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ASYMMETRIC RANDOM WALK ON A RANDOM THUE–MORSE LATTICE ☆

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Dedicated to Benoit Mandelbrot on the occasion of his 65th birthday

We study the behavior of an asymmetric random walk in a one-dimensional environment whose nonuniformity is in between that of quasi-periodic and random. We construct the environment from arithmetic subsequences of the Thue–Morse sequence. The construction induces in a natural way a measure μ on the space of environments which is invariant and ergodic with respect to translations but is not mixing and has zero entropy. The behavior of the random walk is rather similar to that found by Sinai for the Bernoulli case, when μ is a product measure for which the entropy has its maximum value; i.e. the particle motion is subdiffusive, the displacement growing in time as $(\log t)^{1/\beta}$, $\beta = \log 3/\log 4$. The nature of the dramatic Sinai–Golosov "localization" is however quite different, exhibiting an interesting fractal structure whose nature depends upon the time scale of observation.

1. Introduction

We study the behavior of an asymmetric random walk in a one-dimensional environment whose nonuniformity is in between that of quasi-periodic and random. We will specify the environment by a "spin" configuration $\boldsymbol{\xi} = \{\xi_j\}$, $\xi_j = \pm 1$, $j \in \mathbb{Z}$. Given $\boldsymbol{\xi}$ and some $0 < \epsilon < 1$, the random walk has a transition probability at site j to the right, p_j (and to the left, $1 - p_j$), of the simple form $p_j = \frac{1}{2}(1 + \epsilon\xi_j)$.

We will consider environments $\boldsymbol{\xi} = \{\xi_j\}$ which are obtained from arithmetic subsequences of the Thue–Morse substitutional sequence [1,2], in a manner to be described in section 3. The construction will induce in a natural way a unique measure μ on the space of configurations (possible environments) $\{-1, 1\}^{\mathbb{Z}}$ [2]. This measure μ is invariant and ergodic with respect to translations but is not mixing, and has zero entropy.

We shall later see that despite this lack of randomness in the environment, the behavior of the random walk is rather similar to that found by Sinai [3] for the Bernoulli case, when μ is a product measure for which the entropy has its maximum value; i.e. the particle motion is subdiffusive, the displacement growing as a power of a logarithm in time. To see how this comes about we now discuss briefly the general setting of the Sinai theorem, while retaining the simple relation $p_j = \frac{1}{2}(1 + \epsilon\xi_j)$.

In one dimension it is always possible to define a potential energy function $U(j)$ so that the transition probabilities satisfy the detailed balance condition with respect to the (non-normalized) measure $\exp[-U(j)]$, i.e.

$$p_j/(1 - p_{j+1}) = \exp[U(j) - U(j+1)]. \qquad (1)$$

Hence the measure $\exp[-U(j)]$ is stationary. Also, it follows that for a translation-invariant ergodic measure μ on the ξ_j, the condition of no drift is simply $\langle \xi_j \rangle_\mu = 0$.

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Let us now consider the position X_t of the random walk at integer times t, in an ergodic statistically translation-invariant random environment ξ, starting at $X_0=0$. Sinai [3] considers the case in which the ξ_j's are independent random variables. As we shall see in section 2, the potential $U(n)$ grows in proportion to $\sum_{j=1}^{n}\xi_j$; consequently, the fluctuations in $U(n)$ will grow like $\sqrt{n}$ for a typical environment. Now the time necessary for a particle to diffuse over a potential barrier of height H grows (asymptotically) like $\exp(H)$. Therefore the time for the random walk to get to n should grow like $\exp(c\sqrt{n})$ (for some constant c) and the displacement $X_t(\xi)$ of the random walk in time t should grow like $(\log t)^2$,

$$X_t(\xi) \sim (\log t)^2 \tag{2}$$

(and $\langle X_t^2(\xi)\rangle \sim (\log t)^4$), for almost all ξ with respect to the product measure [3].

It is clear from the heuristic discussion given here that the essential feature of the environment leading to the behavior (2) is the fluctuation behavior of $\sum\xi_j$ in (2). When this sum goes like n^β then the power of $(\log t)$ on the right-hand side of (2) will be $1/\beta$. In the model we shall consider $\beta=\log 3/\log 4>\frac{1}{2}$, so the random walk will be even more confined than in the random case.

In addition to the "confinement" expressed by (2), Sinai also showed that for the Bernoulli environment the particle exhibits a more striking form of localization: Sinai proves that the environment ξ determines a function $m(t)=m(t,\xi)$ which is a good indicator of the position X_t of the particle in the sense that as $t\to\infty$, $X_t-m(t)=o[(\log t)^2]$. More striking still, Golosov [4] has shown that $X_t-m(t)=\mathcal{O}(1)$, i.e., that up to an error of order unity, $m(t)$ gives the particle's position at time t, *for arbitrarily large t*. In other words, for any $\epsilon>0$, almost all of the randomness in X_t arises from the environment, for t sufficiently large.

The localization of our random walk exhibits a richer structure than in the case of a Bernoulli environment. For a typical environment and large t there are three possibilities for the position X_t:

(i) X_t is near the center $m(t)$ of a "V-shaped valley". In this case $X_t-m(t)=\mathcal{O}(1)$.

(ii) X_t is near the center $m(t)$ of a "W-shaped valley". In this case $X_t-m(t)=\mathcal{O}(\log t)^{1/\beta}$ and has an asymptotically singular distribution on a Cantor set.

(iii) X_t is near the center of a random valley.

Here $m(t)=m(t,\xi)$ is determined by the environment ξ; the environment also determines which of the three cases occurs at time t; for a typical environment all three cases occur repeatedly.

We also analyze the behavior of our random walk on "macroscopic" time scales, again finding several possibilities. In particular, on "critical" time scales, we find a jump process of transitions between valleys. We also analyze more refined asymptotic distributions.

In section 2 we describe some relevant generalities concerning one-dimensional random walks: the potential function and escape times from valleys. In section 3 we define our random environment, and analyze its hierarchical structure. Section 4 contains the results on the asymptotics of the random walk in our random environment.

