Spectral and stability aspects of quantum chaos

H. R. Jauslin and J. L. Lebowitz
Department of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903

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The relation between the spectrum of a generalized quasienergy operator and the stability of quantum systems driven by quasiperiodic time-dependent forces is discussed.

I. INTRODUCTION

The long time behavior of quantum systems subject to time-dependent—not necessarily periodic—forces is a problem of considerable theoretical and practical interest.[1-8] A natural framework for the study of typical behavior of such systems, which include both periodic and random potentials as special cases, is to write the Hamiltonian in the form

$$H = H_0(x) + V[x;\underline{\theta}(t)]. \tag{1}$$

Here, x stands for the internal dynamical variables of the system and $\underline{\theta}(t) \in \Omega$ is a time-dependent "external" classical variable, corresponding to the trajectory of a classical dynamical system on a domain Ω, having an invariant ergodic measure μ. One then considers typical or averaged behavior with respect to μ.

This situation can be thought of as a limiting case of the system being in contact with an external bath whose relevant time dependence, described by $\underline{\theta}(t)$, is independent of the state of the system evolving according to the Hamiltonian (1).

The behavior of systems with time-dependent perturbations can be qualitatively different from the one of isolated quantum systems in a bounded spatial domain, or confined by a potential going rapidly to infinity. Since the energy levels of the latter are discrete, all correlation functions will, of necessity, be almost periodic in time.[9] This rules out any good ergodic properties or other forms of classical chaoticity, which require decay of correlations. Of course, as the size of the system becomes large and the spacing between levels becomes very small, the ensemble averages of some classes of quantum variables may exhibit good decay properties over long time periods. These can become, in suitable limits, indistinguishable from those given by chaotic dynamics.

Systems described by time-dependent Hamiltonians, on the other hand, can have an evolution associated with a continuous spectrum, leading to decaying correlations. Our interest here is however primarily in a different aspect of the behavior of such systems, namely, their long time stability. This may be formulated as follows: Given a system initially in a state that is localized in "phase space," does the time evolution under (1) lead to delocalization? This would correspond in some cases, like the kicked rotator, to an unbounded growth of the energy, and in others, like Rydberg atoms, to ionization. In still other cases, such as those described by a finite-dimensional Hilbert space, e.g., spin systems, for which neither of these behaviors is possible, the question of stability can still be formulated, as we will see, in terms of boundedness of certain quantities associated with a suitably defined quasienergy operator (QEO).

The case of time-periodic external force, which corresponds to $\underline{\theta}(t) = \underline{\theta} + \omega t$ with Ω being a circle of length $T = 1/\omega$ and $d\mu = d\underline{\theta}/T$, has been studied most extensively both for classical and quantum systems. The stability problem can then be expressed in terms of the properties of the Floquet operator $U(T,0)$ which gives the evolution of the system over one period. Classically, U is a canonical volume preserving Poincaré map in phase space, while quantum mechanically it is a unitary operator on the Hilbert space $\mathscr{H}$ of the unperturbed system. The long time behavior of the system and its stability is determined by iteration of this map. Classically, the system can be studied by standard methods, including direct numerical simulation. It can exhibit varieties of behavior, ranging from integrable to chaotic, i.e., positive Lyapounov exponents. Quantum mechanically the problem is reduced to the study of the spectrum of the Floquet operator. One of the outstanding questions in the field is the relation between the behavior of corresponding quantum and classical systems. There are models, such as the periodically kicked rotator, which classically show a linear growth of the energy, while the quantum analogs have been found to saturate, leading to the conjecture that there is a "quantum limitation of diffusion."[10-12]

This situation is different in the case of nonperiodic perturbations. Existing studies indicate that the quantum limitation of diffusion is weaker or nonpresent for nonperiodic perturbations. This has been observed numerically for the quasiperiodically kicked rotator, with two or three incommensurate frequencies,[13,14] and proven rigorously for the randomly kicked rotator[15] (see also Refs. 16–21). The latter is, in fact, the case for a general class of quantum systems, in which the time dependence is given by a Markov process.[22,23] It appears from that analysis that quantum systems may be even more unstable than classical systems under such random perturbations.

The problem of stability was investigated in Ref. 24 in the context of a harmonic oscillator subjected to a general time-dependent force. The simplicity of the model, in which the classical and quantum behavior are similar, made it possible to establish some general relations between the ergodic properties of the dynamical system $\underline{\theta}(t)$ and the growth of the energy of the oscillator. It was found that in general when the autocorrelation of the force decays fast enough, the system behaves chaotically and the

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energy grows linearly. In fact, it was shown there for an explicit example with good ergodic properties, that the energy behaves like the square of a Gaussian random variable with variance proportional to time. We expect that this type of behavior occurs generically whenever $\underline{\theta}(t)$ has a positive Lyapounov exponent. When the correlations do not decay rapidly the situation is more complicated. The study of some examples in Ref. 24 indicate that the asymptotic behavior depends on the fine details of the particular model. There are examples both of bounded and of unbounded growth of the energy.

