqq G(\phi(t))$, where the function G is continuous and increasing on $[0, \infty)$ and satisfies $G(0) = 0$, $G(\phi_0) = 2$ for $\phi_0 = 16\pi^2\sqrt{3}/(1+\sqrt{3})$. On the other hand, from (1.4) and an argument as in the proof of Proposition 3.1, or directly from the proof of that proposition, we find that

$$\frac{d}{dt}\int_{S^1} y(t)\,dx = \bar{g}^{1/2}\phi(t)\,. \tag{5.5}$$

Since $\int_{S^1} y(t)\,dx \leqq \|y(t)\|_{L^2} = \bar{g}^{1/2}$, (5.5) implies that for any $T < T^*$, $\int_0^T \phi(s)\,ds \leqq 1$; in particular, for some $t_0 \in [0, T]$, $\phi(t_0) \leqq 1/T$ and hence

$$\psi(t_0) \leqq G(1/T)\,. \tag{5.6}$$

Now suppose that condition (iii) holds. Then we may choose T so that $\phi_0^{-1} < T < T^*$, and (5.6) implies that $\psi(t_0) \leqq 2$, so that $T^* = \infty$ by condition (i) applied on the interval $[t_0, \infty)$. Finally, if $T^* = \infty$ we may take T arbitrarily large in (5.6), so that, from the monotonicity of $\psi(t)$ and the condition $G(0) = 0$, $\lim_{t\to\infty}\psi(t) = 0$, which implies that $w(t) \to \bar{g}$ in H^1.

6. Higher Dimensions

Among possible generalizations of (1.1) to higher dimensions the most interesting appears to be

$$w_t = -\Delta^2 w + \partial_i\partial_j\left(\frac{\partial_i w \partial_j w}{w}\right), \tag{6.1}$$

or equivalently

$$w_t = -\partial_i\partial_j(w\partial_i\partial_j \log w) . \tag{6.2}$$

Here $x \in X$, with X taken to be Euclidean space $\mathbb{R}^d$ or the d-torus $(S^1)^d$, and we have adopted the convention of summing over repeated indices and written $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial_i\partial_i$. We choose (6.2) as our generalization of (1.3), rather than the apparently more natural equation $w_t = -\triangle(w \triangle \log w)$, for two reasons. First, the resulting modification of the equation (1.4) for the variable $y = \sqrt{w}$,

$$y_t = -\triangle^2 y + \frac{(\triangle y)^2}{y} , \tag{6.3}$$

has a simple form which again suggests the possible presence of a mechanism to preserve positivity of solutions. Second, as we will see shortly, (6.2) implies that certain of the functionals of Proposition 3.1 — specifically, the integrals of w and y, the entropy, and the Fisher information — are again Lyapunov functionals.

In this section we discuss briefly the extent to which our one-dimensional results generalize to the equation (6.1). As indicated above, some of the monotonicity results of Proposition 3.1 remain valid; moreover, we can give a result on existence, uniqueness, and regularity which is quite similar to Theorem 4.2. It is of course still necessary to control the denominator of the non-linear term in (6.1) by controlling the uniform norm of w; for $d > 1$, this (and other considerations in the proof) seem to necessitate working in a more restrictive space than H^1. On the other hand, the monotonic behavior of the Fisher information gives control only on the H^1 norm of the solution. As a result, we are not able to establish any connection between global existence and positivity preservation which would generalize Theorem 5.1.

We take $X = (S^1)^d$ throughout the remainder of this section and adapt to this case, without further elaboration, the notations for function spaces and the definitions of mild and classical solutions introduced earlier. In particular, a mild solution of (1.1) with initial condition $w(0) = g$ is a solution of the integral equation

$$w(t) = e^{-At}g + \partial_i\partial_j \int_0^t e^{-A(t-s)}F_{ij}(w(s))\, ds , \tag{6.4}$$

with $A = \triangle^2$ and $F_{ij}(f) = \partial_i f \partial_j f/f$. We begin with the generalization of Proposition 3.1; it is natural in view of the regularity result to be proved below to state it for spaces of continuous functions.

