RIGOROUS RESULTS ON SOME STOCHASTIC CELLULAR AUTOMATA

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INTRODUCTION

Computer simulations on cellular automata show that they reproduce patterns observed in real physical fluids evolving according to the solutions of the hydrodynamic equations. This is remarkable, even astonishing, since at a microscopic-particle level the dynamics seems at best a caricature of the interactions between real molecules. It shows that the macroscopic behavior of a fluid does not depend on the detailed features of the particle interactions: systems which microscopically look completely different may give rise to the same type of macroscopic equations.

Scale separation is responsible, for the existence of this kind of *universality phenomena*. The purpose of this paper is to explain, by a rigorous analysis, the origin of such collective behavior in a class of stochastic cellular automata; a stochastic variant of the HPP, [HPP], and FHP, [FHP] models. We shall see that these models have a very interesting and complex macroscopic structure. There are in fact several space time regimes, and when the state of the microscopic system is correspondingly suitably *prepared* at the initial time, then, in that specific regime, it behaves according to some macroscopic equation. There is a *window* in space time through which we see a kinetic behavior, described by a Boltzmann like equation.Looking through a different *lens*, focused on a longer time scale, we see hydrodynamical behavior described by an Euler like equation. Focusing on still longer times and suitably choosing the initial state we can observe the analogue of the incompressible Navier-Stokes and Euler equations.

In this paper we state the results without proofs, for which we refer to [DELP] and [DEL].

THE MODEL.

Particles move on the square lattice, Z^2. Analogous definitions can be given for the triangular lattice, but since our proofs do not distinguish between the two cases we state them for Z^2 which is notationaly simpler. We call east the positive direction of the x-axis, north that of the y-axis, west and south are then determined accordingly.

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The unit vectors in these directions are denoted by e_i, $i = 1, \cdot, 4$, respectively. We consider four copies of Z^2; it is convenient to think of them as imbedded in three dimensional space as four parallel planes obtained one from the other by a vertical translation. Each of them has a label $\sigma \in \{1,2,3,4\}$. We call q the generic site in Z^2 so that (q,σ) specifies a site in Z^2 and the plane σ on which q is located. Particles move on each plane and jump from plane to plane. Particles have *velocities* denoted by $v \in \{e_1, \cdots, e_4\}$. There is an exclusion rule according to which there can be at most one particle on each point $\gamma \equiv (q,\sigma,v)$ of the phase space Γ. A particle configuration is therefore an element $\eta \in \{0,1\}^\Gamma$.

We consider three types of updating of a configuration: the true evolution will be a suitable combination of applications of these three steps. The first one is the *streaming updating* according to which all particles move one step in the direction of their velocity remaining on the same plane [σ is conserved], i.e. the configuration $\eta \equiv \eta(q,\sigma,v)$ is changed into $\eta'(q,\sigma,v) = \eta(q-v,\sigma,v)$. The second one is the *collision updating* the same as in the HPP model. Both σ and q are conserved [particles do not move] in the collision updating while velocities change at all sites (q,σ) which have exactly two particles with opposite velocities; e.g. south-north or east-west. The two particles then change velocities, if previously they were north-south they become east-west and vice versa. The third rule defines the *stirring updating*. This consists of two consecutive steps. The first one is stochastic. Independently for each q we choose with equal probability a number in the set $\{1,2,3,4\}$. Assume that for the site q the chosen number is a, then if $\eta(q,\sigma,v)$ were the occupation number before the updating, then in the first step they are changed into $\eta'(q,\sigma,v) = \eta(q,\sigma-a,v)$ [$\sigma - a$ is defined modulo 4]. This means that this step consists of independent rigid vertical rotations of all the vertical segments of four sites. The second step of the *stirring updating* is a rigid translation of each of the four planes, in the plane σ all particles move one step in the direction e_σ: a configuration η' goes into the configuration η'' defined as $\eta''(q,\sigma,v) = \eta'(q-e_\sigma,\sigma,v)$.