To go beyond the harmonic oscillator we need a general formalism for dealing with the stability of systems with general time-dependent forces analogous to the one provided by the Floquet and quasienergy operators in the periodic case.[25–29] A scheme for a generalization of the QEO was proposed by Bellissard,[30] on the basis of work by Yajima.[27] In this article we will describe the connection between the spectrum of this QEO and stability properties for the case of quasiperiodic time dependence. The generalized QEO is defined on an enlarged Hilbert space, in which there is an auxiliary degree of freedom for each fundamental frequency of the force. We shall see that the relation between the spectrum of this generalized QEO and the growth of energy is not as direct as in the case of periodic forces. Most results involve an average over the initial phases of the dynamical system. We find that point spectrum implies that the time evolution is almost periodic, and thus the energy is bounded. On the other hand, continuous spectrum does not necessarily imply unbounded energy H_0, as seen in the example of a spin-1/2 system in a quasiperiodic magnetic field, for which there are examples with continuous spectrum, although the energy is always bounded.

In Sec. II we define the generalized QEO. Some results for the periodic case are summarized in Sec. II, to be used as background for the treatment of quasiperiodic forces. In Sec. IV we consider the case of quasiperiodic Hamiltonians. In analogy with the periodic case, we define in Sec. IV A a generalized Floquet operator and show that its spectrum is in one-to-one correspondence with the one of the QEO. In Secs. IV B and C we discuss, respectively, the stability properties associated with point and continuous spectrum. Finally, in Sec. V we review some general results on the relation between the spectrum of the QEO and the asymptotic behavior of the correlation functions.

II. GENERALIZED QUASIENERGY OPERATOR

We consider the time-dependent Hamiltonian (1) with $\underline{\theta}(t)=g_t\underline{\theta}\in\Omega$, where g_t is an invertible flow, i.e., one considers a family of Hamiltonians depending parametrically on the initial point $\underline{\theta}\in\Omega$.

Let $U(t,s;\underline{\theta})$ be the corresponding propagator, which we assume to be unitary and strongly continuous in t and s. It satisfies

$$U(t+a,s+a;\underline{\theta})=U(t,s;g_a\underline{\theta});\quad a\in\mathbf{R}. \tag{2}$$

One then considers, in analogy with the usual construction for the periodic case, the enlarged space

$$\mathscr{K}=\mathscr{H}\otimes L_2(\Omega,d\mu), \tag{3}$$

and the one-parameter family of operators $W(t)$ acting on $\Psi\in\mathscr{K}$ by

$$\begin{aligned}[W(t)\Psi](\underline{\theta})&=U(0,-t;\underline{\theta})\mathscr{T}_{-t}\Psi(\underline{\theta})\\ &\equiv\mathscr{T}_{-t}U(t,0;\underline{\theta})\Psi(\underline{\theta}),\end{aligned} \tag{4}$$

with

$$[\mathscr{T}_{-t}\Psi](\underline{\theta})=\Psi(g_{-t}\underline{\theta}). \tag{5}$$

Under suitable conditions on the Hamiltonian, $W(t)$ is a strongly continuous family of unitary operators.[27,30] By Stone's theorem it can therefore be represented as

$$W(t)=e^{-iKt}. \tag{6}$$

The self-adjoint generator K is the quasienergy operator. It acts as

$$[K\Psi](\underline{\theta})=-i\left.\frac{d\Psi(g_t\underline{\theta})}{dt}\right|_{t=0}+H(\underline{\theta})\Psi. \tag{7}$$

The problem is then to determine the relation between spectral characteristics of K and stability properties of the time evolution of the system.

III. PERIODIC HAMILTONIANS

In this case $\Omega = S^1_T$ is a circle of length T with $g_t\theta = \theta + \omega t$, and $d\mu=d\theta/T$, and the QEO can be written as

$$K=-i\frac{\partial}{\partial\theta}+H(\theta) \tag{8}$$

in the space $\mathscr{H}\otimes L_2\,(S^1_T,d\theta/T)$. The spectrum of the QEO in the Hilbert space $\mathscr{K}$ is directly related to the spectrum of the Floquet operator $U_F\equiv U(T,0;0)$ in the Hilbert space $\mathscr{H}$. The eigenvalues and eigenvectors of the two operators are related by

$$\begin{aligned}&K\psi=\lambda\psi,\\ &U(T,0;0)\phi=e^{-i\lambda T}\phi,\\ &\psi(x,\theta)=e^{i\lambda\theta}U(\theta,0;\,0)\phi(x).\end{aligned} \tag{9}$$

The following result was proven in Ref. 24 using previous results of Ref. 28: The energy remains bounded if the spectrum is pure point, and it grows unbounded if the spectrum is continuous. To make this statement more precise, one can introduce the following subspaces: Let $\mathscr{H}_{pp}$ and $\mathscr{H}_c$ be the pure point and continuous spectrum subspaces, respectively, corresponding to the Floquet operator $U(T,0)$. The subspace $\mathscr{H}_{be}$ of state trajectories with bounded energy is defined as

$$\mathscr{H}_{be}=[\psi\in\mathscr{H}|\lim_{E\to\infty}\sup_{t>0}||F(H_0>E)\,U(t,0;\,0)\psi||_{\mathscr{H}}=0], \tag{10}$$

where $F(H_0 > E)$ is the spectral projection on the eigenspace of H_0 corresponding to energies larger than E. This definition is chosen such that $\mathscr{H}_{be}$ contains all states that are localized in energy space uniformly for all times, without specifying, however, how they decay at large energies. As the following theorem shows, these are precisely the states associated with point spectrum of the QEO. As an intermediate step and for later generalization it is useful to consider also the subspace of states with precompact trajectories, which is defined as

$$\mathscr{H}_{pc}=\{\psi\in\mathscr{H} \mid \text{closure of } [U(t,0;0)\psi,\ t\geqslant 0] \text{is compact}\}, \quad (11)$$

i.e., the trajectories in $\mathscr{H}_{pc}$ evolve in a space of finite dimension except for a small correction.