2. Potentials, valleys and escape times for one-dimensional random walks

The potential function $U(n)$ which satisfies (1) is most conveniently written as a linear interpolation of a function U defined at the midpoints between sites. Specifically, for $n\in\mathbb{Z}$,

$$U(n)=\tfrac{1}{2}[U(n-\tfrac{1}{2})+U(n+\tfrac{1}{2})], \tag{3}$$

where

$$
\begin{aligned}
U(n+\tfrac{1}{2}) &= -\log(1+\epsilon\xi_0)-\alpha\sum_{j=1}^{n}\xi_j, \qquad \text{for } n\geq 0,\\
&= -\log(1-\epsilon\xi_0)+\alpha\sum_{j=n+1}^{-1}\xi_j, \quad \text{for } n<0,
\end{aligned}
\tag{4}
$$

where $\alpha=\log[(1+\epsilon)/(1-\epsilon)]$ and where $\log[(1+\epsilon\xi_j)/(1-\epsilon\xi_j)]=\alpha\xi_j$ has been used. This corresponds to the normalization $U(0)=-\frac{1}{2}\log(1-\epsilon^2)$.

The potential U defines a stationary (in fact, reversible) measure for the random walk X_t in the random environment $\boldsymbol{\xi}$. All the basic probabilistic characteristics of this process, such as exit probabilities, expected exit times from an interval, and stationary probabilities for an interval with reflecting endpoints (which can be explicitly obtained by solving the corresponding Dirichlet problem) can be naturally expressed in terms of U. Moreover, the structural features of the graph of U, i.e. the nature of the "valleys", play the critical role in the asymptotic analysis of the process X_t. We say that $[a, b]$ (for a, $b\in\mathbb{Z}+\frac{1}{2}$) is a valley for the potential energy function $U(n)$ if there exists a $d\in\mathbb{Z}+\frac{1}{2}$ such that $\min_{n\in[a,b]}U(n)=U(d)$, $\max_{n\in[a,d]}U(n)=U(a)$, and $\max_{n\in[d,b]}U(n)=U(b)$ [3]. The height of such a valley is $H=\min\{U(a), U(b)\}-U(d)$.

Let T_H denote the time required to escape from the bottom of a valley of height H. Now as $H\to\infty$, T_H becomes exponential with mean

$$
\bar{t}\equiv\langle T_H\rangle\sim \mathrm{e}^{H}. \tag{5}
$$

Hence

$$
\begin{aligned}
\operatorname{Prob}[T_H<t] &\sim 1-\mathrm{e}^{-t/\bar{t}}\approx t\,\mathrm{e}^{-H}, \quad \text{for } t\ll\bar{t},\\
&\approx 1, \qquad\qquad\qquad\quad\ \, \text{for } t\gg\bar{t}.
\end{aligned}
\tag{6}
$$

Therefore the probability of escaping by time t from a valley of size $c\log t$ is, for $c>1$, proportional to $t\,\mathrm{e}^{-c\log t}=t^{1-c}\to 0$ as $t\to\infty$, while, for $c<1$, this probability approaches 1 as $t\to\infty$. Thus we have for the "critical potential barrier for time t", $H_{\text{cr}}\sim\log t$. Hence if the fluctuations in $U(m)\sim m^{\beta}$, we have $H_{\text{cr}}\sim m_{\text{cr}}^{\beta}$, and X_t should be near $m_{\text{cr}}\sim(\log t)^{1/\beta}$.

3. A random environment developed from the Thue–Morse sequence

For any non-negative integer n, let $b(n)$ denote the sum of the binary digits of n, and let $a_n=(-1)^{b(n)}$. The sequence a_0, a_1, a_2, ..., is the well-known Thue–Morse sequence [1]. Equivalently, this sequence may be generated by the substitution rule γ: $+1\mapsto+1, -1$; $-1\mapsto-1, +1$. γ maps any finite ±1 sequence to another; we observe that $\gamma^k(+1)$ generates the first 2^k terms of the Thue–Morse sequence. The dynamical systems resulting from substitutions of this form are studied extensively by Queffélec [2].

The Thue–Morse sequence possesses an important block structure. For $k\geq 0$, let A_k denote the block a_0, a_1, ..., a_{4^k-1}, and let $\bar{A}_k$ denote $-a_0$, $-a_1$, ..., $-a_{4^k-1}$. Then, due to the substitution rule, the block A_{k+1} of size 4^{k+1} may be formed by concatenating blocks of size 4^k:

$$
A_{k+1}=A_k\bar{A}_k\bar{A}_kA_k. \tag{7}
$$

We refer to the blocks A_k and $\bar{A}_k$ as the k-blocks, since any block of size 4^k (starting from a multiple of 4^k) in the sequence will be either A_k or $\bar{A}_k$.

Define $\Omega=\{\text{limit points of } S^n(\gamma^\infty(+1))\,|\,n\to\infty\}$, where S is the left shift on $\{+1,-1\}^{\mathbb{Z}}$, i.e. Ω is the set of doubly infinite sequences obtained by shifting the semi-infinite Thue–Morse sequence to the left by n steps and then taking limits $n\to\infty$ along subsequences. Let μ be the unique S-invariant probability measure on Ω. (μ, S) is ergodic, but not even weakly mixing (in fact, S^2 is not ergodic), and it has entropy 0 [2].

The block structure of the Thue–Morse sequence allows a more "constructive" definition of Ω and μ. Let ω_0, η_1, η_2, ... be a sequence of independent random variables, with ω_0 taking on values $+1$ and -1 with equal probability, and each η_j taking on values 1, 2, 3, and 4 with equal probability. This sequence determines a (random) element $\omega=\{\omega_n | n\in\mathbb{Z}\}\in\Omega$ as follows: For every $k\geq 1$, ω is built out of a doubly infinite sequence of k-blocks. η_1 gives the position of the origin in the 1-block containing the origin: $\eta_1=i$ if the origin is the ith site in this block. Similarly, η_k gives the position of the $(k-1)$-block containing the origin in the k-block containing the origin. Finally ω_0 is of course the value of $\boldsymbol{\omega}$ at the origin. Note that ω_0 and η_1 determine whether the 1-block containing the origin is A_1 or $\bar{A}_1$; these together with η_2 determine whether the 2-block containing the origin is A_2 or $\bar{A}_2$; and so on. Moreover, the product measure distribution on ω_0, η_1, η_2, ... gives rise to the previously mentioned distribution μ on Ω. (Sequences with $\eta_k=1$ for all $k>N$ or with $\eta_k=4$ for all $k>N$, for some $N>0$, do not correspond to elements of Ω because they define only a semi-infinite sequence. But there are exactly two elements of Ω to which such a sequence can be extended. The set of all such sequences (which of course has probability zero) is just $\pm$ the Thue–Morse sequence, preceded by $\pm$ its reflection, together with all of its translates. Some of these sequences will play an important role in the asymptotics.)

