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Incompressible Navier-Stokes and Euler Limits of the Boltzmann Equation

A. DE MASI
Universita dell'Aguila

R. ESPOSITO
Universita di Roma Tor Vergata

AND

J. L. LEBOWITZ
Rutgers University

Abstract

We consider solutions of the Boltzmann equation, in a d-dimensional torus, $d = 2, 3$,

$$(*) \qquad \partial_\tau f(r, v, \tau) + v \cdot \nabla_r f = Q(f, f),$$

for macroscopic times $\tau = t/\varepsilon^N$, $\varepsilon \ll 1$, $t \geqq 0$, when the space variations are on a macroscopic scale $x = \varepsilon^{N-1} r$, $N \geqq 2$, x in the unit torus. Let $u(x, t)$ be, for $t \leqq t_0$, a smooth solution of the incompressible Navier Stokes equations (INS) for $N = 2$ and of the Incompressible Euler equation (IE) for $N > 2$.

We prove that $(*)$ has solutions for $t \leqq t_0$ which are close, to $O(\varepsilon^2)$ in a suitable norm, to the local Maxwellian $[\bar{\rho}/(2\pi \bar{T})^{d/2}]\exp\{-[v - \varepsilon u(x, t)]^2/2\bar{T}\}$ with constant density $\bar{\rho}$ and temperature $\bar{T}$. This is a particular case, defined by the choice of initial values of the macroscopic variables, of a class of such solutions in which the macroscopic variables satisfy more general hydrodynamical equations. For $N \geqq 3$ these equations correspond to variable density IE while for $N = 2$ they involve higher-order derivatives of the density.

1. Introduction

The Boltzmann Equation (BE) describes the time evolution of the positional and velocity density $f(r, v, \tau)$ of a gas. An old question is the relation of the Boltzmann equation to the Euler and Navier-Stokes equations of fluid dynamics. (For an historical background see Chapman and Cowling [1].) In particular, under what conditions does the BE lead to closed hydrodynamical type equations for the locally conserved quantities (collision invariants) which depend only on r and τ. These are the five moments of f corresponding to mass, momentum and

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energy densities,

$$\rho(r,\tau) = \int_{\mathbb{R}^d} f(r,v,\tau)\,dv, \qquad \rho u(r,\tau) = \int_{\mathbb{R}^d} v f(r,v,\tau)\,dv, \tag{1.1}$$
$$\rho T(r,\tau) = \frac{1}{d}\int_{\mathbb{R}^d} (v - u(r,\tau))^2 f(r,v,\tau)\,dv.$$

The possibility of such a "contracted description" is due to the big disparity (in many situations) between the microscopic scale of the BE, represented by the mean free path and mean free time and the macroscopic scales on which the (conserved) hydrodynamical variables change. This can be expected to lead to a state close to local equilibrium, i.e., $f(r,v,\tau)$ should be close to a local Maxwellian,

$$M(\rho,u,T;v) = \rho/(2\pi T)^{d/2}\exp\{-(v-u)^2/2T\}, \tag{1.2}$$

with the hydrodynamical variables changing "slowly" in space and time. M is singled out by the fact that it is the only function for which $Q(M,M) = 0$ (see [1]–[3]).

The disparity between microscopic and macroscopic scales is the central ingredient in the derivation of a variety of macroscopic laws from microscopic model systems (see [4]). In all these cases one follows the path initiated by Hilbert [5] and Enskog [6] who introduced the ratio of microscopic to macroscopic length scales, the Knudsen number ε, as an explicit parameter in the BE changing (*) to

$$\partial_\tau f_\varepsilon(r,\tau;v) + v\cdot\nabla_r f_\varepsilon = \varepsilon^{-1}Q(f_\varepsilon,f_\varepsilon), \tag{1.3}$$

where we have changed the order of the variables in f_ε to emphasize that v is to be thought of as a parameter.

It is now possible to make various formal expansions of f_ε in powers of ε about some local or global Maxwellian. In particular the well-known Hilbert-Chapman-Enskog method [1] leads to a sequence of equations for the hydrodynamical variables corresponding, respectively, to the (non-dissipative) Euler equations, the Navier-Stokes equations, etc. From now on we always consider a collision operator Q corresponding, a la Grad [2], to cut-off hard potentials.

An important step in the mathematical development of the theory was taken by Caflisch [7] (and by Caflisch and Papanicolau [8] for a discrete velocities model) who proved the following: let ρ, u, T be smooth solutions of Euler equations in a torus for $t \in [0, t_0]$. Then there exists a solution $f_\varepsilon(r,\tau;v)$ of the

BE (1.3) such that, for $\tau < t_0$, we have, in a suitable norm,

$$\|f_\varepsilon - M(\rho, u, T; v)\| < C\varepsilon. \tag{1.4}$$

This result has been extended by Lachowicz [9] to more general initial data $f_\varepsilon(r, 0; v)$ and more general domains, see also Bardos [10] and Bardos and Golse [11].

We now observe that equation (1.3) is a special case of scaled equations which can be obtained directly from (∗) by looking explicitly at solutions which vary on different macroscopic space-time scales. More precisely we let ε^{-1} be the macroscopic scale, i.e., the size of the domain (torus) in which the system is confined and we let $r = \varepsilon^{-1}x$, $\tau = \varepsilon^{-\alpha}t$. The time evolution of $F^\varepsilon_{(\alpha)}(x, t; v) = f(\varepsilon^{-1}x, v, \varepsilon^{-\alpha}t)$ will satisfy the $(\mathrm{BE})_\alpha$ equation

$$\frac{\partial F^\varepsilon_{(\alpha)}}{\partial t} + \varepsilon^{(1-\alpha)} v \cdot \nabla_x F^\varepsilon_{(\alpha)} = \varepsilon^{-\alpha} Q\big(F^\varepsilon_{(\alpha)}, F^\varepsilon_{(\alpha)}\big), \tag{1.5}$$

with some initial condition $F^\varepsilon_{(\alpha)}(x, 0; v)$. Equation (1.5) is nothing more than a rewriting of (∗) in terms of the new variables x and t. Any new results valid for "ε small" must therefore follow form (∗) and the assumptions, if any, on the initial state $F^\varepsilon_{(\alpha)}(x, 0; v)$. This will be discussed further in Section 5; see also [12].