The time evolution of our system is a Markov chain with the following transition probability: at each integer time we choose with probability ϵ the collision updating while with probability $1-\epsilon$ first the stirring and then the streaming updatings are applied. The law of this Markov chain is denoted by P_μ^ϵ, where μ denotes the initial distribution (the distribution at time 0), E_μ^ϵ is the corresponding expectation. We shall be interested in the limiting behavior when $\epsilon \to 0$. For notational convenience it is often useful to consider continuous times, not necessarily integers, it should then be understood that the corresponding time in the Markov chain is its integer part.

Remarks

The HPP evolution is determined by consecutively applying the streaming and the collision rules and then looking at what happens on a single plane (i.e. for a fixed value of σ, σ does not change in this case). We have added the stirring evolution, a two dimensional version of the process introduced by Boghosian and Levermore in [BL], to have more stochasticity in the system. Its properties can be better understood by considering first the case when there is just one single particle in the whole space.

The effect of the stirring is to make it jump from one plane to the other and then move by one step in the direction determined by the plane where it is staying. By only looking at its q position, i.e. forgetting about σ, this is a symmetric random walk. On the other hand, if only streaming were present, it would move one unit at each time step in the direction of its velocity. When we consider both streaming and stirring we have a random walk with a drift. When more particles are present the motion due to stirring and streaming can be proven to be in some suitable sense close to that of independent random walks with drift. It is not exactly independent because when particles have the same q position they undergo the same vertical rotation appearing in the first step of the stirring updating, but the correlations built up by this *interaction* can be controlled. This process is in many respects analogous to the simple exclusion process, in particular to a realization of this process called by the probabilists working on interacting particle systems the stirring process. By an abuse of notation we have adopted the same name for our process.

We need the stirring to act *more often* than the collisions, however the choice of having a streaming after each stirring does not seem so necessary and we hope to relax it in subsequent works. What is surely needed is the introduction of the stirring updating.

We shall analyze the process in different time regimes and different space scales [determined by the initial measure which also will depend on ϵ], this is the scale separation we discussed in the Introduction. We start by a characterization of the typical trajectories of the process.

TYPICAL CONFIGURATIONS

Given a cylinder function f on $\{0,1\}^\Gamma$ [i.e. a function which depends on a configuration η only through finitely many of its entries] we denote by $f_{(q'\sigma')}$ the function f first shifted on each plane by q' and then vertically by σ', modulo 4. Furthermore we let

$$\mathcal{E}(f|q,\delta)(\eta) = |\Lambda(q,\delta)|^{-1} \sum_{(q',\sigma')\in\Lambda(q,\delta)} f_{(q'\sigma')}(\eta) \tag{1}$$

where $\Lambda(q,\delta) = \{(q',\sigma) : |q'-q| \le \epsilon^{-\delta}\}$ and $|\Lambda(q,\delta)|$ is the number of sites in $\Lambda(q,\delta)$. We then denote by $\mathcal{E}(v|q,\delta)$ the value of $\mathcal{E}(f|q,\delta)$ when $f = \eta(0,1,v)$. $\mathcal{E}(v|q,\delta)$ is therefore the (empirical) density of particles in $\Lambda(q,\delta)$ with velocity v. Finally let $\rho \equiv \rho(v)$, $v = 1,\cdots,4$, $0 \le \rho(v) \le 1$, and denote by ν_ρ the product measure such that

$$\nu_\rho(\eta(q,\sigma,v)) = \frac{1}{4}\rho(v) \tag{2}$$

[We denote by $\mu(f)$ the μ-expectation of f]. We then have

THEOREM 1. *Let μ^ϵ be an arbitrary sequence of probabilities on Γ and let k be any fixed positive integer. Then for any $\delta < \frac{1}{2}$ there is $\zeta > 0$ such that for all cylinder*

functions f

$$\lim_{\epsilon\to 0} E^{\epsilon}_{\mu^\epsilon}\Big(\sup_{|q|\le\epsilon^{-k},\ \epsilon^{-2\delta}\le t\le\epsilon^{-k}} |\mathcal{E}(f|q,\delta)(\eta_t) - \nu_{\mathcal{E}(\cdot|q,\delta)(\eta_t)}(f)| > \epsilon^{\varsigma}\Big) = 0 \qquad (3)$$