We now describe the random environment we shall consider for the rest of this paper. Since the behavior of a random walk depends heavily upon the fluctuation behavior of $\sum\xi_j$, we base our environment upon its arithmetic subsequences of difference 3, i.e. we define the set of all possible environments to be $\Xi=\{\xi\colon \xi_j=\omega_{3j},\ \omega\in\Omega\}$. Now, since every $\boldsymbol{\omega}\in\Omega$ has the block structure described in the previous paragraph, it suffices to consider the fluctuations associated with the rarefactions $(a_i, a_{i+3}, a_{i+6}, ...)$, $i\in\{0, 1, 2\}$, of the Thue–Morse sequence itself. Unlike the original sequence, these rarefied sequences exhibit significant fluctuations, cf. Newman [5] and Coquet [6].

More precisely, to determine the fluctuations, we calculate rarefied sums of the form

$$S_i(n)=\sum_{0\leq j<n} a_j,\quad j\equiv i \pmod 3. \tag{8}$$

The following relations are easily derived from the block structure of the Thue–Morse sequence, and the fact that $4^k\equiv 1 \pmod 3$: First, the block A_k (of size 4^k) is symmetric, so for $k\geq 0$

$$S_1(4^k)=S_2(4^k). \tag{9}$$

Next, since $A_{k+1}=A_k\bar{A}_k\bar{A}_kA_k$,

$$S_i(4^{k+1})=S_i(4^k)-S_{i+2}(4^k)-S_{i+1}(4^k)+S_i(4^k). \tag{10}$$

Applying these relations, and the fact that $S_0(4^1)=2$ and $S_1(4^1)=-1$, we see that for $k>0$

$$S_0(4^k)=2\cdot 3^{k-1},\qquad S_1(4^k)=S_2(4^k)=-3^{k-1}, \tag{11}$$

corresponding to fluctuations in $\sum_{j=0}^{n}\xi_j$ on the order of n^{β}, where $\beta=\log 3/\log 4$.

We will now analyze the structure of the potential $U(n)$ by examining the detailed graphs of the $S_i(n)$. We claim that (11), together with the hierarchical structure of our environment, completely determines U. As suggested by (3) and (4), it is convenient to focus on the values of U on $\mathbb{Z}+\frac{1}{2}$, for which we define a sequence of piecewise linear approximations. These approximations will be built up from pieces $\sum\xi_j$ over basic blocks of the environment. We begin by defining, for each $k>0$, linear functions $R_0^{(k)}$ on $[0, \frac{1}{3}(4^k+2)]$, and $R_1^{(k)}$ and $R_2^{(k)}$ on $[0, \frac{1}{3}(4^k-1)]$ by specifying their values at their endpoints: $R_i^{(k)}(0)=S_i(0)=0$ (for $i=0, 1, 2$); $R_0^{(k)}(\frac{1}{3}(4^k+2))=S_0(4^k)=2\cdot 3^{k-1}$; and $R_i^{(k)}(\frac{1}{3}(4^k-1))=S_i(4^k)=-3^{k-1}$ (for $i=1, 2$). Clearly, $R_i^{(k)}(n)$ is a linear approximation to $S_i(3n)$ on the interval of definition; we may obtain a better approximation on the same interval by "concatenating" several $R_i^{(k-1)}$ functions:

The "concatenation" of two continuous functions is the operation of putting the graphs of the functions end to end. Suppose f_1: $[0, x_1]\to\mathbb{R}$ and f_2: $[0, x_2]\to\mathbb{R}$. We define a function $f_1\square f_2$ (the "concatenation" of f_1 and f_2) on $[0, x_1+x_2]$ as follows:

$$\begin{aligned}(f_1\square f_2)(x) &= f_1(x), &&\text{for } 0\le x<x_1,\\ &= f_1(x_1)+f_2(x-x_1), &&\text{for } x_1\le x\le x_2.\end{aligned} \tag{12}$$

We now recall that, for $k>0$, $A_k=A_{k-1}\bar{A}_{k-1}\bar{A}_{k-1}A_{k-1}$, and $4^{k-1}\equiv 1 \pmod 3$. Hence we obtain the following "refined" approximation of $S_i(3n)$:

$$S_i(3n)\approx[R_i^{(k-1)}\square(-R_{i+1}^{(k-1)})\square(-R_{i+1}^{(k-1)})\square R_i^{(k-1)}](n), \tag{13}$$

with equality holding at all endpoints and concatenation points. We call this approximation a "refinement" of $R_i^{(k)}$. Now, each of the terms of (13) may be further refined, yielding successively better approximations, until after k iterations, we obtain a function which coincides with $S_i(3n)$ for all integers n in the interval of definition.

Figs. 1a–1c show the first refinements of $R_i^{(k)}$, for $i\in\{0, 1, 2\}$. These three graphs, together with their opposites ($-R_i^{(k)}$, for $i\in\{0, 1, 2\}$), form the "building blocks" for successive refinements of each $R_i^{(k)}$. Fig. 2 illustrates this process through the construction of the next refinement of $R_1^{(k)}$.