Equation (1.3) thus corresponds to $\alpha = 1$, i.e., space and time are scaled in the same way. That this is the right choice for the Euler equations is clearly related to the fact that these equations are invariant under this scaling. The reduction to Euler type equations under uniform space-time scaling can also be proven for some simple microscopic models (see [13]) and shown to occur, under the assumption of local equilibrium, for real particle systems evolving according to Hamiltonain equations of motion (see [4]).

A natural question to ask then is whether there is any different micro-macro scaling of space-time which produces the Navier-Stokes equations. An inspection of those equations shows that they are not invariant under any rescaling of space-time. This implies that there are no space-time scalings which produce the full Navier-Stokes equations as a well-defined limit. They have to be thought of instead as a correction to the Euler equations. On the other hand, if one also scales the macroscopic velocity u, i.e., one considers u's very small compared to microscopic velocities (or the Mach number), then there is a scaling which leaves a "subset" of the Navier Stokes equations, and in particular the INS equations, invariant; c.f. [14]. Similarly there are scalings of u which make the dissipative term in the INS vanish but leave a "subset" of the Euler equations, including the IE, invariant; see [15]. This suggests a derivation of these equations directly from the BE with proper scaling of the initial u's. We carry out this program for the INS and IE in Sections 2, 3, 4. We leave for Section 5 the discussion of the larger class of hydrodynamical evolutions, obtained by imposing some constraint on the $d + 2$ variables occurring in the compressible hydrodynamic equations, which can be obtained from the BE by choosing suitable initial conditions.

2. Statement of Results

Let $u(x, t)$ be the solution of the INS or IE in a unit d-dimensional torus Π, i.e., u satisfies the equations

$$\bar{\rho}\partial_t u + \bar{\rho} u \cdot \nabla_x u = -\nabla_x p + \eta_N \Delta u, \tag{2.1a}$$

$$\operatorname{div} u = 0, \tag{2.1b}$$

$$u(x,0) = u_0(x), \tag{2.1c}$$

where $\bar{\rho}$ is the constant mass density and η_N is the viscosity coefficient for $N = 2$ (INS), while $\eta_N = 0$ if $N > 2$ (IE). The vector field $\nabla_x p$ which corresponds physically to the pressure gradient is not specified a priori but is to be considered formally as an unknown function also determined from (2.1). The parameter N will be defined later. It is known (see for example [16]) that if u_0 is sufficiently smooth there is a t_0 depending on u_0 such that equation (2.1) has a unique smooth solution $(u(x, t), p(x, t))$ for $t \leqq t_0$. In two dimensions, t_0 can be taken arbitrarily large.

In order to define precisely the smoothness we require, we introduce the Sobolev space $H_s(\Pi)$ of order s with scalar product

$$(h, g)_s = \sum_{j=0}^{s} \int_\Pi dx\, \nabla^j h(x) \cdot \nabla^j g(x). \tag{2.2}$$

Furthermore, we denote by B_{js} the space of the measurable functions on $\Pi \times \mathbb{R}^d$, such that

$$\|h\|_{js} = \sup_{v \in \mathbb{R}^d} (1 + v^2)^{j/2} |h(\cdot, v)|_s, \qquad |h|_s^2 = (h, h)_s, \tag{2.3}$$

is finite.

We wish to prove now that any smooth velocity and pressure field satisfying (2.1) can in fact be obtained from the solution to the $(\mathrm{BE})_\alpha$ equation (1.5) for suitable initial distribution and choices of the scaling α. The case $\alpha = 1$ is covered by the Caflisch theorem, equation (1.4). We shall now consider the case $1 < \alpha \leqq 2$, and observe that by letting $\varepsilon^{\alpha-1} \to \varepsilon$ we can rewrite equation (1.5) as follows:

$$\frac{\partial F_N^\varepsilon}{\partial t} + \varepsilon^{-1} v \cdot \nabla_x F_N^\varepsilon = \varepsilon^{-N} Q(F_N^\varepsilon, F_N^\varepsilon), \tag{2.4}$$

with $N = \alpha/(\alpha - 1)$, so that $N \geqq 2$. We assume that N is an integer: our proofs presumably work also for any rational number N, but we did not treat that case.

We prove the following

THEOREM 1. *Given a divergence-free field* $u_0 \in H_s(\Pi)$, $s \geqq 4$, $\bar{\rho} > 0$, $\bar{T} > 0$, $N \geqq 2$, *there are fixed* $t_0 > 0$ *and* $\varepsilon_0 > 0$ *such that equation* (2.4) *has a solution* $F_N^\varepsilon(x, t; v) \in B_{j2}$ *for* $t \leqq t_0$, $j > 4$, $\varepsilon < \varepsilon_0$, *satisfying*

$$\left\| F_N^\varepsilon(x, t; v) - M(\bar{\rho}, \varepsilon u(x, t), \bar{T}; v) \right\|_{j2} < C\varepsilon^2, \tag{2.5}$$

whenever $u(x, t)$ *is a solution to equation* (2.1) *in* $H_s(\Pi)$ *for* $t \leqq t_0$. *In the case* $N = 2$, *the viscosity coefficient* $\eta = \eta(\bar{\rho}, \bar{T})$ *is given by the equation* (3.32) *below.*

Remarks. (i) The bounds in the above statement are not optimal, but they are sufficient for $d = 2, 3$.

(ii) Note that u in equation (2.5) is not the same as the u in equation (1.1), $u \to \varepsilon u$.

(iii) We actually prove that there is a solution $F_N^\varepsilon(x, t; v)$ to equation (2.4) of the following form: we consider for concreteness the case $N = 2$ and write $F_2^\varepsilon = F^\varepsilon$,

$$F^\varepsilon(x, t; v) = M(\rho_\varepsilon, u_\varepsilon, T_\varepsilon) + \varepsilon^2 g(x, v, t) + \varepsilon^3 h_\varepsilon(x, t, v), \tag{2.6}$$

where

$$\rho_\varepsilon = \bar{\rho} + \varepsilon^2 \rho^{(2)}(x, t), \tag{2.7a}$$

$$u_\varepsilon = \varepsilon u(x, t) + \varepsilon^2 u^{(2)}(x, t), \tag{2.7b}$$

$$T_\varepsilon = \bar{T} + \varepsilon^2 T^{(2)}(x, t), \tag{2.7c}$$

$g(x, v, t)$ is a function orthogonal to the space generated by the collision invariants (see (3.25)) and all the functions ρ_ε, u_ε, T_ε, $g(x, v, t)$ and $h_\varepsilon(x, t, v)$ are uniformly (in ε) bounded in B_{j2}.