Furthermore

$$\lim_{\epsilon\to 0} \sup_{\sigma,|q|\le\epsilon^{-k},\ \epsilon^{-2\delta}\le t\le\epsilon^{-k}} |E^{\epsilon}_{\mu^\epsilon}(f_{(q,\sigma)}(\eta_t)) - E^{\epsilon}_{\mu^\epsilon}\nu_{\mathcal{E}(\cdot|q,\delta)(\eta_t)}(f)| = 0 \qquad (4)$$

Remarks

Roughly speaking the state in a *neighborhood* of each point is at all times [except for an initial time *layer*] a superposition of Bernoulli measures, with the same parameters in the different planes. This happens, in the limit $\epsilon \to 0$, over regions of the order of $\epsilon^{-\delta}$. The law of such a superposition is determined by that of the velocity densities, i.e. the density of particles in the region considered, which have a given velocity. Therefore after a time $\epsilon^{-2\delta}$ the microscopic structure of the process is lost and the state space is essentially described by partitioning the space into cells of size $\epsilon^{-\delta}$ and looking only at the velocity densities in each cell. The law of such densities varies a priori on a time scale of the order of $\epsilon^{-\delta}$, because particles have finite velocities. On such a time scale the phenomena connected with the collision updatings are completely missing.

The proof of Theorem 1 is in fact based on the observation that on times of the order of $\epsilon^{-2\delta}$ the collision updating most probably never occurred:they occur on a time scale ϵ^{-1} and $2\delta < 1$. Only streaming and stirring are then effective and it is the stirring which produces the nice properties stated in the theorem. This holds for general initial conditions and arbitrarily long times, since, loosely speaking, after each collision we have a long time, with only streaming and stirring which re-establishes, as before, the Bernoulli measure. The collisions do not modify the local structure of the typical configurations, they determine however the way the local velocity-densities change and the nature of the hydrodynamical equations.

A complete description of the macroscopic behavior of the model is achieved once we know at all times the joint distribution of the velocity densities in each cell of the above partitioning of the lattice. We are not able to answer this question, we shall only obtain results on given time scales assuming suitable conditions on the initial measure μ^ϵ. Namely we shall ask for some *smoothness properties* related to the time scale we wish to analyze. For instance to study the time scale ϵ^{-1} we shall need a μ^ϵ such that $\mu^\epsilon(\eta(q,\sigma,v))$ also varies on the spatial scale ϵ^{-1}. We do not know whether this is really necessary or whether it is the evolution which on each time scale produces the correct space smoothness. We note finally that if we had more stirring, namely if we had stirring with probability $1-\epsilon-\epsilon^2$, streaming with probability ϵ and collisions with probability ϵ^2, we would then derive a parabolic diffusion equation on the time scale ϵ^{-2}, analogous to that obtained by Guo-Papanicolaou and Varadhan, [GPV], for a system of diffusive interactions.

THE KINETIC REGIME

The mean free path of a particle between two consecutive collision updatings is of the order of ϵ^{-1} since the typical times between two collisions is of this order and particles have non zero velocities. The kinetic regime is characterized by a Knudsen number, the ratio between the mean free path and the typical distances on which the macroscopic densities vary, of the order of 1. We therefore choose the initial measure μ^ϵ as a product measure such that

$$\mu^\epsilon(\eta(q,\sigma,v)) = \rho(\epsilon q, v) \tag{5}$$

where $\rho(r,v)$ is for each v a non negative smooth function of $r \in \Re^2$ which is not larger than 1. In this way we have imposed a velocity density profile whose typical variations are of the order of ϵ^{-1} on the *microscopic scale* on which particles move. Therefore q is a microscopic point and ϵq the corresponding macroscopic point. Since the space scale goes like ϵ^{-1} and in the system there are finite velocities we need to rescale time too, namely the macroscopic time τ corresponds to the microscopic time $\epsilon^{-1}\tau$. We then have the following result.