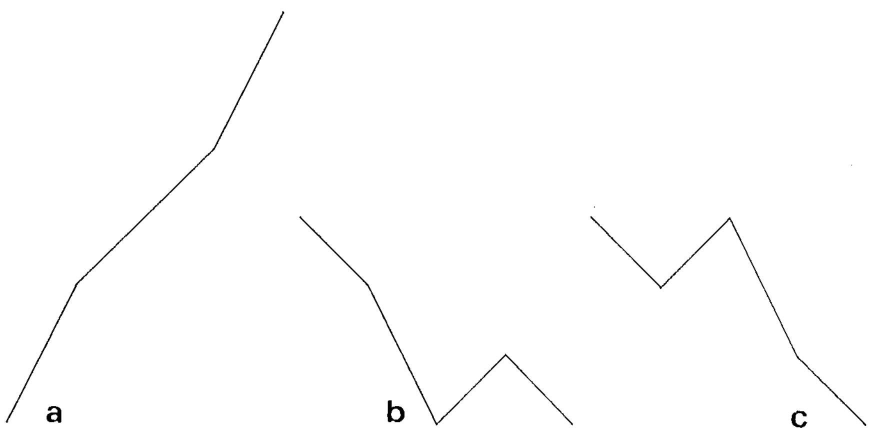

Fig. 1. (a)–(c) Refinements of the graphs of $R_i^{(k)}$ (for $i=0, 1, 2$ respectively), as given by (13).

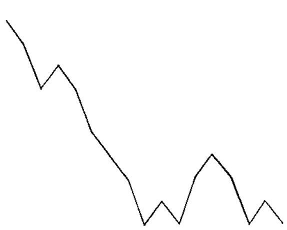

Fig. 2. Second refinement of the graph of $R_1^{(k)}$.

Since every sequence $\xi \in \Xi$ possesses the same block structure as the rarefied Thue–Morse sequence, we can describe the fluctuations of U corresponding to any ξ by using the linear pieces $R_i^{(k)}$ defined above. Given ξ, let $\{b_j^{(k)} \equiv b_j^{(k)}(\xi)\}_{j=-\infty}^{\infty} \subset \mathbb{Z}+\frac{1}{2}$, $b_j^{(k)} < b_{j+1}^{(k)}$, denote the set of points in between the rarefied k-blocks of ξ. (It is not hard to see that this set is uniquely determined by ξ.) We shall call the graph of the piecewise linear function which interpolates U on $\{b_j^{(k)}\}$ the *k-graph* of U. This coincides (up to a translation and multiplication by α) with a doubly infinite concatenation of $\pm R_i^{(k)}$'s. In fact, this concatenation will be of the form

$$\ldots\square \pm R_i^{(k)} \square \pm R_{i+2}^{(k)} \square \pm R_{i+1}^{(k)} \square \pm R_i^{(k)} \square \ldots, \tag{14}$$

with the signs determined by the corresponding $\omega \in \Omega$. Inspection of figs. 1a–1c demonstrates that absolute extrema of U on $[b_j^{(k)}, b_{j+1}^{(k)}]$ occur at the endpoints. Thus, in order to find the valleys of U of height, say, 3^k, it suffices to check the k-graph of U, and hence to check this concatenation for local minima at the concatenation points. We find that three of the possible concatenations lead to local minima at the concatenation point:

$$\ldots\square R_2^{(k)} \square - R_1^{(k)} \square \ldots, \tag{15}$$

$$\ldots\square R_1^{(k)} \square R_0^{(k)} \square \ldots, \tag{16}$$

$$\ldots\square - R_0^{(k)} \square - R_2^{(k)} \square \ldots. \tag{17}$$

The type of minimum seen in (15) behaves completely differently from the other two during subsequent refinements. Figs. 1b and 1c reveal that the first type of local minimum is conserved, always remaining the unique deepest point in a symmetric, roughly V-shaped valley.

On the other hand, one may conclude from fig. 1 and (13) (see also fig. 2) that the minima which arise from (16) and (17) are doubled at each successive refinement, thus leading to approximate Cantor-set-like minima. We will refer to these valleys as W-shaped valleys.

Note that further refinement of a valley of height 3^k in the k-graph of U will produce subvalleys of height at most 3^{k-1}.

We require some terminology for describing the valleys in our structure. Since all V- and W-shaped valleys will have height 3^k for some $k \in \mathbb{Z}$, we refer to such valleys as "valleys of order k", or simply, "k-valleys". Also, when we specify the endpoints $[a, b]$ (where $a, b \in \mathbb{Z}+\frac{1}{2}$) of a k-valley, we will require that a and b be the endpoints of rarefied k-blocks in the environment. (This distinction will become important in our discussion of escape times from k-valleys.)

4. Results

We present here some results concerning the long time behavior of the random walk in our Thue–Morse environment ξ chosen according to the measure μ. Most of our results deal with the motion of a particle in a fixed environment ξ which is typical for the measure μ: We let P_ξ denote the probability distribution on random walk trajectories in the environment ξ. Our results are consequences of the recursive nature of our construction, and of the structure of valleys in our model.

We begin with the most basic result about the growth behavior of the random walk.

Proposition 1. There exists a positive constant C such that for μ-almost every ξ

$$\lim_{t\to\infty} {\textstyle\sup \atop \textstyle\inf} X_t/(\log t)^{1/\beta} = \pm C \tag{18}$$

P_ξ almost surely, where $\beta = \log 3/\log 4$.

This proposition is a consequence of several of our later results, as well as the ergodic theorem, applied to the renormalization group transformation $\varphi: \Xi \to \Xi$ defined by $\varphi((\lambda, \omega_0, \eta_1, \eta_2, \ldots)) = (\lambda', \pm\omega_0, \eta_2, \eta_3, \ldots)$ where the sign of $\pm\omega_0$ is $+$ if $\eta_1 = 1$ or 4 and $-$ if $\eta_1 = 2$ or 3, where $\lambda' = \lambda + \eta_1 - 1$ and the "λ" coordinate is the integer (mod 3) indicating the "type" of rarefied block containing the origin ($\pm R_0$, $\pm R_1$, or $\pm R_2$) on the relevant scale, with initially, on the microscopic scale, $\lambda = 0$, and where $(\omega_0, \eta_1, \eta_2, \ldots)$ encodes the environment $\boldsymbol{\omega}$ as in section 3.

Our next proposition deals with our ability to predict the location of a particle after a long time. Proposition 2 will imply that for most choices of time t and environment ξ, we can predict the location and type of valley in which the particle will be trapped. The deterministic functions $m(t)$ and $u(t)$, which depend on ξ, and which we are about to define, will give respectively the center of the valley and the type of valley (V- or W-shaped) in which the particle will be trapped, for most choices of ξ and t.