Finally, $u(x, t)$ in (2.7b) is the solution to the INS and the term proportional to ε^2 in $\rho_\varepsilon T_\varepsilon$, i.e., $\bar{T}\rho^{(2)}(x, t) + \bar{\rho}T^{(2)}(x, t)$ is "the pressure" $p(x, t)$ associated to the solution $u(x, t)$ of equation (2.1) with $N = 2$.

(iv) Our results involve choosing an initial distribution $F_N^\varepsilon(x, 0; v)$ which is of the "right form". This may in fact not be necessary. As was shown by Lachowitz [9] for the case $\alpha = 1$ considered by Caflisch (and by Ellis [17] and Ellis and Pinsky [12] in some simple linear case), choosing a "slightly wrong" distribution at $t = 0$ has the effect of producing an "initial layer" which then has to be "matched" to the hydrodynamic solution. Presumably this is also the case for $\alpha > 1$, at least within a certain class of distributions. We discuss this and other questions, including extensions of Theorem 1 to the case in which the gas is subject to a suitably scaled "smooth" external force, in Section 5. We also discuss there choices of initial conditions which lead to variable density IE and to some "higher-order" Navier-Stokes equations in which the pressure is constant up to order ε^2.

The idea of the proof of (2.5) is similar to that of Caflisch [7]; perform a truncated (Hilbert) series expansion in ε and control the remainder equation. All the estimates of the terms of the expansion and of the remainder are due to the good control of the appropriate hydrodynamic equations in the interval $[0, t_0]$. In the next section we develop the expansion, while Section 4 is devoted to the control of the remainder.

3. The Expansion

We first consider the

Case $N = 2$. In this case equation (2.1) becomes

$$\partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla F^\varepsilon = \varepsilon^{-2} Q(F^\varepsilon, F^\varepsilon). \tag{3.1}$$

We look for a solution of (3.1) of the form

$$F^\varepsilon = m_\varepsilon + \sum_{n=2}^{8} \varepsilon^n f_n + \varepsilon^4 f_R, \tag{3.2a}$$

where

$$m_\varepsilon(x, t, v) = M\big(\rho_0(x, t), \varepsilon u(x, t), T_0(x, t); v\big), \tag{3.2b}$$

for some functions u, ρ_0, T_0 to be determined. The functions f_n will be chosen to coincide with those obtained from the Hilbert expansion while the remainder term f_R will satisfy the equation below. We note that m_ε is such that $Q(m_\varepsilon, m_\varepsilon) = 0$ and that the first correction to m_ε is $O(\varepsilon^2)$.

We write

$$m_\varepsilon = m_0 + \sum_{n=1}^{k} \varepsilon^n \varphi_n + \varepsilon^{k+1} r_{k+1}, \tag{3.3}$$

where

$$\varphi_n = \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n} m_\varepsilon \bigg|_{\varepsilon = 0}, \qquad n \geqq 1, \tag{3.4}$$

and r_{k+1} is a remainder of order $k + 1$,

$$\varphi_1 = m_0 \frac{u \cdot v}{T_0}, \qquad \varphi_2 = \frac{1}{2} m_0 \frac{\big(v_i v_j - T_0 \delta_{ij}\big)}{T_0^2} u_i u_j, \quad i, j = 1, \cdots, d. \tag{3.5}$$

In (3.5) and below we sum over all repeated indices.

Substituting (3.2a) and (3.3) in equation (3.1) and equating terms of the same order in ε not containing f_R we get

$$\varepsilon^{-1}: \quad v \cdot \nabla m_0 = 0, \tag{3.6a}$$

$$\varepsilon^{0}: \quad \partial_t m_0 + v \cdot \nabla \varphi_1 = Lf_2, \tag{3.6b}$$

$$\varepsilon^{1}: \quad \partial_t \varphi_1 + v \cdot \nabla (f_2 + \varphi_2) = Lf_3 + L^{(1)} f_2 \tag{3.6c}$$

and, for $2 \leqq k \leqq 6$,

$$\begin{aligned} \varepsilon^{k}: \quad & \partial_t (f_k + \varphi_k) + v \cdot \nabla (f_{k+1} + \varphi_{k+1}) \\ & = Lf_{k+2} + L^{(1)} f_{k+1} + \sum_{\substack{n,\, m \geqq 2 \\ n+m=k+2}} Q(f_n, f_m). \end{aligned} \tag{3.7}$$

Moreover we have the remainder equation

$$\partial_t f_R + \varepsilon^{-1} v \cdot \nabla f_R = \varepsilon^{-2} L_\varepsilon f_R + \widetilde{L_\varepsilon} f_R + \varepsilon^2 Q(f_R, f_R) + \varepsilon^3 A. \tag{3.8}$$

In equations (3.6), (3.7) and (3.8) we have used the following definitions:

$$L_\varepsilon f = 2Q(m_\varepsilon, f), \tag{3.9}$$

$$Lf = 2Q(m_0, f), \tag{3.10}$$

$$L^{(1)} f = 2Q(r_1, f), \tag{3.11}$$

$$\widetilde{L_\varepsilon} f = 2Q\left(\sum_{n=2}^{\infty} \varepsilon^{n-2} f_n, f \right), \tag{3.12}$$

$$\begin{aligned} A = & -\partial_t [(f_7 + \varphi_7) + \varepsilon (f_8 + r_8)] - v \cdot \nabla (f_8 + r_8) + L^{(1)} f_8 \\ & + \sum_{k \geqq 9} \varepsilon^{k-9} \sum_{\substack{n,\, m \geqq 2 \\ n+m=k}} Q(f_n, f_m). \end{aligned} \tag{3.13}$$

Now we analyze the consequences of equations (3.6) and put $d = 3$: equation (3.6a) implies,

$$\nabla \rho_0 = 0, \qquad \nabla T_0 = 0. \tag{3.14}$$

Equation (3.6b) implies

$$P(\partial_t m_0 + v \cdot \nabla \varphi_1) = 0, \tag{3.15}$$

$$(1 - P)(\partial_t m_0 + v \cdot \nabla \varphi_1) = Lf_2, \tag{3.16}$$

where

(3.17)

$$Pf(v) = \int_{\mathbb{R}^3} f(v')\, dv' + v \int_{\mathbb{R}^3} v' f(v')\, dv' + (v^2 - 3T_0) \int_{\mathbb{R}^3} (v'^2 - 3T_0) f(v')\, dv'.$$

