



The Einstein relation for the displacement of a test particle in a random environment[☆]

Joel L. Lebowitz^a, Hermann Rost^{b,*}

^aDepartment of Mathematics and Physics, Rutgers' University, New Brunswick NJ 08903, USA

^bInstitut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

Received June 1993; revised March 1994

Abstract

Consider a stochastic system evolving in time, in which one observes the displacement of a tagged particle, $X(t)$. Assume that this displacement process converges weakly to d -dimensional centered Brownian motion with covariance D , when space and time are appropriately scaled: $X^\varepsilon(t) = \varepsilon X(\varepsilon^{-2}t)$, $\varepsilon \rightarrow 0$. Now perturb the process by putting a small "force" εa on the test particle. We prove on three different examples that under the previous scaling the perturbed process converges to Brownian motion having the same covariance D , but an additional drift of the form $M \cdot a$. We show that M , the "mobility" of the test particle, and D are related to each other by the Einstein formula

$$M = (1/2 \beta) D, \quad (1)$$

where $\beta = 1/kT$ (T being temperature and k Boltzmann's constant) is defined in such a way that the reversible state for the modified dynamics gets the correct Boltzmann factor.

The method used to verify (1) is the calculus of Radon–Nikodym derivatives of measures in the space of trajectories (Girsanov's formula). Scaling simultaneously force and displacement has also a technical advantage: there is no need to show existence, under the perturbed evolution, of an invariant measure for the process "environment seen from the test particle" such that it is equivalent to the invariant measure under the unperturbed evolution.

Keywords: Girsanov formula; Interacting particle system; Random environment; Central limit theorem; Boltzmann factor; Einstein relation

1. Introduction

In recent years, some rather general central limit theorems have been established for additive functionals of reversible stationary Markov processes; see e.g. Kipnis and

*Corresponding author.

[☆]Supported by NSF, Grant DMR 89-18903 and Deutsche Forschungsgemeinschaft (SFB 359: Reaktive Strömungen, Diffusion und Transport).

Varadhan (1986), De Masi et al. (1989). A typical example of such a functional is the displacement of a particle moving in \mathbb{R}^d under the influence of a random environment; quite different types of models are covered by the results mentioned above, such as diffusion of a single particle in a frozen (constant in time) random environment or motion of a test particle in a “heat bath” of infinitely many other particles. In all those cases the displacement process of the particle converges in the usual central limit scaling to a centered Brownian motion. The question we want to study in the present article is whether the covariance matrix D characterizing the limiting Brownian motion satisfies a so-called *Einstein relation*, i.e. an identity of the form

$$D = 2kT \times \text{mobility}, \quad (1.1)$$

if one uses a physically reasonable definition of what, for a given concrete model, should be called “mobility” and “Boltzmann factor” $\beta = 1/kT$. (Differently from common physics notation we write a factor 2 at the right hand side of (1.1), due to the normalization chosen: we characterize a diffusion process with independent increments by the covariance matrix of its increment per unit of time; if $D = (D_{ij})$ is this matrix the generator of the process is equal to $\frac{1}{2} \sum D_{ij} \partial_i \partial_j$.)

In order to define β we imbed the Markovian semigroup (P_t) governing the evolution of the complete system, particle plus environment, into a family (P_t^a) , $a \in U$, where U is a neighborhood of 0 in \mathbb{R}^d , such that $P_t = P_t^0$. We want to interpret a as an external force acting on the particle. Let us describe the general principle of our construction (for concrete examples, see the subsequent Sections 2–4): We start with a stationary, time reversible system undergoing a Markovian evolution governed by (P_t) ; this system is supposed to consist of two components, \mathbb{R}^d (or \mathbb{Z}^d) and some abstract space E , corresponding to the position of the particle and the state of the environment, respectively; the group of spatial shifts acts in a natural way on either component. We assume that ρ , the distribution of the system at any fixed time, is invariant under spatial shifts; it is infinite, since its projection on the first component is (a multiple of) Lebesgue measure. We exhibit for each $a \in U$ a Markovian semi-group (P_t^a) in such a way that for some a priori given constant β , independent of a , the measure ρ^a , defined by

$$d\rho^a = d\rho \exp(\beta a \cdot x), \quad (1.2)$$

will be invariant and reversible for (P_t^a) . (The dot denotes the usual scalar product in \mathbb{R}^d or the action of a matrix on a vector.) The constant β is interpreted as Boltzmann’s $1/kT$ and a as force acting on the test particle. The reason for choosing this terminology is clear: ρ^a , the equilibrium state for the dynamics (P_t^a) considered here, is related to ρ , the equilibrium state in the force free case, by a factor $\exp(-\beta H(x))$, where $H(x) = -a \cdot x$ is the Hamiltonian corresponding to the field of constant force a . The concrete form of the perturbed dynamics (P_t^a) is to some extent arbitrary; the simple requirement that it be reversible for a certain measure does not result in a unique canonical choice. The typical example of perturbation we have in mind is the following: the unperturbed trajectory of the particle is given as solution of an Itô equation containing some centered Brownian motion as its driving noise; the

perturbed trajectory is defined by adding to the noise a drift term proportional to the force (see e.g. Section 2).