The positions of our particle at large times will typically be near the "centers" $m_k = m_k(\xi) \in \mathbb{Z} + \frac{1}{2}$ of certain k-valleys, where $k = k(t)$ and m_k depend on t (and ξ) in a manner which we will now specify. Consider the k-graph of U. Starting at the point $(0, U(0))$ on this graph, proceed "downhill" to the first minimum; this point will be a minimum of the k-valley we wish to specify. Suppose this point is $(s, U(s))$; then we define m_k according to the type of concatenation at that point. If the concatenation is of type (15), s is the unique minimum in a V-shaped valley, so we set $m_k = s$. If the concatenation is of type (16) or (17), s is a minimum of a W-shaped valley, but this minimum is not unique. Since we want m_k to represent the center of the W-shaped valley, we set $m_k = s - \frac{1}{3}(4^{k-1} - 1)$ for type (16) valleys, and $m_k = s + \frac{1}{3}(4^{k-1} - 1)$ for type (17) valleys. Furthermore, we denote by $u_k \equiv u_k(\xi)$ the type of k-valley at m_k; i.e., $u_k = \mathrm{V}$ or W depending upon whether m_k is the center of a V- or W-shaped k-valley.

Next, we define a sequence of "critical" times $t_k \equiv t_k(\xi)$ for our process. First, we define t_k^{V} and t_k^{W}, the "escape times" from V- and W-shaped k-valleys, in the following manner. Let $[a, b]$ be a V-shaped k-valley (with endpoints specified according to the remark at the end of section 3). Then $t_k^{\mathrm{V}} = 2E_{(a+b)/2 \pm 1/2}(T_{\{a,b\}})$, i.e. twice the expected time of arrival at either endpoint of the valley, having started from the bottom. (We multiply by 2 here because we want $(t_k^{\mathrm{V}})^{-1}$ to represent the rate of "one-sided escape".) Taking into account the detailed, fractal-like structure near the top of the V-valley, we find that $t_k^{\mathrm{V}} = c_k^{\mathrm{V}} \mathrm{e}^{\alpha 3^k}$, where c_k^{V} is of order 2^k, and $\alpha = \log[(1+\epsilon)/(1-\epsilon)]$.

Now, let $[a, b]$ be a W-shaped k-valley. To be definite, suppose $[a, b]$ is a k-valley formed by a concatenation of type (17). Then $t_k^{\mathrm{W}} = 2E^{(\mathrm{r})}_{(a+b)/2+1/2}(T_{\{b\}})$, twice the expected time of arrival at the right (i.e. "lower") endpoint, assuming that we impose, say, a reflecting boundary condition on the left end of the interval. We multiply by 2 to account for the fact that once the particle reaches point b, it is equally likely (by symmetry) to proceed down from the peak at b in either direction. (The reader should convince himself that b is at a concatenation point of the form $\ldots\square - R_2^{\{k\}} \square R_1^{\{k\}} \square \ldots$, which gives the aforementioned symmetry.) Now, $t_k^{\mathrm{W}} = c_k^{\mathrm{W}} \mathrm{e}^{\alpha 3^k}$, where C_k^{W} is of order unity.

For k large, the one-sided escape times T_k^{V} and T_k^{W} from (the bottom of) V- and W-shaped k-valleys are approximately exponential with means t_k^{V} and t_k^{W}. More precisely, as $k \to \infty$, $T_k^{\mathrm{V}}/t_k^{\mathrm{V}}$ and $T_k^{\mathrm{W}}/t_k^{\mathrm{W}}$ converge in distribution to exponential random variables of mean 1.

Finally, we define the critical time $t_k \equiv t_k(\xi)$ by $t_k = t_k^{u_k}$.

We may now define $k \equiv k(t)$, the order of the valley in which our particle is likely to be trapped at time t, by setting $k(t) = k$ whenever $t_{k-1} < t \le t_k$. Using $k(t)$, we define (with a slight abuse of notation) $m(t)$ and $u(t)$ by $m(t) = m_{k(t)}$ and $u(t) = u_{k(t)}$. For "most" times t, our particle will be near $m(t)$, the center of a $k(t)$-valley of type $u(t)$. (In this regard, the reader should convince himself that when a W-shaped k-valley "destabilizes" into a V-shaped k-valley, the latter valley in fact lies at the center of a V-shaped $(k+1)$-valley.)

We note, however, that if the origin of our random walk is in a V-shaped $(k-1)$-valley resulting from a single further refinement of either $\ldots\square R_0^{(k)}\square R_2^{(k)}\square\ldots$ or $\ldots\square -R_1^{(k)}\square -R_0^{(k)}\square\ldots$, then m_k is not a very good indicator of the position of the particle, even at times t for which $k(t)=k$, since the particle could have descended from the V-shaped $(k-1)$-valley in either direction with probability of order unity. Let $\Xi_k\subset\Xi$ be the complement (in Ξ) of the set of environments of the type just described. A simple computation shows that $\mu(\Xi_k)=\frac{17}{18}+o(1)$ (as $k\to\infty$). Again abusing notation, we define $\Xi(t)$ by setting $\Xi(t)=\Xi_k$ if $t_{k-1}^{\mathrm{V}}<t\le t_k^{\mathrm{V}}$. (If $\xi\notin\Xi_k$, then, typically, $t_{k-1}=t_{k-1}^{\mathrm{V}}$.) For environments $\xi\in\Xi(t)$, the position X_t of the random walk should be close to $m(t)$.

In stating the following propositions, we require the function $[\log](t)\equiv 3^{k(t)}$. (Note that this depends, through $k(t)$, on ξ.) $[\log](t)$ is a left-continuous step function with jumps at t_k, $k=1, 2, \ldots$, and for $t=t_k$, $[\log](t)=\alpha^{-1}\log(t/c_k^{u_k})$.