THEOREM 2. *Let μ^ϵ be as above, then for all integers n and all $t \geq 0$*

$$\lim_{\epsilon \to 0} \sup_{\gamma_1, \cdots, \gamma_n} |E^\epsilon_{\mu^\epsilon}[\prod_{i=1}^n \eta(\gamma_i, \epsilon^{-1}t)] - \prod_{i=1}^n \rho(\epsilon q_i, v_i, t)| = 0 \tag{6}$$

where the sup is over all distinct n-tuplets $(\gamma_1, \cdots, \gamma_n)$ of phase space points, $\gamma_i \equiv (q_i, \sigma_i, v_i)$, $i = 1, \cdots, n$. Furthermore $\rho(r,v,t)$ satisfies the Boltzmann-like equation

$$\frac{\partial}{\partial t}\rho + (v \cdot \nabla_r)\rho = Q(\rho) \tag{7}$$

with initial condition $\rho(r,v,0) = \rho(r,v)$. The collision kernel $Q(\rho)$ is given by

$$\begin{aligned} Q(\rho)(r,v) = {} & \rho(r,v^\perp)\rho(r,-v^\perp)(1-\rho(r,v))(1-\rho(r,-v)) \\ & - \rho(r,v)\rho(r,-v)(1-\rho(r,v^\perp))(1-\rho(r,-v^\perp)) \end{aligned} \tag{8}$$

where $v^\perp$ is obtained from v by a counterclockwise rotation of $\frac{\pi}{2}$.

Remarks

The **proof** of a weaker version of Theorem 2 can be obtained by using the iterative scheme introduced by Lanford, [La], to derive the Boltzmann equation for a gas of hard spheres in the Boltzmann-Grad limit. The limitation to *short times* in Lanford's result can here be lifted because of the exclusion condition built into the dynamics, which gives an a-priori bound on the correlation functions. To get the uniformity as stated in (6) we need a more precise analysis. This is based on the study of the so called *V-functions*, defined as

$$V^\epsilon(\underline{\gamma}, t) = E^\epsilon_{\mu^\epsilon}[\prod_{i=1}^n [\eta(\gamma_i, t) - \rho_\epsilon(\gamma_i, t)]] \tag{9}$$

where $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n)$ and ρ_ϵ is the solution to the equation

$$\rho_\epsilon(q, \sigma, t+1) = (1-\epsilon)\frac{1}{4}\sum_{\sigma'} \rho_\epsilon(q - v - e_\sigma, \sigma', v, t) + \epsilon\frac{1}{4}Q(\rho_\epsilon(q, \sigma, v, t)) \qquad (10)$$

where Q is defined as in (8) σ being just a parameter: (10) is just the discretization of (7) which is the same as the expression obtained by computing the expected updating of the occupation numbers assuming propagation of chaos.

The main point is then the following estimate: there is a $\delta > 0$ such that the following holds. For any n there is a c such that uniformly in $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n)$

$$|V^\epsilon(\underline{\gamma}, t)| \leq c\epsilon^{\delta n} \qquad (11)$$

This estimate is obtained by writing down the expression for the time derivative of the V-function. This is a linear combination of new V-functions. The corresponding integral equation can then be solved iteratively in a way similar to that appearing in [LOP], we omit the details. In fact the estimate (11) also holds for the V-functions defined starting from an arbitrary configuration, the proofs of Theorems 1 and 2 easily follow from such an extension of (11).