Equation (3.15) thus becomes

$$\int_{\mathbb{R}^3} (\partial_t m_0 + v \cdot \nabla \varphi_1)\, dv = 0, \tag{3.18a}$$

$$\int_{\mathbb{R}^3} v(\partial_t m_0 + v \cdot \nabla \varphi_1)\, dv = 0, \tag{3.18b}$$

$$\int_{\mathbb{R}^3} (v^2 - 3T_0)(\partial_t m_0 + v \cdot \nabla \varphi_1)\, dv = 0. \tag{3.18c}$$

Using (3.14) and the relations

$$\int_{\mathbb{R}^3} m_0\, dv = \rho_0, \quad \int_{\mathbb{R}^3} v_i v_j m_0\, dv = \rho_0 T_0 \delta_{ij}, \quad \int_{\mathbb{R}^3} v_i \varphi_1\, dv = \rho_0 u_i, \tag{3.19}$$

(3.18a) and (3.18c) give

$$\partial_t \rho_0 + \operatorname{div} \rho_0 u = 0, \tag{3.20}$$

$$3\rho_0 \partial_t T_0 + 2 \operatorname{div} T_0 \rho_0 u = 0. \tag{3.21}$$

Integrating equations (3.20) and (3.21) on the volume Π it follows by (3.14) that $\partial_t \rho_0 = 0$ and $\partial_t T_0 = 0$. This, together with (3.14) implies that

$$\rho_0 = \bar{\rho}, \qquad T_0 = \bar{T}, \tag{3.22}$$

i.e., m_0 is a global Maxwellian,

$$m_0(x, t; v) = \bar{\rho} g_0 \equiv M(\bar{\rho}, 0, \bar{T}; v). \tag{3.23}$$

Moreover, by (3.22) and (3.20) we must have, for the expansion to be consistent,

$$\operatorname{div} u = 0 \qquad \text{(incompressibility)}. \tag{3.24}$$

Finally (3.18b) is satisfied by the properties of the Gaussian integrals.

Equation (3.16) yields, for all $t \leqq t_0$, including the initial distribution at $t = 0$,

$$\begin{aligned} f_2 = {} & L^{-1}\left(\bar{T}^{-1} g_0\left(v_i v_j - \tfrac{1}{3} v^2 \delta_{ij}\right)\right) \bar{\rho} \partial_j u_1 \\ & + g_0\left[\rho^{(2)} + \rho^{(2)} u^{(2)} \cdot v + e^{(2)}\left(v^2 - 3\bar{T}\right)\right], \end{aligned} \tag{3.25}$$

where $\rho^{(k)}$, $u^{(k)}$, $e^{(k)}$ will denote here and below the components of f_k along the collision invariants. Since they are in the null space of L, they are not fixed by equation (3.25) and will be fixed later. We find (see for example [3])

$$L^{-1}\left(g_0\left(v_i v_j - \tfrac{1}{3} v^2 \delta_{ij}\right)\right) = -\Psi\left(v^2\right)\left(v_i v_j - \tfrac{1}{3} v^2 \delta_{ij}\right) g_0, \tag{3.26}$$

with Ψ a known positive function. Therefore,

$$\begin{aligned} f_2 = {} & -\frac{\bar{\rho}}{\bar{T}} \Psi\left(v^2\right)\left(v_i v_j - \tfrac{1}{3} v^2 \delta_{ij}\right) g_0 \partial_j u_i \\ & + g_0\left[\rho^{(2)} + \rho^{(2)} u^{(2)} \cdot v + e^{(2)}\left(v^2 - 3\bar{T}\right)\right]. \end{aligned} \tag{3.27}$$

The stress tensor and the heat current vector associated to f_k are defined by

$$\pi_{ij}^{(k)} = \int_{\mathbb{R}^3} v_i v_j f_k \, dv, \qquad q_i^{(k)} = \int_{\mathbb{R}^3} v_i\left(v^2 - 3\bar{T}\right) f_k \, dv. \tag{3.28}$$

To compute the ones associated to f_2 we note that

$$\begin{aligned} & \int_{\mathbb{R}^3} dv\, \Psi\left(v^2\right)\left(v_i v_j - \tfrac{1}{3} v^2 \delta_{ij}\right) v_k v_l g_0(v) \\ & \quad = \mu(\bar{T})\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right)\left(1 - \delta_{ij}\right) + \sigma(\bar{T}) \delta_{ij} \delta_{kl}, \end{aligned} \tag{3.29}$$

with positive $\mu(\bar{T})$ and $\sigma(\bar{T})$. Moreover,

$$\begin{aligned} & \int_{\mathbb{R}^3} \rho^{(2)} u_i^{(2)} v_i v_k v_l g_0(v) \, dv = 0, \qquad \int_{\mathbb{R}^3} \rho^{(2)} v_k v_l g_0(v) \, dv = \rho^{(2)} \bar{T} \delta_{kl}, \\ & \int_{\mathbb{R}^3} e^{(2)} v_k v_l\left(v^2 - 3\bar{T}\right) g_0(v) \, dv = 2 e^{(2)} \bar{T}^2 \delta_{kl} \equiv \bar{\rho} T^{(2)} \delta_{kl}. \end{aligned} \tag{3.30}$$

Hence, using (3.24),

$$\pi_{ij}^{(2)} = -\tfrac{1}{2}\eta\left(\partial_i u_j + \partial_j u_i\right) + p\delta_{ij}, \tag{3.31}$$

where the viscosity coefficient and the correction to the pressure are given by

$$\frac{1}{2}\eta = \frac{\mu(\bar{T})\bar{\rho}}{\bar{T}}, \qquad p = \left(\rho^{(2)} + 2\bar{T}e^{(2)}\right)\bar{T} = \rho^{(2)}\bar{T} + \bar{\rho}T^{(2)}. \tag{3.32}$$