The definition of mobility is less obvious. One possible approach is the following: define first the velocity as displacement per unit time of the test particle under the perturbed evolution (P_t^a), be it pathwise in the limit $t \rightarrow \infty$ (assume the system is ergodic) or be it as expectation of that quantity; denote the velocity by $w(a)$; define the mobility matrix M by

$$M_{ij} = \partial_j w_i(a)|_{a=0}. \tag{1.3}$$

Such a procedure, however, requires not only differentiability but, even more fundamentally, existence of $w(a)$. If one tries to prove the latter using Birkhoff’s ergodic theorem, one has to show first existence of an invariant probability measure under the perturbed evolution for the “system seen from the particle”; more precisely: it is not meaningful to have *any* invariant measure; one has to demand that it is equivalent to the invariant measure for the unperturbed evolution. Unfortunately, we are not able to produce such an existence theorem in sufficient generality; to our knowledge it only holds in some rather restricted cases, like frozen environment in dimension one or frozen periodic environment in arbitrary dimension (see Ferrari et al. (1985)); it is not clear to us whether one should expect it to be true in general within the framework of our hypotheses. A second argument against this attempt of defining mobility is of physical nature: the passage from micro- to macroscale is performed by choosing ε^{-1} and ε^{-2} as new spatial resp. temporal units. Adding a non-zero drift to the motion of the particle leads after a macroscopic time to an additional displacement of order ε^{-1} on the macroscale; thus, after perturbation, the macroscopic displacement of the particle at finite times would become arbitrarily large.

Hence we are led to a different approach for defining mobility, which is in better agreement with physical intuition, see Ferrari et al. (1985). We fix $a \in \mathbb{R}^d$, take for each $\varepsilon > 0$ the dynamical system defined by ($P_t^{a\varepsilon}$), rescale space and time by the factors ε^{-1} and ε^{-2} , respectively, and consider the macroscopic effect due to the change of dynamics. If we are able to show that the law of the displacement process under the new dynamics, as ε tends to 0, converges to a Brownian motion with diffusion matrix D (the same as in the unperturbed case) and drift b , where b is a linear function of a :

$$b = M \cdot a, \tag{1.4}$$

then we say that M is the mobility matrix of the test particle; it describes how, in a linear régime, the displacement process responds to an external force. Technically speaking, the difference between the first and second attempt to define mobility lies in the order in which the limits involved are carried out: first $t \rightarrow \infty$ and then $a \rightarrow 0$ or both limits simultaneously at an appropriate speed.

Let us neglect for a moment the mathematical problem whether the two ways of passing to the limit are equivalent. We claim that our approach of defining mobility and Boltzmann factor β and trying to establish the relation (1.1) is in perfect agreement with Einstein’s original idea. In the modern probabilistic language, this idea can simply be expressed by saying that for Brownian motion with diffusion matrix D and drift b the measure $dx \exp(c \cdot x)$ is reversible and invariant if and only if

the relation

$$b = \frac{1}{2} D \cdot c \tag{1.5}$$

holds. (The point in Einstein's article, however, was not (1.5) as a mathematical statement, but the physical explanation of the behavior of real Brownian particles which one observes through the microscope. Einstein furthermore argued that measuring b and D for such particles provides a method of obtaining the microscopic Boltzmann's constant k by macroscopic observations. This was a strong argument in favor of the atomic structure of matter, see e.g. Einstein (1905, 1956), Nelson (1967).)

The purpose of the present paper is to show, for a wide class of models of self diffusion of the type considered in De Masi et al. (1989) or Papanicolaou and Varadhan (1985), that under the scaling described above convergence of the perturbed displacement process to a non-centered Brownian motion holds, where the drift is a linear function of the "force", with a mobility, in the sense of (1.4), satisfying the Einstein relation (1.1). The main tool used in the proofs is Girsanov's formula: one passes from the unperturbed to the perturbed dynamical system through a change of measures in the underlying probability space; it can be applied as long as the two measures are equivalent or as long as the observation times are of order ε^{-2} . The method relies heavily on the stochastic and Markovian character of the dynamics; it fails completely in a deterministically evolving purely "mechanical" system in which all randomness stems from the initial data. The work of Calderoni and Dürr (1989) even shows that an Einstein relation does *not* hold for some peculiar initial states with a highly degenerate velocity distribution (where only values ± 1 occur). One may expect, however, that it is true for the class of Gibbs or equilibrium measures as initial states, i.e. under a Gaussian velocity distribution; see Ferrari et al. (1985). At present, we do not know of any method how to establish an Einstein relation in that class of models in desirable generality.

The organization of the paper is as follows: In Section 2 we prove the result for a tagged particle in a system of interacting diffusions. Section 3 then shows the analogous result for a single particle performing a random walk in a frozen random environment on the lattice. (There is no intrinsic reason for stating the first result in a spatially continuous and the second one in a spatially discrete version; we just want to show that our method works in either set up.) Section 4 treats the case of a single particle performing an Ornstein–Uhlenbeck process in a random potential. We remark that the case of a tagged Ornstein–Uhlenbeck particle belonging to an infinite system can be treated in the same way – provided one has first established a central limit theorem for the unperturbed dynamics.