Theorem 3.1: If the spectrum of the unperturbed H_0 is discrete and bounded from below (e.g., having a confining potential, or defined on a compact manifold) and the propagator $U(t,s)$ exists as a strongly continuous family of unitary operators, then

$$\mathscr{H}_{pp}=\mathscr{H}_{pc}=\mathscr{H}_{be}; \quad \mathscr{H}_c=\mathscr{H}_{be}^{\perp}, \quad (12)$$

where $^{\perp}$ denotes the orthogonal complement.

We remark that point spectrum does not guarantee that the expectation value of the energy $\langle\varphi(t),H_0\varphi(t)\rangle_{\mathscr{H}}$ is bounded. It is in principle possible that a state is well localized in energy space (i.e., $\in\mathscr{H}_{be}$), but has a tail that decays too slowly and therefore has an infinite expectation of the energy. An equivalent statement is that not all of the eigenfunctions of K are necessarily in the domain of the energy operator. This phenomenon can become manifest already in finite time or only in the limit $t\to\infty$. To our knowledge there are no general results concerning this behavior.

For systems like spin-1/2, described in a finite-dimensional Hilbert space, the spectrum is evidently always pure point, since the Floquet operator is a finite matrix, and the energy of course is bounded. As we will see, this is not the case for quasiperiodic forces and more complicated behavior is possible.

We note here incidentally that there appears to be no criterion for classical systems of a generality comparable to that of Theorem 3.1. Indeed classically it is very difficult to make statements applicable to all trajectories—we can only expect results for typical ones.

IV. QUASIPERIODIC HAMILTONIANS

We consider (1) with H depending quasiperiodically on time with two incommensurate fundamental frequencies ω_1, ω_2, i.e., the domain Ω is a torus $S^1_{T_1}\times S^1_{T_2}$, $T_j=1/\omega_j$, and the flow $g_t(\theta_1,\theta_2)=(\theta_1+t,\theta_2+t)$. The enlarged space can be represented as

$$\mathscr{K}=\mathscr{H}\otimes L_2(S^1_{T},d\theta_1/T_1)\otimes L_2(S^1_{T_2},d\theta_2/T_2)$$

and the QEO acts as

$$[K\Psi](\theta_1,\theta_2)=\left(-i\frac{d}{d\theta_1}-i\frac{d}{d\theta_2}\right)\Psi(\theta_1,\theta_2) + H(\theta_1,\theta_2)\Psi(\theta_1,\theta_2). \quad (13)$$

We restrict ourselves to only two frequencies for notational simplicity. The extension to the case of a finite number of fundamental frequencies is straightforward.

A. Generalized Floquet operator

In the quasiperiodic case there is a natural generalization of the Floquet operator and of the relation (9) with the QEO: Define the generalized Floquet operator

$$U_F:\mathscr{K}_1\to\mathscr{K}_1; \quad \mathscr{K}_1=\mathscr{H}\otimes L_2\left(S^1_{T_1},\frac{d\theta_1}{T_1}\right) \quad (14)$$

as the unitary operator

$$U_F(\theta_1)=\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1,0)\equiv U(0,-T_2;\theta_1,0)\mathscr{T}^1_{-T_2}, \quad (15)$$

where

$$[\mathscr{T}^1_{-T_2}\phi](\theta_1)=\phi(\theta_1-T_2). \quad (16)$$

Theorem 4.1: (i) If $\phi\in\mathscr{K}_1$ is an eigenfunction of U_F,

$$U_F\phi=e^{-i\lambda T_2}\phi, \quad (17)$$

then the function ψ defined by

$$\psi(\theta_1,\theta_2)=e^{i\theta_2\lambda}U(0,-\theta_2;\theta_1,\theta_2)\phi(\theta_1-\theta_2), \quad (18)$$

belongs to $\mathscr{K}$ and is an eigenfunction of the QEO, with eigenvalue λ:

$$e^{-iKt}\psi=e^{-i\lambda t}\psi, \quad \forall t\in\mathbf{R}. \quad (19)$$

(ii) Conversely, if $\psi\in\mathscr{K}$ is an eigenfunction of (19), then there is a function $\phi\in\mathscr{K}_1$ such that the ψ can be represented by (18), and ϕ is an eigenfunction of the generalized Floquet operator (17).

Remark: This theorem implies that only the relative phase is relevant. The essential difference between this and the periodic case is that here the generalized Floquet operator does not act on the Hilbert space $\mathscr{H}$ of the unperturbed problem. It carries a dependence on the difference between the initial phases $\theta_2-\theta_1$. This is the origin of the richer spectral structure even in the case of finite dimensional $\mathscr{H}$.