On the other hand, since

(3.33)

$$\int_{\mathbb{R}^3} dv\, \Psi(v^2)\left(v_i v_j - \tfrac{1}{3}v^2\delta_{ij}\right) v_k v^2 g_0(v) = 0, \qquad \int_{\mathbb{R}^3} dv\, v_k v^2 g_0(v) = 0,$$

and

$$\int_{\mathbb{R}^3} (v^2 - 3\bar{T}) v_i v_j g_0(v)\, dv = 2\bar{T}^2\delta_{ij}, \tag{3.34}$$

we have

$$q_i^{(2)} = 2\bar{T}^2\rho^{(2)}u_i^{(2)}. \tag{3.35}$$

Using the relations

$$\int_{\mathbb{R}^3} \varphi_2 v\, dv = 0, \qquad \int_{\mathbb{R}^3} \varphi_2 v(v^2 - 3\bar{T})\, dv = 0, \tag{3.36}$$

and

$$\int_{\mathbb{R}^3} \varphi_2 v_i v_j\, dv = \bar{\rho} u_i u_j, \tag{3.37}$$

we have by (3.6), to order ε,

$$\operatorname{div} \rho^{(2)} u^{(2)} = 0, \tag{3.38}$$

$$\partial_t \bar{\rho} u_1 + \partial_j \pi_{ij}^{(2)} + \partial_j \bar{\rho} u_i u_j = 0, \tag{3.39}$$

$$\begin{aligned} f_3 = {} & L^{-1}\left[(1 - P)(\partial_t \varphi_1 + v \cdot \nabla f_2) - L^{(1)} f_2\right] \\ & + g_0\left[\rho^{(3)} + \rho^{(3)} u^{(3)} \cdot v + e^{(3)}(v^2 - 3\bar{T})\right], \end{aligned} \tag{3.40}$$

with $\rho^{(3)}$, $u^{(3)}$, $e^{(3)}$ to be fixed later. Using the expression (3.31), equation (3.39) becomes the incompressible Navier-Stokes equation.

By (3.7), we have, for $2 \leqq k \leqq 6$,

$$P\left(\partial_t(f_k + \varphi_k) + v \cdot \nabla(f_{k+1} + \varphi_{k+1})\right) = 0, \tag{3.41}$$

$$\begin{aligned} f_{k+2} = L^{-1}\Bigg\{&(1-P)\left[\partial_t(f_k + \varphi_k) + v \cdot \nabla(f_{k+1} + \varphi_{k+1})\right] \\ &- L^{(1)} f_{k+1} - \sum_{\substack{n,\, m \geqq 2 \\ n+m=k+2}} Q(f_n, f_m)\Bigg\} \\ &+ g_0\left[\rho^{(k+2)} + \rho^{(k+2)} u^{(k+2)} \cdot v + e^{(k+2)}(v^2 - 3\overline{T})\right]. \end{aligned} \tag{3.42}$$

After equation (3.41) is satisfied, equation (3.42) determines f_{k+2} up to the projection on the space of the collision invariants. On the other hand, we can use the arbitrary functions $\rho^{(k)}$, $u^{(k)}$, $e^{(k)}$ to satisfy equation (3.41). The analysis of the conditions for $\rho^{(k)}$, $u^{(k)}$, $e^{(k)}$ is simple and we omit it: the functions f_k can be determined and will be considered as known functions of x, v, t below. It is a classical result of Grad [2] that they have the same spatial regularity as u and satisfy the bounds

$$|f_k(x, t, v)| \leqq \mathscr{P}_k(x, t; v) M_0(v), \tag{3.43}$$

where the $\mathscr{P}_k(x, t; v)$ are some polynomials in v with coefficients depending on x, through $u(x, t)$.

In consequence of the expansion, if u is a solution of the incompressible Navier-Stokes equation and the f_k are chosen as above, equation (3.1) becomes equivalent to equation (3.8). This equation is much simpler than the original one, since the nonlinear term is multiplied by a power of ε. This fact allows then for control of the remainder term.

Case $N > 2$. We use the same procedure as before. We look for a solution of (2.1) of the form

$$F_N^\varepsilon = m_\varepsilon + \sum_{n=0}^{2N+4} \varepsilon^{n+N} f_{n+N} + \varepsilon^{N+2} f_R, \tag{3.44}$$

where

$$m_\varepsilon(x, v, t) = M\left(\rho_\varepsilon(x, t), u_\varepsilon(x, t), T_\varepsilon(x, t); v\right). \tag{3.45}$$

We assume that $\rho_\varepsilon(x,t)$ and $T_\varepsilon(x,t)$ have the following expansions:

$$\rho_\varepsilon(x,t) = \bar\rho + \sum_{n=2}^{N-1} \rho_n(x,t)\varepsilon^n,$$

(3.46)

$$T_\varepsilon(x,t) = \bar T + \sum_{n=2}^{N-1} T_n(x,t)\varepsilon^n,$$

and $u_\varepsilon = \varepsilon u$ for $N = 3$ having the expansion

$$u_\varepsilon(x,t) = \varepsilon u + \sum_{n=3}^{N-1} u_n(x,t)\varepsilon^n \tag{3.47}$$

for $N > 3$.

We could have chosen m_ε as in (3.45) also in the previous case: this would have only changed the definitions of the f_k in (3.2) for $k \geqq 2$.

We have (see equations (3.4) and (3.5))

$$m_\varepsilon = m_0 + \varepsilon\varphi_1 + \sum_{n=2}^{N+5} \varepsilon^n(\varphi_n + G_n) + \varepsilon^{N+6} r_{N+6}, \tag{3.48}$$

where we denote by $\varphi_n + G_n$ the coefficients of the power series of m_ε, while the φ_n still denote the quantities (3.5). We now proceed as in the case $N = 2$ and consider the equations coming from the orders ε^{-1}, ε^0 and ε^1. The ε^{-1}-order is exactly the same, and therefore from it we get the same informations as before. The ε^0-order gives

$$\partial_t m_0 + v \cdot \nabla_{\varphi_1} = Lf_N. \tag{3.49}$$

Equation (3.49) is equal to (3.6b) and so as before it gives the incompressibility condition and the expression for f_N as in (3.25). The ε^1-order is

$$\partial_t \varphi_1 + v \cdot \nabla(G_2 + \varphi_2) = Lf_{N+1} + L^{(1)}f_N. \tag{3.50}$$

If we multiply by v_j both sides of (3.50) and take the v-integral, we get (see equation (3.37)),

$$\bar\rho\partial_t u + \bar\rho u \cdot \nabla_x u + \nabla_x p = 0, \qquad p = \bar T\rho_2 + \bar\rho T_2. \tag{3.51}$$

Equation (3.51) is the incompressible Euler equation. The functions f_n are determined as before and the functions ρ_n, T_n, u_n, $n = 3,\cdots, N-1$, are determined in such a way that the solvability conditions for f_{N+n-1} are satisfied,

$$P[\partial_t(\varphi_{n-1} + G_{n-1}) + v \cdot \nabla(\varphi_n + G_n)] = 0, \quad n = 3,\cdots, N-1. \tag{3.52}$$

Once all the functions appearing in the expansion are determined, we are left with the following equation for the remainder f_R.