2. Interacting diffusions

We start by verifying the validity of the Einstein relation in the case of a tagged particle in a system of interacting diffusion particles. Such a system is defined as the

solution $X = (X_i, i \geq 0)$ of the coupled Itô equations

$$X_i(t) = y_i + \int_0^t B_i(X(s)) ds + \sigma W_i(t), \quad i = 0, 1, 2, \dots,$$

$$B_i(x) = - \sum_{j \neq i} \nabla U(x_i - x_j), \tag{2.1}$$

with W_i being independent standard Brownian motions. In Lang (1977) conditions on the function U ("pair potential"), like smoothness, finite range, superstability, are given which ensure existence and pathwise uniqueness of solutions to (2.1), if the initial configuration y is chosen at random, in such a way that the law of the point set $\{y_i\}$ is the Gibbs measure, formally written as $\mu = c \cdot \exp(-\beta H)$, for the Hamiltonian

$$H(y) = \sum_{i < j} U(y_i - y_j) - \lambda \sum_i 1 \tag{2.2}$$

with Boltzmann factor $\beta = 2\sigma^{-2}$ and suitable values of the chemical potential λ . It is also shown there that after symmetrization, i.e. after forgetting the labels i of the individual particles, the process X evolves as a reversible Markov process with μ as its law at each fixed time instant. For our purpose less is needed, namely unique solvability of (2.1) in distributional sense.

To be precise: let $\mathcal{C}' = \mathcal{C}([0, \infty), \mathbb{R}^d)$ be the space of possible trajectories of one particle, and \mathcal{C} the Cartesian product of countably many copies of \mathcal{C}' . Denote by $x = (x_i, i \geq 0)$ an element of \mathcal{C} ; let \mathcal{F}_t be the sigma-field on \mathcal{C} generated by $x_i(s), i \geq 0, 0 \leq s \leq t$, and \mathcal{F} the supremum of the \mathcal{F}_t . (As usual, we will also write X or X_i instead of x or x_i if we want to stress the aspect that we consider "random elements" of \mathcal{C} .) Since we are interested in the motion of a test particle we give that particle the label $i = 0$ and locate it initially at the origin. The random set formed by the other particles gets as initial distribution μ_0 , the Palm measure associated to μ ; this measure has the explicit form

$$\mu_0 = c\mu \exp\left(- \sum_{i \geq 1} U(x_i)\right),$$

with c being a normalizing constant. Hence we look for a probability measure \mathbb{P} on $(\mathcal{C}, \mathcal{F})$ such that

(2.3a) $\mathbb{P}(x_0(0) = 0) = 1;$

(2.3b) under \mathbb{P} , the random set $\{x_i(0), i \geq 1\}$ is a point process with law $\mu_0;$

(2.3b) under \mathbb{P} , the processes Z_i with

$$Z_i(t) = x_i(t) - x_i(0) - \int_0^t B_i(x(s)) ds, \quad i \geq 0,$$

are independent Brownian motions with respect to the filtration (\mathcal{F}_t) , having diffusion constant σ^2 .

We make the following assumption on U and μ :

(2.4) the measure \mathbb{P} solving (2.3) exists and is unique up to permutation of indices.

Then Theorem 6.2 of De Masi et al. (1989) becomes applicable; it states that under \mathbb{P} the process $\varepsilon X_0(\varepsilon^{-2})$ converges, as $\varepsilon \rightarrow 0$, in law to a Brownian motion with some diffusion matrix D given by

$$D = \sigma^2 I - 2 \int dt \left\{ \int B_0(x(0)) \cdot B'_0(x(t)) d\mathbb{P} \right\}. \quad (2.5)$$

(Here B'_0 denotes the transpose of the vector B_0 and I the identity matrix.)

In order to model the perturbed dynamical system with a constant force acting on the test particle, we define the probability measure \mathbb{P}^a by the following properties:

$$(2.6a) \quad \mathbb{P}^a = \mathbb{P} \quad \text{on } \mathbb{F}_0;$$

$$(2.6b) \quad \text{under } \mathbb{P}^a, \text{ the processes}$$

$$t \rightarrow Z_i(t) - \delta(i, 0)ta, \quad i \geq 0,$$

are independent Brownian motions with respect to the filtration (\mathcal{F}_t) , with diffusion constant σ^2 .

The measure \mathbb{P}^a exists and is unique, by Girsanov's formula (Stroock and Varadhan (1979), Theorem 6.4.2). For each finite t the measures \mathbb{P} and \mathbb{P}^a are equivalent on \mathcal{F}_t with Radon–Nikodym derivative

$$d\mathbb{P}^a/d\mathbb{P} = \exp\{\sigma^{-2}(Z_0(t) \cdot a - a \cdot at/2)\}. \quad (2.7)$$

After symmetrizing the non-tagged particles (the “environment”) we get under \mathbb{P} a stationary reversible dynamical system if we consider the process (“environment seen from the particle”) whose state at time t is the point set $\{X_i(t) - X_0(t): i \geq 1\}$; its law at each time instant is equal to μ_0 ; see De Masi et al. (1989), Section 6.