Proof: (i) (a) We start by showing that $\psi\in\mathscr{K}$: The periodicity $\psi(\theta_1+T_1,\theta_2)=\psi(\theta_1,\theta_2)$ is evident from the definition. The periodicity $\psi(\theta_1,\theta_2+T_2)=\psi(\theta_1,\theta_2)$ can be shown as follows:

$$\psi(\theta_1,\theta_2+T_2)=e^{i\lambda\theta_2}e^{i\lambda T_2}U(0,-\theta_2-T_2;\theta_1,\theta_2+T_2) \times\phi(\theta_1-\theta_2-T_2). \quad (20)$$

We use the identities

$$U(0,-\theta_2-T_2;\theta_1,\theta_2+T_2) =U(0,-\theta_2;\theta_1,\theta_2)U(-\theta_2,-\theta_2-T_2;\theta_1,\theta_2) \quad (21)$$

and

$$U(-\theta_2,-\theta_2-T_2;\theta_1,\theta_2)$$
$$=\mathscr{T}^1_{-T_2}U(-\theta_2,-\theta_2-T_2;\theta_1+T_2,\theta_2)$$
$$=\mathscr{T}^1_{-T_2}U(0,-T_2;\theta_1-\theta_2+T_2,0)$$
$$=\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1-\theta_2,0)=U_F(\theta_1-\theta_2) \quad (22)$$

to obtain

$$\psi(\theta_1,\theta_2+T_2)$$
$$=e^{i\lambda\theta_2}e^{i\lambda T_2}U(0,-\theta_2;\theta_1,\theta_2)U_F(\theta_1-\theta_2)\phi(\theta_1-\theta_2)$$
$$=e^{i\lambda\theta_2}U(0,-\theta_2;\theta_1,\theta_2)\phi(\theta_1-\theta_2)=\psi(\theta_1,\theta_2). \quad (23)$$

(b) ψ is an eigenfunction of K:

$$[e^{-iKt}\psi](\theta_1,\theta_2)$$
$$=\mathscr{T}_{-t}U(t,0;\theta_1,\theta_2)e^{i\lambda\theta_2}U(0,-\theta_2;\theta_1,\theta_2)\phi(\theta_1-\theta_2)$$
$$=e^{-i\lambda t}e^{i\lambda\theta_2}\mathscr{T}_{-t}U(t,-\theta_2;\theta_1,\theta_2)\phi(\theta_1-\theta_2)$$
$$=e^{-i\lambda t}e^{i\lambda\theta_2}U(t,t-\theta_2;\theta_1-t,\theta_2-t)\phi(\theta_1-\theta_2)$$
$$=e^{-i\lambda t}e^{i\lambda\theta_2}U(0,-\theta_2;\theta_1,\theta_2)\phi(\theta_1-\theta_2)$$
$$=e^{-i\lambda t}\psi(\theta_1,\theta_2). \quad (24)$$

(ii) To prove the converse we start by embedding the space $\mathscr{K}$ into the following covering space:

$$\mathscr{K}_L=\mathscr{H}\otimes L_2\left(S^1_{T^1},\frac{d\theta_1}{T_1}\right)\otimes L_{2,\mathrm{loc}}\left(\mathbf{R},\frac{d\theta_2}{T_2}\right), \quad (25)$$

where $L_{2,\mathrm{loc}}$ denotes the space of functions that are locally L_2. Here, $U(0,-\theta_2;\theta_1,\theta_2)$ and its inverse $U(-\theta_2,0;\theta_1,\theta_2)$ are operators in $\mathscr{K}_L$. The equation

$$\phi_2(\theta_1,\theta_2)=e^{-i\theta_2\lambda}U(-\theta_2,0;\theta_1,\theta_2)\psi(\theta_1,\theta_2) \quad (26)$$

defines a function $\phi_2\in\mathscr{K}_L$ that, as we will show, depends only on the difference of the arguments:

$$\phi_2(\theta_1+t,\theta_2+t)$$
$$=\mathscr{T}_t\phi_2(\theta_1,\theta_2)$$
$$=e^{-i\theta_2\lambda}e^{-it\lambda}U(-t-\theta_2,0;\theta_1+t,\theta_2$$
$$+t)\mathscr{T}_t\psi(\theta_1,\theta_2). \quad (27)$$

Using the identities

$$U(-t-\theta_2,0;\theta_1+t,\theta_2+t)\mathscr{T}_t$$
$$=U(-\theta_2,t;\theta_1,\theta_2)\mathscr{T}_t$$
$$=U(-\theta_2,0;\theta_1,\theta_2)U(0,t;\theta_1,\theta_2)\mathscr{T}_t$$
$$=U(-\theta_2,0;\theta_1,\theta_2)e^{iKt}, \quad (28)$$

we can write

$$\phi_2(\theta_1+t,\theta_2+t)$$
$$=e^{-i\theta_2\lambda}e^{-it\lambda}U(-\theta_2,0;\theta_1,\theta_2)e^{iKt}\psi(\theta_1,\theta_2)$$
$$=\phi_2(\theta_1,\theta_2). \quad (29)$$