Proposition 2. For every $\xi\in\Xi$, as "$t\to\infty$" (in a sense explained after this proposition),

$$P_\xi\{[X_t-m(t)]/B_{u(t)}(t)<x\}-F_{u(t)}(x)\to 0, \tag{19}$$

where

$$F_u=\mathscr{V}\quad\text{if } u=\mathrm{V},$$
$$\quad=\mathscr{W}\quad\text{if } u=\mathrm{W},$$

with $\mathscr{V}$ and $\mathscr{W}$ denoting specific probability distribution functions which will be described later, and

$$B_u(t)=1\qquad\qquad\text{if } u=\mathrm{V},$$
$$\quad=4^{k(t)}\;[=([\log](t))^{1/\beta}]\quad\text{if } u=\mathrm{W}.$$

In the statement of the proposition, by "$t\to\infty$" we mean that $t\to\infty$ in such a way that $\xi\in\Xi(t)$, and t stays away from the times t_k, in the sense that $\mathrm{dist}(\log t, \{\log t^k\}_{k=1}^{\infty})\to\infty$.

For V-shaped valleys, the random walk exhibits a strong, Golosov-type localization, while for W-shaped valleys the random walk spreads out around $m(t)$ on a scale of order $([\log]t)^{1/\beta}$. Figs. 3 and 4 illustrate the distribution of a particle during random walks in V- and W-shaped valleys respectively.

We remark that while for $\xi\notin\Xi(t)$, $m(t)$ is not a good indicator of X_t, it is nonetheless possible to give a complete description of the situation. (For brevity, we omit the details.)

Proposition 3. As $t\to\infty$,

$$\mu(\{m(k)/4^k<x\,|\,\xi\in\Xi_k\})\to\mathscr{M}(x), \tag{20}$$

where $\mathscr{M}$ is a probability distribution function.

The following proposition describes the behavior of our process X_t when observed on "macroscopic" length and time scales; after suitable rescaling we obtain a process $X_t^{(T)}$ where T defines the time scale and $t\in\mathbb{R}^+$ is the time variable on time scale T. Parts (ii) and (iii) follow from proposition 2 and the fact that if $k(T)=k$, then on time scale T our particle quickly escapes from valleys of order $k-1$ or smaller. Part (i) is an immediate consequence of part (iii).

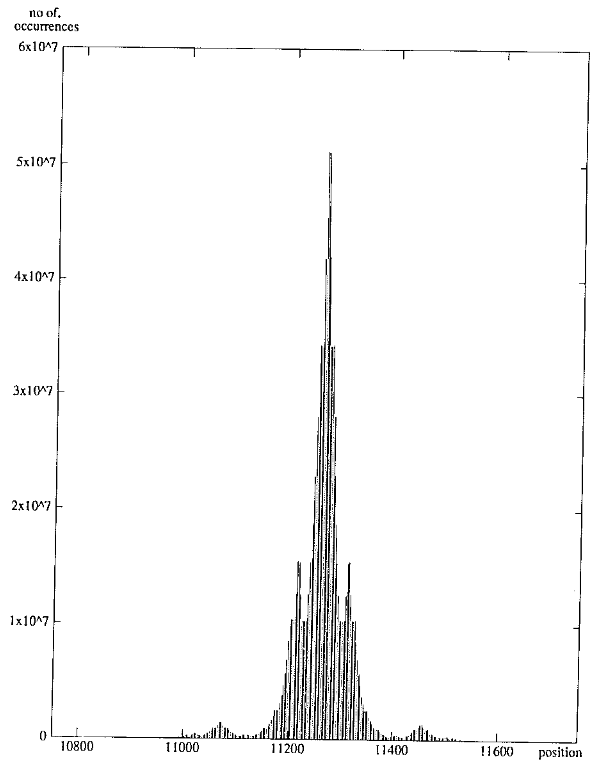

Fig. 3. Distribution of a random walk of 10^9 steps in a V-shaped valley ($\epsilon=0.1$).

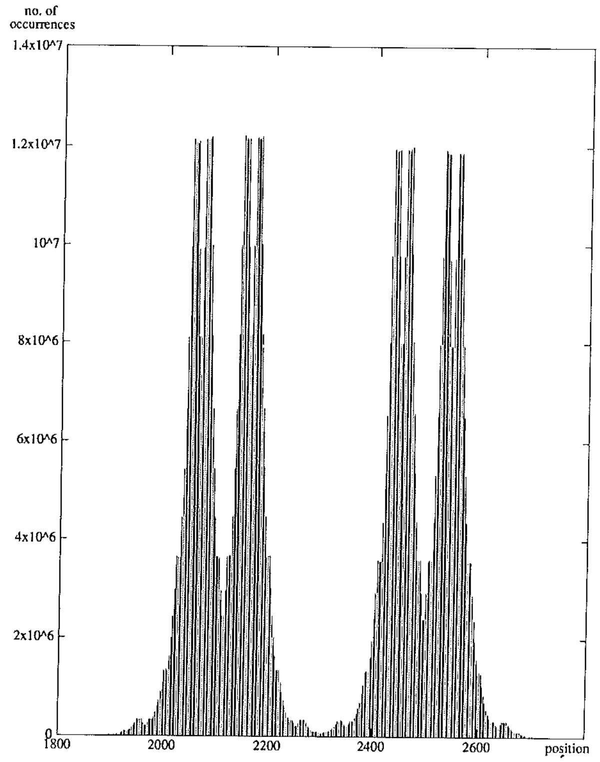

Fig. 4. Distribution of a random walk of 10^9 steps in a W-shaped valley ($\epsilon=0.1$).

Proposition 4. Consider the rescaled process (for $t\geq 0$, $T\in\mathbb{Z}^+$)

$$X_t^{(T)} := [X_{Tt} - m(T)]/([\log]T)^{1/\beta}. \tag{21}$$

Then for every $\xi\in\Xi$ the following assertions are valid:

(i) As "$T\to\infty$" (in the same sense as in proposition 2) in such a way that $u(T)=\mathrm{V}$, we have $X_t^{(T)}\to 0$ in P_ξ probability.

(ii) As "$T\to\infty$" in such a way that $u(T)=\mathrm{W}$, we have $\{X_t^{(T)}\}_{t>0}\to\{Y_t\}_{t>0}$, where the convergence is to be understood in the sense of finite-dimensional distributions, and where $\{Y_t\}_{t>0}$ is a process with independent, identically distributed values whose common distribution is $\mathscr{W}$ (see proposition 2).

(iii) As "$T\to\infty$" in such a way that $u(T)=\mathrm{V}$, we have $\{X_{Tt}-m(T)\}_{t>0}\to\{Z_t\}_{t>0}$, where the sense of convergence is as in (ii), and where $\{Z_t\}_{t>0}$ is a process with independent, identically distributed values whose common distribution is $\mathscr{V}$.