Consider now the model where stirring, streaming and collision updatings are applied consecutively at each time step. In this case there is no ϵ in the transition probability, the scale separation is not built in already in the definition of the evolution and the model is therefore much more difficult to analyze. In this case the mean free path is of order unity. To have it larger we need to take the limit of very rarefied densities. Namely we choose an initial measure μ^ϵ such that

$$\mu^\epsilon(\eta(q, \sigma, v)) = \epsilon\rho(\epsilon q, v) \qquad (12)$$

Again, at least heuristically, it will take a time of the order of ϵ^{-1} before a given particle undergoes a collision, hence the Knudsen number is again of order unity. Notice that for hard spheres the analogue of the above scaling is exactly the same, up to an isomorphism, as the usual Boltzmann-Grad limit. In that limit the radius of the spheres is decreased while increasing the total number of particles, keeping the mean free path constant, cf. [La]. Both limits lead to the same B.E. For our lattice gas the above scaling should lead to a different Boltzmann Equation than (7). We expect that for this system

$$\lim_{\epsilon \to 0} \sup_{\gamma_1, \cdots, \gamma_n} |E^\epsilon_{\mu^\epsilon}[\prod_{i=1}^n \eta(\gamma_i, \epsilon^{-1}t)] - \epsilon^{-n}\prod_{i=1}^n \rho(\epsilon q_i, v_i, \epsilon t)| = 0 \qquad (13)$$

where ρ satisfies (7) with collision kernel

$$Q(\rho)(r, v) = \rho(r, v^\perp)\rho(r, -v^\perp) - \rho(r, v)\rho(r, -v) \qquad (14)$$

This is just the well known Broadwell equation. Such an equation has been derived in [CDPP] from a stochastic particle model with no exclusion condition: in fact, in

the limit of low densities, we do not expect that the exclusion present in our model plays any role. We are indebted to Herbert Spohn for discussions on this point.

A proof of (13) and (14) involves a modification of the Lanford's procedure [La]. The combinatorial factors appearing in Lanford's perturbative series are in this case exploding because the interaction is a 4-body one. We hope that using the techniques introduced in [CDPP] we will be able to deal with this problem.

THE EULER REGIME

The hydrodynamical equations describe the behavior of the system in a regime where the Knudsen number is small. Since in our model the mean free path is of the order of ϵ^{-1}, then, to have a Knudsen number which vanishes when $\epsilon \to 0$, we choose an initial state which varies on a space scale $\lambda(\epsilon)\epsilon^{-1}$ where $\lambda(\epsilon) \to \infty$ as $\epsilon \to 0$. Therefore the initial measure μ^ϵ is a product measure such that

$$\mu^\epsilon(\eta(q,\sigma,v)) = \rho(\epsilon\lambda(\epsilon)^{-1}q, v) \tag{15}$$

where $\rho(r,v)$ is for each v a non negative smooth function of $r \in \Re^2$ not larger than 1 and such that for any r there are two smooth functions $h(r) \in \Re$ and $c(r) \in \Re^2$ such that

$$\rho(r,v) = \frac{1}{1 + exp[h(r) + c(r)\cdot v]} \tag{16}$$

Notice that the collision operator $Q(\rho)$ vanishes identically when ρ is given by (16), we will come back to this point later. Since the macroscopic scale is now $\epsilon^{-1}\lambda(\epsilon)$ we need to change, in the same way, also the macroscopic time scale. We have the following result:

THEOREM 3. *Let μ^ϵ be as above and choose $\lambda(\epsilon) = clog(\epsilon^{-1})$. Then for c sufficiently small the analogue of (6) [replacing ϵ^{-1} by $\lambda(\epsilon)\epsilon^{-1}$] holds, for all integers n and all $t \geq 0$. Now however $\rho(r,v,t)$ satisfies the relation (16) for all r and t and furthermore*

$$\frac{\partial}{\partial t}\rho(r,t) + \nabla \cdot j(r,t) = 0 \tag{17a}$$

$$\frac{\partial}{\partial t} j_i(r,t) + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\Pi_{i,j} \tag{17b}$$

where $\rho(r,t) = \sum_v \rho(r,v,t)$, $r \equiv (x_1, x_2)$, $j(r,t) = \sum_v v\rho(r,v,t)$ and the momentum flux tensor Π is given by

$$\Pi_{i,j}(r,t) = \delta_{i,j}\left[\rho(r,v_i,t) - \rho(r,-v_i,t)\right] \tag{17c}$$

We have used the notation $v \equiv (v_1, v_2)$.

Remarks

Eq. (17) has the structure of a conservation law for the density and momentum flux, it is an Euler-like equation. Its proof is based on the following two facts. Firstly

starting from the Boltzmann equation (7) and scaling by $\lambda(\epsilon)$ the initial datum considered above, we derive (17), in the limit $\epsilon \to 0$. This is achieved by using the Hilbert expansion and Caflisch techniques, [Ca]. This requires that the initial state has suitable smoothness properties, i.e. it scales proportionally to the time. Furthermore we need that for all r the initial density is the analogue of a Maxwellian, i.e. a *local equilibrium assumption*. This is achieved in our case by imposing (16).

The proof of Theorem 3 is then based on controlling the closeness between the actual particle densities and those given by the solution to the Boltzmann equation (7). Indeed the same estimates used to prove Theorem 2 ensure this whenever $\lambda(\epsilon)$ is as in Theorem 3.

THE INCOMPRESSIBLE NAVIER-STOKES AND EULER EQUATIONS

The scaling to be used to derive a macroscopic equation is dictated by the scaling symmetries of the equation to be derived. In this way the Euler equation was obtained by scaling space and time by the same parameter. The Navier-Stokes equation does not have any such symmetry, except for special cases, as for instance for pure heat diffusion. Traditionally, in kinetic theory, the Navier-Stokes equation is presented as a second order term in a Chapman-Enskog expansion but this does not seem to specify in which regime and in what sense, mathematically speaking, the behavior of the system is better described by the Navier Stokes equation. The incompressible Navier-Stokes equation, however, has a scaling symmetry obtained by scaling time as the square of the space [diffusive scaling] and, at the same time, scaling velocities like their ratio: $r \to r' = \epsilon^{-1} r$; $t \to t' = \epsilon^{-2} t$ and $u \to u' = \epsilon u$. To exploit such a symmetry in our setup we recall from Theorems 2 and 3 that the microscopic model becomes close to the *Boltzmann* equation (7) after a time ϵ^{-1}. On the other hand it was shown in [DEL]that the behavior of the solution to the *classical Boltzmann equation* under the above scaling corresponds for suitable initial conditions,to the incompressible Navier-Stokes equation. As before the result is obtained using the Hilbert expansion and the Caflisch techniques. The connection between the Boltzmann equation and the incompressible Euler and Navier-Stokes equations was also discussed in the present workshop by Levermore and investigated by Bardos, Golse and Levermore in [BGL]. As for the derivation of the Euler equation we exploit the closeness of the microscopic system to the solution of (7) up to times of the order of $c\log\epsilon^{-1}$ and the results obtained in [DEL].

For the incompressible Navier-Stokes equation we have the following setup: μ^ϵ is given (up to terms of order $\lambda(\epsilon)^{-2}$) by (15) and (16) where $h(r) \equiv h_0$, h_0 being any given constant, while $c(r)$ does depend on ϵ and is given by

$$c(r) = \lambda(\epsilon)^{-1} c_0(r) \tag{18a}$$

Let then ρ_0 be the constant density and $\rho_0 u_0(r)$ the corresponding momentum flux $j(r)$ at $t = 0$. Assume that u_0 is divergenceless, i.e.