By this property the class of equivalence $\{\phi_2(\theta_1+t,t), t\in\mathbf{R}\}$ defines a function $\phi\in\mathscr{K}_1$

$$\phi(\theta_1)=\phi_2(\theta_1+t,t), \quad (30)$$

that contains the information to reconstruct ϕ_2;

$$\phi_2(\theta_1,\theta_2)=\phi(\theta_1-\theta_2). \quad (31)$$

It follows immediately from the definition that ϕ is T_1-periodic and $\phi\in\mathscr{K}_1$. We finally verify that ϕ satisfies (17): The eigenvalue equation in $\mathscr{K}_1$

$$[U_F\phi](\theta_1)\equiv\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1,0)\phi(\theta_1)=e^{-i\lambda T_2}\phi(\theta_1) \quad (32)$$

is equivalent to the following equation in $\mathscr{K}$:

$$\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1,0)\phi_2(\theta_1+t,t)$$
$$=e^{-i\lambda T_2}\phi_2(\theta_1+t,t), \quad \forall t\in\mathbf{R}. \quad (33)$$

By the definition of ϕ, we can write the left-hand side as

$$\mathrm{lhs}=\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1,0)\phi_2(\theta_1+t,t)$$
$$=\mathscr{T}^1_{-T_2}U(T_2,0;\theta_1,0)e^{-it\lambda}U(-t,0;\theta_1+t,t)$$
$$\times\psi(\theta_1+t,t)$$
$$=U(T_2,0;\theta_1-T_2,0)e^{-it\lambda}U(-t,0;\theta_1+t-T_2,t)$$
$$\times\psi(\theta_1+t-T_2,t). \quad (34)$$

Using the identifies

$$U(T_2,0;\theta_1-T_2,0)=U(0,-T_2;\theta_1 0)$$
$$=U(-t,-t-T_2;\theta_1+t,t) \quad (35)$$

and

$$U(-t,0;\theta_1+t-T_2,t)=U(-t-T_2,-T_2;\theta_1+t,t), \quad (36)$$

we obtain

$$\mathrm{lhs}=e^{-it\lambda}U(-t,-t-T_2;\theta_1+t,t)$$
$$\times U(-t-T_2,-T_2;\theta_1+t,t)\psi(\theta_1+t-T_2,t)$$
$$=e^{-it\lambda}U(-t,-T_2;\theta_1+t,t)\mathscr{T}_{-T_2}\psi(\theta_1+t,t). \quad (37)$$

The fact that ψ is an eigenfunction of (19) can be equivalently stated as

$$[\mathscr{T}_s\psi](\theta_1,\theta_2)=e^{i\lambda s}U(s,0;\theta_1,\theta_2)\psi(\theta_1,\theta_2). \quad (38)$$

Expressing $\mathscr{T}_{-T_2}\psi$ in this form in Eq. (37) yields

$$\mathrm{lhs}=e^{-iT_2\lambda}e^{-it\lambda}U(-t,-T_2;\theta_1+t,t)$$
$$\times U(-T_2,0;\theta_1+t,t)\psi(\theta_1+t,t)$$
$$=e^{-iT_2\lambda}e^{-it\lambda}U(-t,0;\theta_1+t,t)(\theta_1+t,t)$$
$$=e^{-iT_2\lambda}\phi_2(\theta_1+t,t). \quad (39)$$

Remark: If $\psi(\theta_1,\theta_2)$ is continuous, the construction can be stated in a more intuitive form: The function $\phi\in\mathscr{K}\ell_1$ can be defined by

$$\phi(\theta_1)=\phi_2(\theta_1,0)\equiv\psi(\theta_1,0). \tag{40}$$

The representation (18) becomes

$$\psi(\theta_1,\theta_2)=e^{i\theta_2\lambda}U(0,-\theta_2;\theta_1,\theta_2)\psi(\theta_1-\theta_2,0), \tag{41}$$

which is reminiscent of the form of the solution of a first-order liner partial differential equation by the method of characteristics.

Then,

$$\begin{aligned}[][U_{\mathrm{F}}\phi](\theta_1)&=\mathscr{T}^{1}_{-T_2}U(T_2,0;\theta_1,0)\psi(\theta_1,0)\\&=U(T_2,0;\theta_1-T_2,0)\\&=U(0,-T_2;\theta_1,0)\mathscr{T}_{-T_2}\psi(\theta_1,0)\\&=[e^{-iKT_2}\psi](\theta_1,0)\\&=e^{-i\lambda T_2}\psi(\theta_1,0)\\&=e^{-i\lambda T_2}\phi(\theta_1).\end{aligned} \tag{42}$$

B. Point spectrum and stability

In this section we give some results linking the point spectrum of the QEO with states of bounded energy. In contrast with the periodic case, the statements here do not refer to all choices of the initial phases θ_1,θ_2 but only to almost all of them (a.a. $\underline{\theta}$).

Theorem 4.2: For initial states $\varphi\in\mathscr{H}$ such that $\varphi\otimes 1$ is in the pure point subspace $\mathscr{K}_{\mathrm{pp}}\subset\mathscr{K}$ associated with the QEO the time evolution $\varphi(t)$ is almost periodic (a.a. $\underline{\theta}$):[31] Let $\varphi(t)=U(t,0;\theta_1,\theta_2)\varphi$. Then for a.a. $\underline{\theta}$ $\forall\epsilon>0$ the set

$$[\tau\in\mathbf{R}\,|\sup_{t\in\mathbf{R}}\|\varphi(t+\tau)-\varphi(t)\|_{\mathscr{H}}\leqslant\epsilon] \tag{43}$$

is relatively dense, i.e., there is an *inclusion length* $L>0$ such that in each interval $(a,a+L)$, $a\in\mathbf{R}$, there is at least one point of the set.