Proposition 4 describes the behavior of the rescaled process $X_t^{(T)}$ for values of T which are not near any of the critical times t_k. According to the proposition, on such time scales T, our particle stays near $m(T)$ on a microscopic scale if $u(T)=\mathrm{V}$, or, if $u(T)=\mathrm{W}$, jumps very rapidly among the sites in a W-valley of a "macroscopic" size $([\log]T)^{1/\beta}$. In the latter case, this motion would be visible only as a glob (a "Cantor glob", see proposition 6). Thus when viewed on macroscopic scales our process defines either point-like or extended structures (see the initial part of fig. 5). These apparent structures, their sizes and their locations depend upon the time scale of observation.

The transition between structures visible on time scales $T' < t_k < T''$ separated by a critical time t_k is best observed on a critical time scale $T = ct_k$, where c is a constant of order unity. On such a time scale, the k-valley at m_k "destabilizes", as does the structure defined by X_t^T, resulting in a Markov process of jumps between k-valleys which ends, after at most a few jumps, either in a V-shaped k-valley (which lies at the bottom of a V-shaped $(k+1)$-valley), or in a W-shaped $(k+1)$-valley, with associated "Cantor glob" (recall that $t_k^{\mathrm{V}} \sim 2^k t_k^{\mathrm{W}}$), or in a state of oscillation between two W-shaped k-valleys (which together form a W-shaped $(k+1)$-valley), corresponding to oscillation between the associated "Cantor globs" (see the initial part of fig. 5). In either case, the essential characteristic of the rescaled process on a critical time scale is that the particle makes a small number of transitions between valleys of height 3^k, and then reaches a valley of height 3^{k+1}, where of course it will remain trapped until times on the next critical time scale. (It is not difficult to give a complete description of all the jump Markov processes which arise from the process $X_t^{(T)}$, for T a critical time scale. There are only a small number of possibilities; we leave them as an exercise for the reader.)

We remark that for the Sinai (Bernoulli) environment a similar picture should hold, the main differences being that in the Bernoulli case there will be no extended structures – only Golosov (microscopic) localization – and no oscillations; in fact, the process of jumps between valleys will involve only a single jump, almost surely.

The remaining propositions concern asymptotic distributions. First consider the distribution function $\mathscr{V}$, first mentioned in proposition 2. As $k \to \infty$, the environment for which a V-shaped k-valley is centered near the origin,

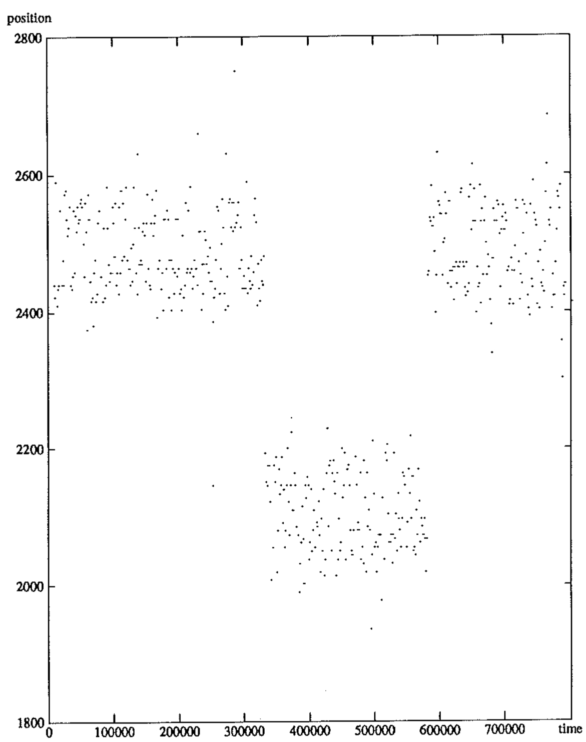

Fig. 5. Path of a typical random walk in a W-shaped valley ($\epsilon = 0.1$; position recorded every 1000 steps).

say at $-\frac{1}{2}$, converges to the environment consisting of the rarefied negative Thue–Morse sequence $\{-a_{3n+1}\}_{n=0}^{\infty}$, preceded by the opposite of the reflection of this sequence about $-\frac{1}{2}$. Since $m(t)\in\mathbb{Z}+\frac{1}{2}$, we easily obtain the following proposition.

Proposition 5. $\mathscr{V}$ is the probability distribution function for the probability measure ρ_V on $\mathbb{Z}+\frac{1}{2}$ given by

$$\rho_V(n+\tfrac{1}{2})=\frac{e^{-U(n)}}{\sum e^{-U(m)}},\quad \text{for } n\in\mathbb{Z}, \tag{22}$$

where U is the potential for the environment

$$\begin{aligned}\xi_j&=-a_{3j+1},\quad j=0,1,2,\dots\\ &=a_{-3j-2},\quad j=-1,-2,\dots\,.\end{aligned}$$

The distribution function $\mathscr{W}$, describing the asymptotic distribution in a W-valley, is somewhat more intricate. Note that the limit of the environments which give rise to the centered W-shaped k-valleys is of little interest. This limit is, in fact, the negative of the V-type environment of proposition 5. This inverted V-type environment is of little interest because our particle will asymptotically be infinitely far from the center, so this environment has little to do with the environment seen by the particle. (The denominator of (22) for this environment would be infinite.)

Proposition 6. $\mathscr{W}$ is the probability distribution function for the Cantor measure ρ_W on the interval $I=[-\frac{1}{12},\frac{1}{12}]$ based on the sequence of removals of middle halves, starting with (Lebesgue measure on) I.