It is convenient for this type of problem to look also at the extended dynamical system which consists of test particle plus environment. Its state space is the Cartesian product of \mathbb{R}^d and the space of point configurations in \mathbb{R}^d : without perturbation, the natural sigma-finite reversible measure is, up to a constant factor,

$$\rho = \int_{\mathbb{R}^d} \delta_u \otimes \tau_u(\mu_0) du. \quad (2.8)$$

(Here δ_u denotes the measure assigning mass one to the point u , and τ_u the shift by the vector u , which acts in \mathbb{R}^d and in a natural way also in the space of point configurations in \mathbb{R}^d and in the space of measures on such configurations.) Under the perturbation corresponding to \mathbb{P}^a , a sigma-finite reversible measure for the extended system is

$$\rho^a = \int_{\mathbb{R}^d} \delta_u \otimes \tau_u(\mu_0) \exp(2\sigma^{-2}u \cdot a) du, \quad (2.9)$$

as one easily checks. Hence, interpreting a as a “force”, equal to the negative gradient of the “potential” $u \rightarrow u \cdot a$, and comparing ρ and ρ^a one sees that the Boltzmann factor has to be chosen according to the formula

$$2kT = \sigma^2. \quad (2.10)$$

We now proceed to the proper rescaling of space, time and force. We fix $a \in \mathbb{R}^d$ and compare for each $t > 0$ the measures $\mathbb{P}^{a\varepsilon}$ and \mathbb{P} on the sigma-field $\mathcal{F}(t\varepsilon^{-2})$; then we pass to the limit $\varepsilon \rightarrow 0$.

Theorem 1. For fixed a , as ε tends to 0, the law of $X_0^\varepsilon = \varepsilon X_0(\varepsilon^{-2})$ under $\mathbb{P}^{a\varepsilon}$ converges weakly in $\mathcal{C}([0, \infty))$ to Brownian motion with diffusion matrix D and drift

$$b = \sigma^{-2} D \cdot a. \quad (2.11)$$

Remark. This is the Einstein formula we have been looking for: the linear relation (2.11) between force and drift identifies the mobility as $\sigma^{-2} D$, which is equal to $2kTD$ in view of (2.10).

Proof. Using the notation $Z_0^\varepsilon = \varepsilon Z_0(\varepsilon^{-2})$ we get from (2.7)

$$d\mathbb{P}^{a\varepsilon}/d\mathbb{P} = \exp\{\sigma^{-2}(Z_0^\varepsilon(t) \cdot a - a \cdot a t/2)\} \quad \text{on } \mathcal{F}(t\varepsilon^{-2}). \quad (2.12)$$

Under \mathbb{P} , the process Z_0^ε is Brownian motion with drift zero and diffusion constant σ^2 . We know further that the law of X_0^ε under \mathbb{P} converges and we know the limiting distribution, see (2.5). In order to prove convergence of X_0^ε under the "perturbed" measure $\mathbb{P}^{a\varepsilon}$ and to identify the limit we only need information on the limiting joint distribution of X_0^ε and the logarithm of the derivative $d\mathbb{P}^{a\varepsilon}/d\mathbb{P}$, which is the same as information on the joint law of X_0^ε and Z_0^ε . This argument is quite common in statistics under the name of Le Cam's third lemma, see e.g. Hájek and Šidák (1967), p. 208. We call $\mathbb{Q}^{a,\varepsilon}$ the law of $(X_0^\varepsilon(s), Z_0^\varepsilon(s))$, $0 \leq s \leq t$, under $\mathbb{P}^{a\varepsilon}$, which is a measure on $\mathcal{C}([0, t])^2$; we denote by v and w the first and second coordinate, respectively, in that space. In De Masi et al. (1989), Theorem 6.2, not only convergence of X_0^ε is proven, but joint convergence of $(X_0^\varepsilon, Z_0^\varepsilon)$, i.e. weak convergence of $\mathbb{Q}^{0,\varepsilon}$. (The reason: both processes are, approximate or exact, martingales for the same filtration.) We denote the limit, if it exists, by \mathbb{Q}^a ; it is computed in that paper for the case $a = 0$ as $2d$ -dimensional centered Brownian motion with diffusion matrix

$$\Sigma = \begin{bmatrix} D & D \\ D & \sigma^2 \end{bmatrix}. \quad (2.13)$$

The three-fold appearance of the block D in the matrix Σ means that in the limit X_0^ε and $X_0^\varepsilon - Z_0^\varepsilon$ become orthogonal. Indeed they are orthogonal for each finite ε . This fact is explained in the paper quoted by a time reversal argument for the underlying dynamical system $\{X_i - X_0: i \geq 1\}$ in the unperturbed case: $X_0(T) - X_0(0)$ is the displacement of the test particle during the interval $[0, T]$ and changes its sign if one passes from the time parameter t to $T - t$; on the other hand, $X_0(T) - Z_0(T)$ is equal to $\int_0^t B_0(x(t)) dt$ and remains unchanged if t is replaced by $T - t$. Because the whole dynamical system is time reversible, "even" and "odd" quantities are orthogonal to each other. Since the ratio

$$d\mathbb{Q}^{a,\varepsilon}/d\mathbb{Q}^{0,\varepsilon} = \exp\{\sigma^{-2}(w(t) \cdot a - a \cdot a t/2)\} \quad (2.14)$$

is trivially uniformly integrable in ε (the projection of $\mathbb{Q}^{0,\varepsilon}$ on the second coordinate is the same for all ε), we obtain weak convergence of $\mathbb{Q}^{a,\varepsilon}$ to a measure \mathbb{Q}^a satisfying

$$\mathbb{Q}^a = \mathbb{Q} \exp\{\sigma^{-2}(w(t) \cdot a - a \cdot a t/2)\}. \tag{2.15}$$

Another application of Girsanov’s formula characterizes the measure \mathbb{Q}^a : under it, the pair (v, w) is Brownian motion with diffusion matrix Σ and drift vector

$$\begin{bmatrix} D & D \\ D & \sigma^2 \end{bmatrix} \begin{bmatrix} 0 \\ a\sigma^{-2} \end{bmatrix} \text{ which is equal to } \begin{bmatrix} D \cdot a\sigma^{-2} \\ D \cdot a \end{bmatrix}. \tag{2.16}$$

In particular, the v -coordinate is Brownian motion with diffusion matrix D and drift $D \cdot a\sigma^{-2}$, as claimed in (2.11). \square

A slight generalization of Theorem 1 is the following variant, where one replaces the constant force a by a position dependent force field $a(\cdot)$ which varies on a macroscopic scale.