Corollary 4.3: If $\varphi\otimes 1\in\mathscr{K}_{\mathrm{pp}}$ then φ has bounded energy (in the average). More precisely,

$$\begin{aligned}\varphi\in\mathscr{H}_{\mathrm{be}}=\{\varphi\in\mathscr{H}|\lim_{E\to\infty}\sup_{t\geqslant 0}||F'(H_0>E)\\\times U(t,0;\theta_1,\theta_2)\varphi||_{\mathscr{H}}=0\},\end{aligned} \tag{44}$$

where $F(H_0>E);\mathscr{H}\to\mathscr{H}$ is the projector into the subspace generated by the eigenfunctions of H_0 with energy larger than E.

Remark: As is the case of periodic force, one cannot conclude that the expectation value of the energy is finite, since the decay of the tail might be too slow.

Proof: We embed φ in $\mathscr{K}$ and expand it in terms of the eigenfunctions of $K\psi_m=\lambda_m\psi_m$ spanning $\mathscr{K}_{\mathrm{pp}}$:

$$\varphi\otimes 1=\sum_m c_m\psi_m. \tag{45}$$

From the definition of the QEO we can write

$$\begin{aligned}\varphi(t)&=U(t,0;\theta_1,\theta_2)(\varphi\otimes 1)\\&=\mathscr{T}_t e^{-iKt}\sum_m c_m\psi_m\\&=\mathscr{T}_t\sum_m c_m e^{-i\lambda_m t}\psi_m,\end{aligned} \tag{46}$$

which is almost periodic in $\mathscr{K}$ since it is a sum of almost-periodic functions,[31] that converges uniformly: $\forall\epsilon>0$ there is an M independent of t such that

$$\begin{aligned}\left\|\varphi(t)-\sum_{m<M}c_m e^{-i\lambda_m t}\mathscr{T}_t\psi_m\right\|^2_{\mathscr{K}}&=\left\|\mathscr{T}_t\sum_{m\geqslant M}c_m e^{-i\lambda_m t}\psi_m\right\|^2_{\mathscr{K}}\\&\leqslant\sum_{m\geqslant M}|c_m|^2<\epsilon^2.\end{aligned} \tag{47}$$

As a consequence there are $\eta_m\in\mathbf{R}$ and $\sigma_m\in\mathscr{K}$ (independent of t) such that[32]

$$\begin{aligned}0&=\left\|\varphi(t)-\sum_m\sigma_m e^{-i\eta_m t}\right\|^2_{\mathscr{K}}\\&\equiv\int_\Omega d\mu(\theta)\left\|\varphi(t)-\sum_m\sigma_m e^{-i\eta_m t}\right\|^2_{\mathscr{H}},\end{aligned}$$

which implies that $\varphi(t)$ is almost-periodic in $\mathscr{H}$ for a.a. $\underline{\theta}$. Corollary 4.3 follows from the fact that the almost periodicity of the trajectory implies that it is precompact, i.e., its closure is compact.[32] This means that $\varphi(t)$ evolves in a space of finite dimension except for a small correction: For any $\epsilon>0$ and a.a. $\underline{\theta}$ there is a decomposition

$$\varphi(t)=\varphi_f(t)+\nu(t), \tag{48}$$

such that $\varphi_f(t)$ is in a finite-dimensional space and $\|\nu(t)|_{\mathscr{H}}<\epsilon$. Therefore,

$$\lim_{E\to\infty}\sup_{t\geqslant 0}\|F(H_0>E)U(t,0;\theta_1,\theta_2)\varphi\|_{\mathscr{H}}<\epsilon, \tag{49}$$

since the contribution from the first component goes to zero.

C. Continuous spectrum, RAGE theorem

As already mentioned, continuous spectrum of the QEO does not necessarily imply unbounded growth of the energy of the system. In this section we give some results that clarify this phenomenon. We consider some energy-like operators that play a role analogous to that of the unperturbed energy operator H_0 in the periodic case. Using the properties of the generalized Floquet operator, we show that continuous spectrum of the QEO implies unboundedness of those quantities.

Let $\mathscr{K}_{1,\mathrm{pp}}$ and $\mathscr{K}_{1,c}$ be the subspaces of $\mathscr{K}_1$ corresponding, respectively, to the pure point and continuous spectrum of the associated generalized Floquet operator U_{F}. We define the following projection operators.

Let $\Pi_0(E_0):\mathscr{H}\to\mathscr{H}$ be the projector into the subspace of $\mathscr{H}$ generated by the eigenfunctions $\varphi^{(0)}$ of H_0 with energy smaller than E_0:

$$H_0\varphi^{(0)}=E\varphi^{(0)};\quad E<E_0. \tag{50}$$

(ii) Let $\Pi_1(E_1)$ be the projector of $L_2(S^1_{T_1},d\theta_1/T_1)$ into the subspace generated by the eigenfunctions $\varphi^{(1)}=\exp(-im\theta_1)$ of $H_1\equiv-\partial^2/\partial\theta_1^2$ with energy smaller than E_1;

$$H_1\varphi^{(1)}=\quad m^2;\quad m^2<E_1. \tag{51}$$

(iii) $\Pi(E_0,E_1)=\Pi_0(E_0)\otimes\Pi_1(E_1)$ is then a projector in $\mathscr{K}_{1,c}$ into the subspace in which both energies are bounded $H_0<E_0$, $H_1<E_1$.