PROPOSITION 6.1. *If w is a (positive) classical solution of (1.1) in $C^0(X)$, then*

(a) *$\int_X w(t,x)\, dx$ is constant in time and $\int_X w(t,x)^{1/2}\, dx$ is nondecreasing; moreover,*

(b) *The entropy $-\int_X w \log(w(t,x))\, dx$ is nondecreasing in time.*

If $w(t,x)$ is also a classical solution of (1.1) in C^1, then

(c) *The Fisher information $\int_X \partial_i w_x(t,x)\partial_i w_x(t,x)/w(t,x)\, dx$ is nonincreasing in time.*

Proof sketch: As in the proof of Proposition 3.1, these results follow by simple manipulations and appropriate integrations by parts. It is convenient to work in terms of the variable y satisfying (6.3) when studying both $\int_X w(t,x)^{1/2}\, dx$ and the Fisher information.

To obtain existence, uniqueness, and regularity we may, according to the Sobolev inequality, control the denominator of $F(w(t)) = \partial_i w \partial_j w/w$ by taking $w(t)$ to lie in any space $W^{1,p}$ with $p > d$.

THEOREM 6.2. *Suppose that $g \in W_+^{1,p}$ for some p satisfying $d < p \leqq \infty$. Then the following hold.*

(a) LOCAL EXISTENCE AND UNIQUENESS OF A MILD SOLUTION: *For some $T > 0$ there exists a unique $w \in C_+([0,T]; W^{1,p})$ satisfying (6.4).*

(b) REGULARITY OF MILD SOLUTIONS: *If $w \in C([0,T]; W^{1,p})$ is a mild solution of (6.1) on $[0,T]$ for some $p > d$ then, for any integer r satisfying $1 \leqq r \leqq p$, $w \in C((0,T]; W^{r,p/r})$. In particular, if $g \in W^{1,\infty}$ then $w \in C((0,T]; C^r)$, and w is a classical solution of (1.1) in C^r, for all r.*

Proof sketch: The proof follows closely the lines of the proof of Theorem 4.2; we begin by noting the necessary changes in the auxiliary lemmas. The estimate (4.3) is replaced by

$$\|A^\rho \partial_x^j e^{-At} f\|_{L^q} \leqq Ct^{-(4\rho+j+d/p-d/q)/4}\|f\|_{L^p} , \tag{6.5}$$

derived in the same way, and the estimates (4.6)–(4.8) by the Sobolev inequalities

$$\|f\|_{C^0} \leqq C\|f\|_{W^{r,p/r}} , \tag{6.6}$$

$$\|f\|_{W^{k,p/k}} \leqq C\|f\|_{W^{r,p/r}} , \qquad 1 \leqq k < r , \tag{6.7}$$

valid for $p > d$ and $p \geqq r \geqq 1$. It follows from the generalization of (4.11) to the higher dimensional case, under these same restrictions on p and r, that F_{ij} is a bounded, Lipschitz mapping from $\{ f \in W^{r,p} \mid f \geqq \nu > 0, \|f\|_{W^{r,p}} \leqq M < \infty \}$ to $W^{r-1,p/(r+1)}$, and thus that if $w \in C_+^\rho((0,T]; W^{r,p/r})$ then $F_{ij}(w) \in C^\rho((0,T]; W^{r-1,p/(r+1)})$. Finally, Lemma 4.5 is modified only in the condition that ρ must satisfy, which is now $4\rho + d/p - d/q < 1$, and Lemma 4.6 is unchanged.

With this machinery in place the proof of (a) is a straightforward application of the fixed point theorem on a ball in $C([0,T]; W^{1,p})$ with center v, where again

$v(t) = e^{-At}g$. To verify regularity we first observe that if w is a mild solution on $[0,T]$ in $W^{1,p}$ then $F_{ij}(w) \in C([0,T]; L^{p/2})$ and hence $w \in C^{\rho}((0,T]; W^{1,p})$ whenever ρ satisfies $4\rho < (1 - d/p)$, and then prove by induction on r that $w \in C^{\rho}((0,T]; W^{r,p/r})$ for all $r \leqq p$. If $p = \infty$ then we conclude immediately from the Sobolev inequalities that $w \in C^{\rho}((0,T]; C^r)$ for all r, and Lemma 4.6 then implies that w is a classical solution.