Theorem 2. *Let $a(\cdot)$ be a bounded continuous vector field in \mathbb{R}^d . Let for each $\varepsilon > 0$ the measure $\mathbb{P}^{a,\varepsilon}$ on $(\mathcal{C}, \mathcal{F})$ satisfy the conditions*

(2.17a) $\mathbb{P}^{a,\varepsilon} = \mathbb{P}$ on $\mathcal{F}(0)$;

(2.17b) under $\mathbb{P}^{a,\varepsilon}$, the processes

$$t \rightarrow Z_i(t) - \delta(i, 0) \int_0^t \varepsilon a(\varepsilon x_i(s)) ds,$$

are independent centered Brownian motions for the filtration $(\mathcal{F}(t))$ with diffusion constant σ^2 .

Then the law of X_0^ε under $\mathbb{P}^{a,\varepsilon}$ converges, as $\varepsilon \rightarrow 0$, to a diffusion with constant diffusion matrix D and drift vector field $M \cdot a(x)$, with $M = \sigma^{-2}D$.

Sketch of proof. The arguments follow exactly the lines of the preceding proof; the only difference lies in the expression for the ratio of \mathbb{Q}^ε and $\mathbb{Q}^{a,\varepsilon}$ on the field of events describable by what happens before the macroscopic time t :

$$d\mathbb{Q}^{a,\varepsilon}/d\mathbb{Q}^\varepsilon = \exp\left\{\sigma^{-2}\left(\int_0^t a(v(s)) \cdot dw(s) - 1/2 \int_0^t a(v(s)) \cdot a(v(s)) ds\right)\right\}. \tag{2.18}$$

Because of the boundedness of $a(\cdot)$ this expression is again uniformly integrable in ε . The joint convergence in law of v and w , together with the continuity of $a(\cdot)$, yields convergence in law of the stochastic and the ordinary Lebesgue integral on the right hand side of (2.18), see Theorem 2.6 in Jakubowski et al. (1989). From there we

conclude that, in the limit $\varepsilon \rightarrow 0$, the measures $\mathbb{Q}^{a,\varepsilon}$ tend to

$$\mathbb{Q}^a := \mathbb{Q} \exp \left\{ \sigma^{-2} \left(\int_0^t a(v(s)) \cdot dw(s) - 1/2 \int_0^t a(v(s)) \cdot a(v(s)) ds \right) \right\}, \tag{2.19}$$

on the field generated by $v(s), w(s), 0 \leq s \leq t$. Another application of Girsanov's theorem shows that \mathbb{Q}^a is the diffusion with the characteristics claimed.

3. Symmetric random walk on a lattice

In this section we study the motion of a single particle on a lattice with time independent random inhomogeneities. The difference between spatially discrete and continuous modelling lies in the fact that the martingale part of the logarithm of the likelihood processes is now no longer an exact Brownian motion and hence unchanged under rescaling, but a compensated jump process, which only in the limit $\varepsilon \rightarrow 0$ becomes Brownian. We have chosen the case of a single particle in a frozen random environment because it simplifies notations; it is clear how the construction has to be modified for the case of a test particle in a lattice gas of moving particles.

The model is described as follows. The particle moves in \mathbb{Z}^d . To each non-oriented bond which connects two neighboring sites x and $y = x + u$ we associate a random variable (jump rate) $c(x, u)$ such that

$$(3.1a) \quad c(x, u) = c(x + u, -u) \quad \text{for all } x \text{ and all } u \text{ with } |u| = 1;$$

$$(3.1b) \quad C_1 \leq c(\cdot, \cdot) \leq C_2 \quad \text{for some constants } 0 < C_1, C_2 < \infty;$$

$$(3.1c) \quad \text{the law of the system } \{c(x, u), x, u \in \mathbb{Z}^d, |u| = 1\} \text{ is invariant and ergodic under the group of spatial translations.}$$

We denote the law in (3.1c) by μ ; it is a measure on $E := \mathbb{R}^B$, B being the set of nearest neighbor bonds. The random walk X on \mathbb{Z}^d is defined by the property that, conditioned on the system $c = \{c(x, u)\}$, its generator is

$$L_c f(x) := \sum_{|u|=1} c(x, u)(f(x + u) - f(x)); \tag{3.2}$$

we will assume that $X(0) = 0$. A central limit theorem for X has been established by many authors, we just mention Künnemann (1983); it also follows from the general results in De Masi et al. (1989), Section 4. We denote again by D the diffusion matrix of the limiting process.