The measure ρ_W is the limit of the sequence of probability measures $\rho^{(k)}$ describing a uniform distribution over the 2^k intervals which remain after k interactions of the removal of middle halves. In order to understand proposition 6, it is useful to note that ρ_W may be characterized as follows: ρ_W is the (unique) probability measure supported on the (middle-half-removal) Cantor set which gives equal weight to each of the 2^k intervals $I_j^{(k)}$ remaining after k iterations. Moreover, if ρ_n is a sequence of probability measures which, for each fixed k, asymptotically (as $n\to\infty$) (i) gives equal weight to each interval $I_j^{(k)}$, and (ii) is supported by $\bigcup_j I_j^{(k)}$, then $\rho_n\to\rho_W$. Such a sequence is provided by the distribution ρ_n of $(X-m)/4^n$, where m is the center of a W-shaped n-valley $[a,b]$ with corresponding potential U_W, and X is distributed on $[a,b]\cap\mathbb{Z}$ according to $\exp[-U_W(j)]/\sum\exp[-U_W(i)]$. Moreover, for $n=n(t)$, $u(t)=W$, and t large, ρ_n is a good approximation to the distribution of $[X_t-m(t)]/4^n$.

Our last two propositions concern more refined asymptotic distributions, namely, the distribution of the "environment seen by the particle" and the joint distribution of this environment and the position of the particle, at times for which the particle is in a W-valley. (These distributions are trivial when $u(t)=V$.)

Proposition 7. Let $S:\Xi\to\Xi$ be a translation by one unit to the left, $S\{\xi_j\}=\{\xi'_j\}$ where $\xi'_j=\xi_{j+1}$. Then for every $\xi\in\Xi$ the pair $([X_t-m(t)]/([\log]t)^{1/\beta}, S^{-X_t}\xi)$ converges in distribution to $(Y,\bar{\xi})$ as "$t\to\infty$" in such a way that $u(t)=W$, where Y has distribution ρ_W (Cantor measure, with distribution function $\mathscr{W}$), $\bar{\xi}$ is a random environment with distribution ν (to be specified in proposition 8), and Y and $\bar{\xi}$ are independent.

The explanation of proposition 7 is as follows: For k large, the (rescaled) position of our particle when in a W-shaped k-valley may be well approximated by the location of the nearest of the 2^n absolute minima of the

$(k-n)$-graph of U over this W-valley. Each of these minima, upon further refinement, yield precisely the same (Cantor-like) local detailed structure for the potential. The convergence in distribution of the proposition refers to the usual weak convergence of measures (i.e. convergence on bounded continuous functions), which for the convergence of the environment distribution amounts to convergence on (microscopically) local functions of the environment. Thus, since each approximate graph of U around an absolute minimum splits upon refinement into 2 pieces which are mirror images of each other, the proposition follows, with ν as described in the next proposition.

Note that independence is achieved only in the limit; in fact, for any finite t, the environment $S^{-X_t}\xi$ seen by the particle is completely determined by X_t (for fixed ξ). Independence is achieved in the limit because the position on the macroscopic scale is well approximated in terms of structures related to refinements from the macroscopic scale down, while the environment seen by the particle is well approximated by structures built up from the microscopic scale.

Proposition 8. For every $\sigma=\{\sigma_k\}_{k=2}^{\infty}$, $\sigma_k=\mathrm{L}, \mathrm{R}$, let $\xi^{(\sigma)}\in\Xi$ be the (unique) environment such that for every $k\geq 2$ the origin is in a W-valley of order k, with an absolute minimum at $-\frac{1}{2}$, which is on the left (right) side of the W-valley if $\sigma_k=\mathrm{L}(\mathrm{R})$. (If σ ends with all L's or all R's, ξ^{σ} is "one-sided" and is not actually an element of Ξ.)

Let $\tilde{\nu}$, on $\tilde{\Xi}_{\mathrm{W}}$, be the image of the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on $\{\sigma\}$ under the map $\sigma\to\xi^{(\sigma)}$. Let $\Xi_{\mathrm{W}}=\bigcup_{j=-\infty}^{\infty} S^j(\tilde{\Xi}_{\mathrm{W}})$, and let ν' be the translation invariant measure on Ξ which agrees with $\tilde{\nu}$ on $\tilde{\Xi}_{\mathrm{W}}$. Then the probability measure ν of proposition 7 satisfies

$$\nu(\mathrm{d}\xi)\equiv \mathrm{d}\nu=\frac{\mathrm{e}^{-U(0)}\,\tilde{\nu}(\mathrm{d}\xi)}{\int \mathrm{d}\tilde{\nu}\,\mathrm{e}^{-U(0)}}=\frac{\mathrm{e}^{-U(0)}\,\mathrm{d}\tilde{\nu}}{\int \mathrm{d}\tilde{\nu}\,\mathrm{e}^{-U(0)}}, \tag{23}$$

where the potential $U(n)\equiv U(n, \xi)$ is now normalized, so that U is, say, 0 at the absolute minima.

We remark that in terms of the coding of section 3, $\tilde{\Xi}_{\mathrm{W}}\subset\Xi$ consists of environments coded by $(\omega_0, \eta_1, \eta_2, \ldots)$ for which

$$\begin{aligned} \eta_{k+1} &= 1 \quad \text{or} \quad 3, \quad \text{for } \eta_k=1, 2,\\ &= 2 \quad \text{or} \quad 4, \quad \text{for } \eta_k=3, 4, \end{aligned} \tag{24}$$

for $k\geq 2$ and $\eta_1=2$, $\omega_0=+1$ when η_2 is even, $\eta_1=3$, $\omega_0=-1$ when η_2 is odd. Note that $\mu(\Xi_{\mathrm{W}})=0$. The environments of Ξ_{W} are precisely those with potential $U(n)$ bounded below, apart from the translates of the V environment of proposition 5. It would be natural to regard the measure ν' on Ξ_{W} as $\mu(\cdot\,|\,\Xi_{\mathrm{W}})$, the conditional distribution given Ξ_{W} arising from μ. However $\nu'(\Xi_{\mathrm{W}})=\infty$, and it is only when ν' is adjusted to reflect the dynamical properties of our random walk that a normalizable measure is obtained.

We conclude by mentioning one aspect of the asymptotics which we have not analyzed in detail, namely, "the motion on a log–log time scale". More precisely, by this we mean the process m_k, $k=1, 2, \ldots$, or the process (m_k, u_k), $k=1, 2, \ldots$, on the probability space $\{\Xi, \mu\}$. The basic structure of these processes should follow from the hierarchical structure of Ξ. Note that the time 1 map $(k\to k+1)$ for these processes is induced by the renormalization group transformation φ described after proposition 1. We leave the computation to the reader.

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