Let

$$\widetilde{L_\varepsilon} f = 2Q\left(\sum_{n=2}^{2N+4} \varepsilon^{n-2} f_n, f\right) \tag{3.53}$$

and

$$\begin{aligned} A = &-\partial_t\left[\varphi_{N+5} + G_{N+5} + \varepsilon r_{N+6} + \sum_{n=N+5}^{2N+4} \varepsilon^{n-N-5} f_n\right] \\ &- v \cdot \nabla\left[r_{N+6^+} \sum_{n=N+6}^{2N+4} \varepsilon^{n-N-6} f_n\right] \\ &+ L^{(1)} f_{2N+4} + \sum_{k \geqq 2N+5} \varepsilon^{k-2N-5} \sum_{\substack{n, m \geqq N \\ n+m=k}} Q(f_n, f_m). \end{aligned} \tag{3.54}$$

Then f_R must satisfy

$$\frac{\partial f_R}{\partial t} + \varepsilon^{-1} v \cdot \nabla_x f_R = \varepsilon^{-N} L_\varepsilon f_R + \widetilde{L_\varepsilon} f_R + \varepsilon^2 Q(f_R, f_R) + \varepsilon^3 A, \tag{3.55}$$

which is completely analogous to equation (3.8).

4. The Remainder Equation

The study of the remainder equation follows closely the works [7], [9]. Therefore we only sketch the proofs in the case $N = 2$. The main difficulty in the study of equation (3.8) comes from the negative powers of ε. The ε^{-1} in front of $v \cdot \nabla f_R$ may be easily managed, because the free stream operator generates an isometry in all the spaces we consider below, independently of ε. A way to control the ε^{-2} in front of L_ε is to use the nonpositivity of L_ε on $L_2[m_\varepsilon^{-1}(x, t; v)\, dv]$. Following Grad [2], one would consider the equation for $\widetilde{f_R} = (m_\varepsilon)^{-1/2} f_R$. Since m_ε depends on x and t, the equation for $\widetilde{f_R}$ would contain polynomials in velocity which are unbounded for large velocities. This difficulty has been solved by Caflisch [7], by means of a decomposition of f_R in low and high velocity parts. The low velocity part is then controlled by the usual Grad technique, while, to control high velocities, one introduces a global Maxwellian $M^*(v)$ with a temperature T^* larger than $\overline{T}$, in order to gain from it a convergence factor in the velocity space.

More precisely, following Caflisch [7], we look for the solution of equation (3.8) in the form

$$f_R = \sqrt{m_\varepsilon}\, g + \sqrt{M^*}\, h, \tag{4.1}$$

where

$$M^*(v) = M(\bar{\rho}, 0, T^*; v), \qquad T^* > \bar{T}, \tag{4.2}$$

and g and h (the low and high velocity parts, respectively) are defined as the solutions of the following coupled system of equations:

$$(\partial_t + \varepsilon^{-1} v \cdot \nabla) g = \varepsilon^{-2} \mathscr{L}_\varepsilon g + \varepsilon^2 \chi \sigma^{-1} \mathscr{K}^* h, \tag{4.3}$$

$$\begin{aligned}(\partial_t + \varepsilon^{-1} v \cdot \nabla) h = {} & - \mu \sigma g - \varepsilon^{-2}(\nu_\varepsilon - \bar{\chi} \mathscr{K}^*) h + \widetilde{L_\varepsilon}(\sigma g + h) \\ & + \varepsilon^2 \nu^* \Gamma^*(\sigma g + h, \sigma g + h) + \varepsilon^3 a,\end{aligned} \tag{4.4}$$

$$g(x, v, 0) = h(x, v, 0) = 0, \tag{4.5}$$

where, given $\gamma > 0$,

$$\chi(v) = \begin{cases} 1, & |v| \leqq \gamma, \\ 0 & \text{otherwise}. \end{cases} \qquad \bar{\chi} = 1 - \chi, \tag{4.6}$$

$$\mu = \tfrac{1}{2} m_\varepsilon^{-1} (\partial_t + \varepsilon^{-1} v \cdot \nabla) m_\varepsilon, \qquad \sigma = \frac{m_\varepsilon}{M^*}, \tag{4.7}$$

$$\nu_\varepsilon = \int_{\mathbf{R}^3} dv_1 \int_{(v_1 - v)\cdot n \geqq 0} dn\, (v_1 - v) \cdot n\, m_\varepsilon(x, t, v_1), \tag{4.8}$$

$$\nu^* = \int_{\mathbf{R}^3} dv_1 \int_{(v_1 - v)\cdot n \geqq 0} dn\, (v_1 - v) \cdot n\, M^*(v_1), \tag{4.9}$$

$$\mathscr{L}_\varepsilon f = \left(\sqrt{m_\varepsilon}\right)^{-1} L_\varepsilon\left[\sqrt{m_\varepsilon} f\right] = -\nu_\varepsilon f + \mathscr{K}_\varepsilon f, \tag{4.10}$$

$$\mathscr{L}^* f = (\sqrt{M^*})^{-1} L_\varepsilon\left[\sqrt{M^*} f\right] = -\nu^* f + \mathscr{K}^* f, \tag{4.11}$$

$$\tilde{\mathscr{L}} f = (\sqrt{M^*})^{-1} \widetilde{L_\varepsilon}\left[\sqrt{M^*} f\right], \tag{4.12}$$

$$\nu^* \Gamma^*(f, g) = (\sqrt{M^*})^{-1} Q\left[\sqrt{M^*} f, \sqrt{M^*} g\right], \tag{4.13}$$

$$a = \frac{A}{\sqrt{M^*}}. \tag{4.14}$$

The term μ in equation (4.4) is unbounded in the velocity, but now it is controlled by the exponentially decaying factor σ.