In order to define the perturbed system we fix a vector $a \in \mathbb{R}^d$ and define the associated generator of the random walk, conditioned on $\{c(x, u)\}$, by

$$L_c^a f(x) := \sum_{|u|=1} c(x, u) \exp(a \cdot u) (f(x + u) - f(x)). \tag{3.3}$$

As in (2.8) we consider also the extended system, position of particle plus environment. The measure

$$\rho = \sum_x \delta_x \otimes \mu \tag{3.4}$$

is reversible for that system under (3.2), whereas under (3.3) the measure

$$\rho^a = \sum_x \exp(2a \cdot x) \delta_x \otimes \mu \tag{3.5}$$

becomes reversible. Hence, by the same reasoning as before, we obtain an admissible choice for the Boltzmann factor interpreting a as “force” and setting $2kT = 1$.

The sample space for the process we consider is $\Omega := E \times S$, where S denotes the set of \mathbb{Z}^d -valued right continuous functions with left limits; an element of Ω will be denoted by (c, x) . On Ω we introduce the filtration $(\mathcal{F}(t))$, where $\mathcal{F}(t)$ is the sigma-field generated by c and $x(s)$, $s \leq t$. The measure \mathbb{P} on Ω corresponding to the unperturbed dynamics (3.2) can be characterized by

(3.6a) the projection of \mathbb{P} on E is μ ;

(3.6b) $\mathbb{P}(x(0) = 0) = 1$;

(3.6c) for each f on \mathbb{Z}^d with finite support the process

$$t \rightarrow f(x(t)) - \int_0^t L_c f(x(s)) ds$$

is an $(\mathcal{F}(t))$ -martingale.

Similarly, one obtains the measure \mathbb{P}^a corresponding to the perturbed dynamics if one replaces L_c in (3.6c) by L_c^a .

Since in our interpretation $2kT$ is equal to 1 and the “force” equal to a , the Einstein relation can here be stated as in the following theorem.

Theorem 3. For any a , the law of $X^\varepsilon := \varepsilon X(\varepsilon^{-2})$ under $\mathbb{P}^{a\varepsilon}$ converges weakly in the Skorohod space $\mathcal{D}(\mathbb{R}^d)$ to Brownian motion with diffusion matrix D and drift b , where

$$b = D \cdot a. \tag{3.7}$$

Proof. By Girsanov’s formula for point processes (see Brémaud (1981), Theorem 3 in Ch. VI.2), the two measures satisfy on $\mathcal{F}(t)$ the relation

$$\log(d\mathbb{P}^a/d\mathbb{P}) = \sum_{|u|=1} \left\{ \int_0^t c(x(s), u) [1 - \exp(a \cdot u)] ds + \sum_{\Delta x(s)=u, s \leq t} a \cdot u \right\}. \tag{3.8}$$

The second term on the right is just $a \cdot x(t)$; hence after rescaling one gets on $\mathcal{F}(t\varepsilon^{-2})$

$$\log(d\mathbb{P}^{a\varepsilon}/d\mathbb{P}) = \varepsilon a \cdot x(t\varepsilon^{-2}) + \sum_u \left\{ \int_0^{t\varepsilon^{-2}} c(x(s), u) [1 - \exp(\varepsilon a \cdot u)] ds \right\}. \tag{3.9}$$

It is convenient to introduce the function $\varphi(y) = y^{-2}(\exp(y) - 1 - y)$ and to write

$$1 - \exp(\varepsilon a \cdot u) = -\varepsilon a \cdot u - \varepsilon^2 (a \cdot u)^2 \varphi(\varepsilon a \cdot u). \tag{3.10}$$

Further, we observe that $t \rightarrow \sum_u \left\{ \int_0^t c(x(s), u) u ds \right\}$ is the compensator under \mathbb{P} of the jump process $x(\cdot)$; hence

$$t \rightarrow Z^\varepsilon(t) := \varepsilon a \cdot \left\{ x(t\varepsilon^{-2}) - \sum_u \int_0^{t\varepsilon^{-2}} c(x(s), u) u ds \right\} \tag{3.11}$$

is a \mathbb{P} -martingale and (3.9) can be rewritten in closer analogy to Girsanov as

$$\log(d\mathbb{P}^{a\varepsilon}/d\mathbb{P}) = Z^\varepsilon(t) - \varepsilon^2 \int_0^t \sum_u c(x(s), u) (a \cdot u)^2 \varphi(\varepsilon a \cdot u) ds. \tag{3.12}$$

Here, because the process “environment seen from the particle” is ergodic and since $\varphi(y)$ has the limit $\frac{1}{2}$ at $y = 0$, the second term converges, as $\varepsilon \rightarrow 0$, in probability to the deterministic process

$$t \rightarrow t/2 \int d\mu \sum_u c(0, u) (a \cdot u)^2, \tag{3.13}$$

which is the common value of the expected quadratic variation for all processes Z^ε . The boundedness of jump sizes and jump rates of X imply uniform (in ε) integrability of all exponentials of linear functions of the compensated X^ε process, in particular of $\exp(Z^\varepsilon(t))$. Further, by Theorem 2.4 in De Masi et al. (1989), the law of the pair $(X^\varepsilon, Z^\varepsilon)$ under \mathbb{P} converges to some $(d + 1)$ -dimensional centered Brownian motion; call Σ its diffusion matrix. Hence, by Le Cam’s third lemma, the law of $(X^\varepsilon, Z^\varepsilon)$ under $\mathbb{P}^{a\varepsilon}$ converges, too, and the limit is Brownian motion with diffusion matrix Σ and drift