Finally, we show that the results of Proposition 6.1 apply to the solutions we have constructed.

COROLLARY 6.3. LYAPUNOV FUNCTIONALS: *If $g \in W_+^{1,p}$ for some $p > d$ and $w \in C_+([0,T]; W^{1,p})$ is a mild solution of* (6.1) *with initial value g then the conclusions of Proposition* 6.1 *hold on* $[0,T]$.

Proof of Corollary 6.3: If $p = \infty$ then this is an immediate consequence of Proposition 6.1 and part (b) of Theorem 6.2 (in fact, as long as $p \geqq 4$ for cases (a) and (b), and $p \geqq 5$ for case (c), then enough derivatives of w exist to justify directly the manipulations used in verifying Proposition 6.1). For general p we may verify that the solutions are continuous functions of the initial data and then approximate the initial value g by elements of $W^{1,\infty}$; we omit details.

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Appendix

In this appendix we record the recursion relations which lead to (1.1) in a scaling limit. Further details are given in [2].

Recall that M_n denotes a random variable taking integer values and that $W_n(m)$ denotes the probability that $M_n = m$. In the recursion for W_n it is helpful to regard M_n as the position, at time n, of a particle undergoing a random walk with a local bias determined by a certain function $H_n(m)$ satisfying $-1 \leqq H_n(m) \leqq +1$:

$$W_{n+1}(m) = \frac{1}{2}[1 + H_n(m-1)]W_n(m-1) + \frac{1}{2}[1 - H_n(m+1)]W_n(m+1)\,. \tag{A.1}$$

The scheme involves also an auxiliary, non-negative, dynamically independent function $U_n(m)$, which satisfies the recursion

$$U_{n+1}(m) = \frac{(W_n(m) + U_n(m+1))(W_n(m) + U_n(m-1))}{U_n(m+1) + 2W_n(m) + U_n(m-1)}\,, \tag{A.2}$$

with the supplementary provision that $U_{n+1}(m) = 0$ if the denominator in (A.2) vanishes. Finally, the bias H is itself determined, non-recursively, by W and U:

$$H_n(m) = \frac{U_n(m+1) - U_n(m-1)}{U_n(m+1) + 2W_n(m) + U_n(m-1)}\,, \tag{A.3}$$

where again $H_n(m) = 0$ if the denominator vanishes. Note that if we define $V_n^{\pm}(m) \equiv W_n(m) + U_n(m \pm 1)$ then (A.3) may be rewritten as $H_n(m) = (V_n^+(m) - V_n^-(m))/(V_n^+(m) + V_n^-(m))$, so that certainly $-1 \leqq H_n(m) \leqq +1$, with strict inequalities as long as $W_n(m)$ or the product $U_n(m-1)U_n(m+1)$ is nonvanishing. It is then clear that the recursive scheme is well defined, preserves the normalization condition $\sum_{m\in\mathbb{Z}} W_n(m) = 1$, and preserves both non-negativity and strict positivity of W.

As indicated in the Introduction, (1.1) is obtained from this recursive scheme when we introduce smooth functions $w(t,x)$, $u(t,x)$, and $h(t,x)$, write $W_n(m) \simeq w(\varepsilon^4 n, \varepsilon m)$, $U_n(m) \simeq u(\varepsilon^4 n, \varepsilon m)$, and $H_n(m) \simeq h(\varepsilon^4 n, \varepsilon m)$, and take a formal $\varepsilon \to 0$ limit. (More precisely, the limiting equation thus obtained differs from (1.1) by a constant rescaling of time: $w_t = -(3/8)(w(\log w)_{xx})_{xx}$.) In the scaling limit $u = w + O(\varepsilon^2)$. At first glance (A.1) might suggest a diffusive scaling $t = \varepsilon^2 n$, $x = \varepsilon m$, leading to a second-order PDE, but in fact the bias H as defined by (A.3) introduces additional cancellations which lead to the fourth-order equation (1.1).

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