The following theorem states that continuous spectrum *implies that either one or both of the energies* H_0, H_1 grows unbounded with time.

Theorem 4.4: Let $\varphi\in\mathscr{H}$ be such that $\varphi\otimes 1\mathscr{K}_{1,c}$. Then, for any fixed energies E_0, E_1

$$\sup_{t\geqslant 0}\|[1_1-\Pi(E_0,E_1)]U(t,0;\theta_1,0)\varphi\|^2_{\mathscr{K}_1}=1. \tag{52}$$

Corollary 4.5: Let $\varphi(t)=U(t,0;\theta_1,0)\varphi$. The expectation value

$$\langle H_0+H_1\rangle(t)\equiv\langle\varphi(t),[H_0+H_1]\varphi(t)\rangle_{\mathscr{K}_1} \tag{53}$$

is unbounded. (The subscript on the scalar product bracket indicates the space in which it acts.) More precisely, $\forall E>0$ there is a time t_1 such that either $\langle H_0+H_1\rangle(t_1)>E$ or $\varphi(t_1)$ is not in the domain of H_0+H_1 (i.e., $\langle H_0+H_1\rangle(t_1)$ is infinite).

These results are a consequence of the following RAGE-type[33] theorem.

Theorem 4.6: Let C be any compact operator in $\mathscr{K}_1$ and $\phi\in\mathscr{K}_{1,c}$, then there is a non-negative function $f(\tau)$ such that $\lim_{|\tau|\to\infty}f(\tau)=0$ and

$$\frac{1}{\tau}\int_0^\tau dt\|CU(t,0;\theta_1,0)\phi\|_{\mathscr{K}_1}\leqslant f(\tau)\|\phi\|_{\mathscr{K}_1}. \tag{54}$$

Proof: The proof is an adaptation of the one for a similar statement for the periodic case in Ref. 28. By the Schwarz inequality we can write

$$\left(\frac{1}{\tau}\int_0^\tau dt\|CU(t,0;\theta_1,0)\phi\|_{\mathscr{K}_1}\right)^2$$
$$\leqslant\frac{1}{\tau}\int_0^\tau dt\|CU(t,0;\theta_1,0)\phi\|^2_{\mathscr{K}_1}$$
$$=\left\langle\phi,\frac{1}{\tau}\int_0^\tau dt\,U^\dagger(t,0;\theta_1,0)C^\dagger CU(t,0;\theta_1,0)P_c\phi\right\rangle$$
$$\leqslant\left\|\frac{1}{\tau}\int_0^\tau dt\,U^\dagger(t,0;\theta_1,0)C^\dagger CU(t,0;\theta_1,0)P_c\right\|\,\|\phi\|_{\mathscr{K}_1}$$
$$\equiv f^2(\tau)\|\phi\|_{\mathscr{K}_1}, \tag{55}$$

where P_c is the orthogonal projection into the continuous spectrum subspace for the generalized Floquet operator U_F, and $f(\tau)$ is defined by the last equality.

To show that $\lim_{|\tau|\to\infty}f(\tau)=0$ we start by decomposing $\tau=\tau_0+nT_2$, $\tau_0\in[0,T_2)$. Then

$$\left\|\frac{1}{\tau}\int_0^\tau dt\,U^\dagger(t,0;\theta_1,0)CU(t,0;\theta_1,0)P_c\right\|_{\mathscr{K}_1}$$
$$\leqslant\frac{T_2}{\tau}\|C\|_{\mathscr{K}_1}$$
$$+\left\|\frac{1}{nT_2}\int_0^{nT_2}dt\,U^\dagger(t,0;\theta_1,0)CU(t,0;\theta_1,0)P_c\right\|_{\mathscr{K}_1}.$$

The first term vanishes for $\tau\to\infty$. The second term can be written as

$$B\equiv\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{T_2}\int_{jT_2}^{(j+1)T_2}dt\,U^\dagger(t,0;\theta_1,0)CU(t,0;\theta_1,0)P_c$$
$$=\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{T_2}\int_0^{T_2}dt\,U^\dagger(t+jT_2,0;\theta_1,0)$$
$$\times CU(t+jT_2,0;\theta_1,0)P_c. \tag{57}$$

Using the identity

$$U(t+jT_2,0;\theta_1,0)$$
$$=U(t+jT_2,jT_2;\theta_1,0)U(jT_2,0;\theta_1,0)$$
$$=U(t,0;\theta_1+jT_2,0)U(jT_2,0;\theta_1,0), \tag{58}$$

we can express it as

$$B=\frac{1}{n}\sum_{j=0}^{n-1}U^\dagger(jT_2,0;\theta_1,0)C'U(jT_2,0;\theta_1,0)P_c, \tag{59}$$

where

$$C'(t)=\frac{1}{T_2}\int_0^{T_2}dt\,U^\dagger(t,0;\theta_1+jT_2,0)CU(t,0;\theta_1+jT_2,0). \tag{60}$$

Expressing $U(jT_2,0;\theta_1,0)$ in terms of iterations of the generalized Floquet operator U_F

$$U(jT_2,0;\theta_1,0)=\mathscr{T}^1_{jT_2}U^j_F, \tag{61}$$

we can write

$$B=\frac{1}{n}\sum_{j=0}^{n-1}(U_F^\dagger)^jC''U_F^jP_c, \tag{62}$$

where

$$C''=\mathscr{T}^1_{-jT_2}C'\mathscr{T}^1_{jT_2}$$
$$=\frac{1}{T_2}\int_0^{T_2}dt\,U^\dagger(t,0;\theta_1,0)CU(t,0;\theta_1,0) \tag{63}$$

is again compact but now independent of j. The end of the proof can now be taken over from Lemma 2.4 of Ref. 28:

By approximating B by finite rank operators it is shown that $\|B\|$ tends to zero as $n\to\infty$.