$$\nabla \cdot u_0 \equiv 0 \tag{18b}$$

THEOREM 4. *Let μ^ϵ be as above and assume that $\lambda(\epsilon) = c\sqrt{log\epsilon^{-1}}$. Then for c sufficiently small and all $t \geq 0$*

$$\lim_{\epsilon \to 0} \sup_{\gamma_1, \cdots, \gamma_n} |E^\epsilon_{\mu^\epsilon}[\prod_{i=1}^n \eta(\gamma_i, \epsilon^{-1}\lambda(\epsilon)^2 t)] - \prod_{i=1}^n E^\epsilon_{\mu^\epsilon}[\eta(\gamma_i, \epsilon^{-1}\lambda(\epsilon)^2 t)]| = 0 \qquad (19)$$

and furthermore:

$$\lim_{\epsilon \to 0} \sup_{\gamma} |E^\epsilon_{\mu^\epsilon}[\eta(\gamma, \epsilon^{-1}\lambda(\epsilon)^2 t)] - \rho_0| = 0 \qquad (20a)$$

where ρ_0 is the constant value of the density per plane at time 0;setting $t_\epsilon = \epsilon^{-1}\lambda(\epsilon)^2 t$, we find :

$$\lim_{\epsilon \to 0} \sup_{q,\sigma,v} \lambda(\epsilon)|E^\epsilon_{\mu^\epsilon}[\eta(q,\sigma,v^\perp,t_\epsilon)\eta(q,\sigma,-v^\perp,t_\epsilon)(1-\eta(q,\sigma,v,t_\epsilon))(1-\eta(q,\sigma,-v,t_\epsilon))$$
$$-\eta(q,\sigma,v,t_\epsilon)\eta(q,\sigma,-v,t_\epsilon)(1-\eta(q,\sigma,v^\perp,t_\epsilon))(1-\eta(q,\sigma,-v^\perp,t_\epsilon)) = 0 \qquad (20b)$$

and

$$\lim_{\epsilon \to 0} \sup_{q,\sigma} \lambda(\epsilon)|\sum_v v E^\epsilon_{\mu^\epsilon}[\eta(q,\sigma,v,t_\epsilon) - \rho_0 u(\epsilon\lambda(\epsilon)^{-1}q,t)| = 0 \qquad (20c)$$

where $u(r,t)$ satisfies the analogue of the incompressible Navier Stokes equation: namely it is divergenceless and

$$\rho_0 \frac{\partial}{\partial t}u_i + \rho_0 g(\rho_0) u_i \frac{\partial}{\partial x_i}u_i = -\frac{\partial}{\partial x_i}p + \nu(\rho_0)\Delta u_i \qquad (21)$$

g and ν are given positive functions and ∇p is uniquely determined by (21), with initial condition $u(r,0) = u_0(r)$

For the incompressible Euler equation we have the same initial condition as above. By looking at times of the order $t_\epsilon = \epsilon^{-1}\lambda(\epsilon)^{1+\alpha}t$, $0 < \alpha < 1$, the analogue of (19) and (20) hold, if we replace the first factor in (20b) and (20c), i.e. $\lambda(\epsilon)$, by $\lambda(\epsilon)^\alpha$. Then $u(r,t)$ satisfies the analogue of the incompressible Euler equation, namely it is divergenceless and

$$\rho_0 \frac{\partial}{\partial t}u_i + \rho_0 g(\rho_0) u_i \frac{\partial}{\partial x_i}u_i = -\frac{\partial}{\partial x_i}p \qquad (22)$$

N.B. The Boltzmann equation (8) is in continuous space - not in a lattice. As a consequence the viscosity $\nu(\rho_0)$ in (21) does not contain the "negative lattice contribution" of FHP. This is due to the fact that in our model the mean free path, and hence the true viscosity, is going to infinity. To obtain a finite value we have to scale things properly and this makes the negative lattice contribution vanish .

For the proofs of the above results and for further details we refer to the paper by DeMasi, Esposito, Lebowitz and Presutti[DELP].

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