To study this problem we consider a linear version of the equations (4.3)-(4.5), i.e., we replace $\nu^*\Gamma^*(f, g) + \varepsilon a$ by a given function b. This yields

$$(\partial_t + \varepsilon^{-1}v \cdot \nabla)g = \varepsilon^2\mathscr{L}_\varepsilon g + \varepsilon^{-2}\chi\sigma^{-1}\mathscr{K}^*h, \tag{4.15}$$

$$\begin{aligned}(\partial_t + \varepsilon^{-1}v \cdot \nabla)h \\ = -\mu\sigma g - \varepsilon^{-2}(\nu_\varepsilon - \bar{\chi}\mathscr{K}^*)h + \tilde{\mathscr{L}}(\sigma g + h) + \varepsilon^2 b,\end{aligned} \tag{4.16}$$

$$g(x, v, 0) = h(x, v, 0) = 0, \tag{4.17}$$

Now we have a system quite similar to (6.1)–(6.3) of [7]. The main difference is that the factor ε^{-1} in the right-hand sides of (6.1) and (6.3) of [7] is replaced by ε^{-2} and that the left-hand sides of (4.15) and (4.16) contain the factor ε^{-1}. Moreover, the x variable is three-dimensional instead of one-dimensional. This requires the control of higher derivatives in x, because eventually the nonlinear term has to be bounded using the algebraic properties of $\mathscr{H}_s$ for $s > \frac{1}{2}d$. The ε^{-2} factor in front of $\mathscr{L}_\varepsilon$ is not relevant because the proof is based on the nonpositivity of $\mathscr{L}_\varepsilon$. Finally the factor ε^{-1} in front of $v \cdot \nabla$ does not make any problem as already mentioned (see Appendix). We therefore prove the following

LEMMA. *There is an $\varepsilon_0 > 0$ such that any solution to equations* (4.15), (4.16) *and* (4.17) *satisfies the bounds*

$$\|g\|_{js} \leqq \varepsilon^{1/4}\|b/\nu_\varepsilon\|_{j+2s}, \tag{4.18}$$

$$\begin{aligned}\|h\|_{js} \leqq \varepsilon^{5/4}\|b/\nu_\varepsilon\|_{js}, \\ \textit{for} \quad j > 3 \quad \textit{and} \quad s \leqq 2 \quad \textit{if} \quad \varepsilon < \varepsilon_0\end{aligned} \tag{4.19}$$

The proof of the lemma is almost a repetition of the proof in [7], [9]; it is given in the appendix for the sake of completeness.

Now we assume

$$b = v^*\Gamma^*(\sigma g + h, \sigma g + h) + \varepsilon a. \tag{4.20}$$

Since in three dimensions the space $\mathscr{H}_s$ is an algebra for $s > \frac{3}{2}$, using the Grad estimate for Γ^*,

$$\|\Gamma^*(f, g)\|_{js} \leqq c\|f\|_{js}\|g\|_{js} \quad \text{for} \quad j > \tfrac{3}{2}, s > \tfrac{3}{2}, \tag{4.21}$$

and the bound (3.43) to estimate a, we get, for $s > \frac{3}{2}$, $j > \frac{3}{2}$,

$$|b/\nu_\varepsilon|_{js} \leqq \left(\|g\|_{0s}^2 + \|h\|_{js}^2 + c\varepsilon\right), \tag{4.22}$$

where we have taken into account the exponentially decaying factor σ. Therefore, by the lemma,

$$\|h\|_{j+2s} \leqq \varepsilon^{5/4}\left(\|g\|_{0s}^2 + \|h\|_{j+2s}^2 + c\varepsilon\right) \tag{4.23}$$

which implies

$$\|h\|_{j+2s} \leqq \|g\|_{0s} + c\varepsilon, \tag{4.24}$$

for ε small enough. Then,

$$\|g\|_{js} \leqq \varepsilon^{1/4}\left(\|g\|_{0s}^2 + \|h\|_{j+2s}^2 + c\varepsilon\right) \leqq 3\varepsilon^{1/4}\left(\|g\|_{js}^2 + c\varepsilon\right) \tag{4.25}$$

and hence the estimate for g. Using this and (4.24) we get the estimate for h,

$$\|h\|_{js} \leqq C\varepsilon, \qquad \|g\|_{js} \leqq C\varepsilon, \tag{4.26}$$

for some constant C.

A standard iteration procedure gives the existence of the solution once the a priori estimate (4.26) has been established. This proves the theorem.

5. Generalizations and Discussion

Our main result, Theorem 1, can be restated, see comment (iii) in Section 2, as follows: Let $F_N^\varepsilon(x, t; v)$ be the solution of the BE (2.4) for the initial distribution

$$F_N^\varepsilon(x,0;v) = M\left(\bar{\rho};\varepsilon u_0(x),\bar{T};v\right) + \sum_{n=N}^{2N+4} \varepsilon^n h_n(x,v), \tag{5.1}$$

$$\operatorname{div} u_0 = 0, \tag{5.2}$$

where the $h_n(x, v)$, $\rho_\varepsilon(x,0)$, $u_\varepsilon(x,0)$, $T_\varepsilon(x,0)$ are determined from the initial values of the hydrodynamical variables satisfying (2.1). (E.g., for $N = 2$, $h_2(x, v)$ is given by the right side of (3.27) for $t = 0$ and ρ_ε, u_ε, T_ε are given by initial values of (2.7) for $N = 2$ and by (3.46), (3.47) for $N > 2$.) Then $F_N^\varepsilon(x, t; v)$ satisfies (2.5) for $t \leqq t_0$. General theory assures (or ought to) that this solution is unique.

A natural question then is, what happens if we replace (5.1) by

$$F_N^\varepsilon(x,0;v) = M(\rho_0(x), \varepsilon u_0(x), T_0(x); v) + \varepsilon h_1(x,v) + \sum_{n=2}^{2N+4} \varepsilon^n h_n(x,v), \tag{5.3}$$

and drop (5.2b)? The answer turns out to be rather complicated and more dependent on N in (2.4) than before. Let us first state the result for the case $N \geqq 4$. We assume, instead of (3.44), a solution of the form

$$F_N^\varepsilon = m_\varepsilon + \sum_{n=1}^{2N+4} \varepsilon^n f_n + \varepsilon^{N+2} f_R. \tag{5.4}$$

In view of (3.46) and (3.47) there is no loss of generality in assuming that the f_n are orthogonal to the collision invariants for $n = 1,\cdots, N-1$. The method of Section 3 then allows us to prove that F_N^ε is a solution to the (2.4) if $f_n = 0$ for $n = 1,\cdots, N-2$, and f_{N-1} satisfies the equation

$$v \cdot \nabla m_0 = Lf_{N-1}. \tag{5.5}$$

The compatibility condition for (5.5) implies that ρ_0 and T_0 may depend on x, but the pressure $p_0 = \rho_0 T_0$ has to be spatially constant. Therefore one can prove that Theorem 1 is still true, but (2.5) is replaced by

$$\|F_N^\varepsilon - M(\rho_0(x,t), \varepsilon u(x,t), T_0(x,t); v)\| \leqq C\varepsilon^2, \tag{5.6}$$

with ρ_0 and u satisfying

$$\begin{aligned} &\partial_t \rho_0 + u \cdot \nabla \rho_0 = 0, \\ &\operatorname{div} u = 0, \\ &\rho_0(\partial_t u + u \cdot \nabla u) = -\nabla p, \end{aligned} \tag{5.7}$$

which is the IE with nonconstant density. Moreover, T_0 is given by $\rho_0 T_0 = p_0$ for some $p_0 > 0$.