$$f = \Sigma \cdot e, \tag{3.14}$$

where e is the vector with the last component equal to 1 and the others equal to 0. The first d components of f form the drift vector b we are looking for; we compute

$$\begin{aligned} b_i &= \Sigma(i, d + 1) = \lim_{\varepsilon \rightarrow 0} \int X_i^\varepsilon(1) Z^\varepsilon(1) d\mathbb{P} = \lim_{\varepsilon \rightarrow 0} \int X_i^\varepsilon(1) \sum_k a_k X_k^\varepsilon(1) d\mathbb{P} \\ &= \sum_k D_{ik} a_k, \quad \text{for } i = 1, \dots, d, \end{aligned} \tag{3.15}$$

where as in the last section we use the fact that the variable $X(t)$ is orthogonal to its compensator at time t : the displacement $X(t)$ changes its sign after time reversal, whereas the compensator, being the time integral of a function on state space, remains unchanged. Remark that, as a consequence of uniform exponential integrability, the second moments under \mathbb{P} of the processes X^ε and Z^ε are conserved in the limit $\varepsilon \rightarrow 0$. Since (3.15) is identical to (3.7) the proof is complete. \square

4. Ornstein–Uhlenbeck process in a random medium

The model we will consider in this section is the same as in Papanicolaou and Varadhan (1985). It consists of a single particle in a random potential whose velocity is

governed by a Langevin equation. That means the following: we are given a process $U = \{U(x), x \in \mathbb{R}^d\}$, stationary and ergodic, which we assume to be bounded together with its first and second derivatives; the law of that process is a probability measure on the space E of continuous functions in \mathbb{R}^d and will be denoted by π . If we denote by X and V the position and velocity of the particle, then conditioned upon U the pair (X, V) satisfies the stochastic differential equation

$$X(t) = X(0) + \int_0^t V(s) ds, \quad (4.1a)$$

$$V(t) = V(0) - \int_0^t \{\alpha V(s) + \nabla U X(s)\} ds + \sigma W(t), \quad (4.1b)$$

where W is standard Brownian motion, independent of U . The numbers α and σ are positive parameters.

The probability space (Ω, \mathbb{P}) on which the processes U, X, V are defined is constructed in the following way. Let (\mathcal{C}, μ) be the space $\mathcal{C}([0, \infty), \mathbb{R}^d)$ equipped with standard Wiener measure; consider \mathbb{R}^d equipped with the Gaussian measure $N(0, \sigma^2/2\rho)$. Denote the elements of $E, \mathcal{C}, \mathbb{R}^d$ respectively by u, w, v . Define Ω as the Cartesian product of these three spaces. On Ω we introduce the filtration $(\mathcal{F}(t))$, with $\mathcal{F}(t)$ generated by u, v and by $w(s), s \leq t$; as usual, \mathcal{F} denotes the supremum of all $\mathcal{F}(t)$. We define the probability measure \mathbb{P} on (Ω, \mathcal{F}) by

$$\mathbb{P} = c\pi \exp(-2\alpha/\sigma^2 u(0)) \otimes \mu \otimes N(0, \sigma^2/2\alpha) \quad \text{with } c \text{ a normalizing constant.} \quad (4.2)$$

The process (X, V) is then defined on the filtered probability space (Ω, \mathbb{P}) as pathwise solution of (4.1) under the initial condition $V(0) = v, X(0) = 0$. An elementary calculation shows that for all t the law of $(U(X(t) + \cdot), V(t))$ is equal to $c\pi \exp(-2\alpha/\sigma^2 u(0)) \otimes N(0, \sigma^2/2\alpha)$. This amounts to saying that in the extended system consisting of position and velocity, conditioned upon $U = u$, a sigma-finite invariant measure is given by

$$dv_u = dx dv \exp(-2\alpha/\sigma^2 \{u(x) + 1/2v^2\}). \quad (4.3)$$

We define in this model the action of an additional force a on the particle by adding the term ta at the right hand side in (4.1b). (This a is “really” a force if the mass of the particle is equal to 1.) This is equivalent to replacing the driving Brownian motion W by $t \rightarrow W(t) + ta/\sigma$, or to replacing the probability measure \mathbb{P} on Ω by \mathbb{P}^a , which is defined as

$$\mathbb{P}^a = \mathbb{P} \exp\{\sigma^{-2}(a \cdot \sigma W(t) - 1/2ta \cdot a)\} \text{ on } \mathcal{F}(t). \quad (4.4)$$

In Papanicolaou and Varadhan (1985) it was shown that (under the probability measure \mathbb{P}) the process $X^\varepsilon := \varepsilon X(\varepsilon^{-2} \cdot)$ converges weakly to a centered Brownian motion with some non-degenerate diffusion matrix D . The problem to solve in our context is the study of X^ε under the law $\mathbb{P}^{a\varepsilon}$ in the limit $\varepsilon \rightarrow 0$. Carrying out the scaling $a \rightarrow \varepsilon a$ and $t \rightarrow \varepsilon^{-2}t$ in (4.4) we see that we must prove a central limit theorem under

the measure \mathbb{P} for the couple

$$(X^\varepsilon, W^\varepsilon) \text{ with } W^\varepsilon = \varepsilon W(\varepsilon^{-2} \cdot), \tag{4.5}$$

and identify the limiting diffusion matrix, because $a \cdot \sigma^{-1} W^\varepsilon - \frac{1}{2} t \sigma^{-2} a \cdot a$ is the logarithm of the Radon-Nikodym derivative on $\mathcal{F}(\varepsilon^{-2} t)$ of $\mathbb{P}^{a\varepsilon}$ with respect to \mathbb{P} .