Proof of Theorem 4.4: Consider $\phi = \varphi\otimes 1\in\mathscr{K}_{1,c}$. Applying Theorem 4.6 with $C=\Pi(E_0,E_1)$ to the first term in the identify

$$1=\|\phi\|_{\mathscr{K}_1}$$
$$=\frac{1}{\tau}\int_0^\tau dt\|\Pi(E_0,E_1)U(t,0;\,\theta_1,0)\phi\|_{\mathscr{K}_1}$$
$$+\|[\mathbf{1}_1-\Pi(E_0,E_1)]U(t,0;\,\theta_1,0)\phi\|_{\mathscr{K}_1}, \qquad (64)$$

we obtain

$$\frac{1}{\tau}\int_0^\tau dt\|[\mathbf{1}_1-\Pi(E_0,E_1)]U(t,0;\theta,0)\phi\|_{\mathscr{K}_1}\geqslant[1-f(\tau)]. \qquad (65)$$

This implies that

$$\sup_{t\in[0,\tau]}\|[\mathbf{1}_1-\Pi(E_0,E_1)]U(t,0;\,\theta_1,0)\phi\|_{\mathscr{K}_1}\geqslant[1-f(\tau)] \qquad (66)$$

and

$$\sup_{t\in[0,\infty)}\|[\mathbf{1}_1-\Pi(E_0,E_1)]U(t,0;\,\theta_1,0)\phi\|_{\mathscr{K}_1}=1 \quad \text{(for any } E_0,E_1). \qquad (67)$$

Proof of Corollary 4.5: From Theorem 4.4 it follows that, given E_0,E_1 and $\epsilon>0$, there is a time t_1 such that

$$\varphi(t_1)=\varphi^\perp+\eta, \qquad (68)$$

with

$$\varphi^\perp=[\mathbf{1}_{\mathscr{K}_1}-\Pi(E_0,E_1]\varphi(t_1) \text{ and } \|\eta\|_{\mathscr{K}_1}<\epsilon. \qquad (69)$$

Assume that $\varphi(t_1)$ (and therefore $\varphi^\perp$ and η) are in the domain of H_0+H_1. Then

$$\langle H_0+H_1\rangle=\langle\varphi^\perp(H_0+H_1)\varphi^\perp\rangle_{\mathscr{K}_1}+\text{const}\,\epsilon^2. \qquad (70)$$

$\varphi^\perp$ can be expanded in terms of eigenfunctions of H_0+H_1 with energies larger than E_0, E_1:

$$\varphi^\perp=\sum_{n,m}c_{n,m}\varphi_n\otimes e^{-im\theta_1} \qquad (71)$$

and therefore

$$\langle H_0+H_1\rangle=\sum_{m,n}c_{n,m}(E^{(n)}+m^2)+\text{const}\,\epsilon^2$$
$$>E_0+E_1+\text{const}\,\epsilon^2. \qquad (72)$$

Since ϵ can be taken arbitrarily small, this completes the proof.

V. CORRELATION FUNCTIONS

We summarize here for completeness the relation between the spectrum of the QEO and the asymptotic behavior of correlation functions (c.f. Bellissard[30,14]), which is valid for general time-dependent Hamiltonians of the form (1): Let $\varphi\in\mathscr{H}$ and consider the correlation function

$$S(t)=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau ds\langle\varphi,U(t+s,s;\,\underline{\theta})\varphi\rangle_{\mathscr{H}}. \qquad (73)$$

The subscript on the scalar product bracket indicates the space in which it acts. Let $\mathscr{K}_c$, $\mathscr{K}_{ac}$, $\mathscr{K}_{pp}$ denote, respectively, the continuous, absolutely continuous and pure point subspaces of $\mathscr{K}$. Then, (i) If $\varphi\otimes 1\in\mathscr{K}_{ac}$ then $S(t)\to_{t\to\infty}0$, for almost all θ. (ii) If $\varphi\otimes 1\in\mathscr{K}_c$ then $S(t)\to_{t\to\infty}0$ in Cesaro mean, for almost all $\underline{\theta}$. (iii) If $\varphi\otimes 1\in\mathscr{K}_{pp}$ then $S(t)$ is almost periodic for almost all $\underline{\theta}$. These statements follow immediately from standard results of ergodic theory and the relation, valid for almost all $\underline{\theta}$:

$$S(t)=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau ds\langle\varphi,U(t+s,s;\,\underline{\theta})\varphi\rangle_{\mathscr{H}}$$
$$=\int_\Omega\mu(\underline{\theta})\langle\varphi,U(t,0;\,\underline{\theta})\varphi\rangle_{\mathscr{H}}=\langle\varphi\otimes 1,e^{iKt}\varphi\otimes 1\rangle_{\mathscr{K}}. \qquad (74)$$

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