First, from Papanicolaou and Varadhan (1985) one sees that for t fixed $X^\varepsilon(t)$ is uniformly (in ε) square integrable; the same is trivially true for $W^\varepsilon(t)$, since W^ε is standard Brownian motion for each ε . Further, in that article, the process X^ε is approximated in quadratic mean by a martingale for the filtration $(\mathcal{F}(t\varepsilon^{-2}))$; since W^ε is Brownian motion for the same filtration, the *jointly* Brownian character of these processes in the limit follows from stationarity of their increments and ergodicity. Let us compute their limiting covariance matrix, which we call Σ . From (4.1) we get the identity

$$\alpha X(t) = V(0) - V(t) - \int_0^t \nabla U X(s) ds + \sigma W(t). \tag{4.6}$$

If we multiply both sides by the transpose of $X(t)$, divide by t and integrate over \mathbb{P} we obtain as $t \rightarrow \infty$

$$\alpha \lim_t t^{-1} \int X(t) X(t)' d\mathbb{P} = \sigma \lim_t t^{-1} \int W(t) X(t)' d\mathbb{P},$$

or

$$\alpha D_{ij} = \sigma \Sigma_{i+d, j} \text{ for } 1 \leq i, j \leq d. \tag{4.7}$$

The reason: since $X(t)$ is of order $t^{1/2}$ and V stationary the scalar product of $V(t) - V(0)$ and $X(t)$ divided by t vanishes in the limit; the integral of $\nabla U(X(s))$ is orthogonal to the displacement $X(t) = \int_0^t V(s) ds$: it is invariant under the time reversal $t \rightarrow -t$, $V \rightarrow -V$, $X \rightarrow X$ whereas the latter quantity changes sign; this property is preserved in the scaling limit.

From the above calculation of Radon-Nikodym derivatives and Girsanov's formula we get for the drift b of X^ε under $\mathbb{P}^{a\varepsilon}$ the expression

$$b = \lim_\varepsilon \int \sigma^{-1} a \cdot W^\varepsilon(1) X^\varepsilon(1)' dP, \tag{4.8}$$

which in view of (4.7) can be restated as

$$b_i = \alpha \sigma^{-2} \Sigma_j D_{ij} a_j \tag{4.9}$$

or

$$b = M \cdot a \text{ with } M = \alpha \sigma^{-2} D, \tag{4.10}$$

so that M is the mobility matrix. From formula (4.3) we can read off immediately the right value of Boltzmann's factor, namely $2kT = \sigma^2/\alpha$. This shows that (4.10) is the Einstein relation we are looking for and proves the following theorem:

Theorem 4. For each a , under the measure $\mathbb{P}^{a\varepsilon}$, the process X^ε converges in the limit $\varepsilon \rightarrow 0$ to Brownian motion with diffusion matrix D and drift $b = M \cdot a$, where $M = \alpha \sigma^{-2} D$.

References

- P. Brémaud, Point Processes and Queues, (Springer, Berlin, 1981).
- P. Calderoni and D. Dürr, The Smoluchowski limit for a simple mechanical model, *J. Statist. Phys.* 55 (1989) 695–738.
- A. De Masi, P. Ferrari, S. Goldstein and D. Wick, An invariance principle for reversible Markov processes. Applications to random motion in random environments, *J. Statist. Phys.* 55 (1989) 787–856.
- A. Einstein, Die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik*, 4. Folge, 17 (1905) 549–560.
- A. Einstein, Investigations on the Theory of the Brownian Movement, R. Fürth, ed., Dover, New York, 1956. (English translation of preceding reference.)
- P. Ferrari, S. Goldstein and J.L. Lebowitz, Diffusion, mobility and the Einstein relation, in: J. Fritz et al. ed., *Statistical Physics and Dynamical Systems* (Birkhaeuser, Boston, 1985) pp. 405–441.
- J. Hájek and Z. Šidák, *Theory of Rank Tests* (Academic Press, New York, 1967).
- A. Jakubowski, J. Mémin and G. Pagès, Convergence en loi des suites d'intégrales stochastiques sur l'espace D de Skorokhod, *Probab. Theory Related Fields* 81 (1989) 11–137.
- C. Kipnis and S.R.S. Varadhan, Central limit theorem for additive functionals of reversible processes and applications to simple exclusion, *Comm. Math. Physics* 104 (1986) 1–19.
- M. Künnemann: The diffusion limit for reversible jump processes in Z^d with ergodic bond conductivities. *Comm. Math. Phys.* 90 (1983) 27–68.
- R. Lang, Unendlich-dimensionale Wienerprozesse mit Wechselwirkung, Teil I, *Z. Wahrscheinlichkeitstheorie verw. Geb.* 38 (1977) 55–72.
- E. Nelson, *The Dynamical Theories of Brownian motion*, (Princeton University Press, Princeton, 1967).
- G.C. Papanicolaou and S.R.S. Varadhan, Ornstein-Uhlenbeck process in a random potential, *Comm. Pure Appl. Math.* 38 (1985) 819–834.
- D.W. Stroock and S.R.S. Varadhan, *Multidimensional Diffusion Processes*, (Springer, Berlin, 1979).