It is clear from (5.5) that f_{N-1} depends on $\nabla\rho_0$. In particular, for $N = 3$, f_2 does not vanish and (3.5) has to be modified accordingly. However, it turns out that f_2 is an odd function of v and the stress tensor $\pi^{(2)}$ vanishes. Therefore, the hydrodynamical equations are the IE with nonconstant density, also in this case. Considering now the case $N = 2$, we find that f_1 does not vanish and (3.21), (3.27) and (3.39) have to be modified. The hydrodynamical equations which can be obtained along the same lines as in Section 3 become more complicated and involve additional terms, containing higher derivatives of ρ_0, which depend on extra transport coefficients. We do not go into details here but simply write them down and then make some remarks:

$$\partial_t \rho_0 + \operatorname{div} \rho_0 u = 0, \tag{5.8a}$$

$$\operatorname{div}\left(\frac{d+2}{d} \rho_0 T_0 u\right) = \partial_i\left(\kappa_{ij}(T_0)\, \partial_j T_0\right), \tag{5.8b}$$

$$\begin{aligned} &\partial_t \rho_0 u_i + \partial_j\left[p_2 \delta_{ij} + \rho_0 u_i u_j\right] \\ &\quad = \partial_j\left(\mu(T_0)\left(\partial_i u_j + \partial_j u_i\right) + \nu(T_0) \delta_{ij} \operatorname{div} u\right) \\ &\qquad + \partial_j\left(\alpha_{ijkl}(T_0)\, \partial_k \partial_l T_0 + \beta_{ijkl}(T_0)\, u_k \partial_l T_0 + \gamma_{ijkl} \partial_k T_0 \partial_l T_0\right), \end{aligned} \tag{5.8c}$$

$$T_0 = \frac{p_0}{\rho_0}, \tag{5.8d}$$

with κ, ν, μ, α, β, γ functions of T_0 which depend on the intermolecular interactions, while p_0 does not depend on x or t.

Remarks. (a) If the density and temperature of the initial distribution are constant up to terms of order ε, then we get the situation discussed in Section 3. The same is true if we consider isothermal or isoentropic flows.

(b) The scaling considered in this paper, for $N = 2$, leaves the new equations invariant. This means that the extra terms cannot be removed by scaling. On the other hand, if one assumes that κ vanishes, then (5.8b) implies

$$\operatorname{div} u = 0.$$

If we assume that ν does not depend on T_0 and that α, β, and γ vanish, we have

$$\operatorname{div} u = 0,$$

$$\partial_t \rho_0 + u \cdot \nabla \rho_0 = 0,$$

$$\rho_0 \partial_t u + \rho_0 u \cdot \nabla u = -\nabla p_2 + \nu \Delta u,$$

which represent the INS with nonconstant density.

There are various directions in which the present paper can or ought to be extended. We discuss some of them here.

5.1. Initial and asymptotic distributions. It would be desirable to prove a theorem of the following type. Given a "smooth" initial distribution $f(r, v, 0)$, the BE (*) will bring f, in times short compared to ε^{-1}, to a local equilibrium state with parameters ρ, u, T satisfying the compressible Euler equations for times $\varepsilon^{-1} < \tau < \varepsilon^{-\alpha}$, $\alpha > 1$, the IE equations for times $\tau \sim \varepsilon^{-\alpha}$, $1 < \alpha < 2$, and finally the INS for times $\tau \sim \varepsilon^{-2}$ leading eventually as $\tau \to \infty$ to the global Maxwellian distribution in which ρ, T, and u are constants determined by the initial state (u can be taken to be zero by a Galileian transformation).

As already mentioned, a first step in that direction was taken by Lachowitz [9] who extended the Caflisch theorem to the case when the initial distribution is not "too far" from a local Maxwellian. His result for the BE (1.5) with $\alpha = 1$ can be stated as follows:

THEOREM 2 (Caflisch-Lachowitz). *Let*

$$\|F^{\varepsilon}_{(1)}(x,0;v) - M(\rho_0, u_0, T_0; v)\| < \delta,$$

for some δ *sufficiently small. Then there is a* $\varepsilon_0 > 0$ *s.t. for any* $\delta' > 0$

$$\|F^{\varepsilon}_{(1)}(x,t;v) - M(\rho, u, T; v)\| \leqq C\varepsilon \quad \text{for all} \quad \varepsilon < \varepsilon_0,$$

for all $t \in (\delta', t_0]$, *where* $\rho(x,t)$, $u(x,t)$ *and* $T(x,t)$ *are the smooth solutions of the compressible Euler equations up to time* t_0 *with initial values* $\rho_0(x)$, $u_0(x)$, $T_0(x)$.

We hope that the work in progress by Bardos, Golse and Levermore [18] which utilizes the recent work of di Perna and Lions [19] proving the global existence of weak solutions of the BE will in fact lead to results along the lines described above. For the moment however, it should be emphasized, we have no uniform bound for arbitrary times t even for the restricted class of initial distributions considered here. We expect however and, in cases where the deviation from a global Maxwellian is small, can prove that as $t \to \infty$ the solution of the INS (2.1) and of the BE (2.4) are $u(x,t) \to 0$ and $F^{\varepsilon}(x,t;v) \to M(\bar{\rho}, 0, \tilde{T}; v)$ as $t \to \infty$. Here

$$\tilde{T} = \bar{T} + \frac{1}{d\bar{\rho}}\left\{\int_{\Pi} dx\left[\int dv\, v^2\left[F^{\varepsilon}(x,0;v) - M(\bar{\rho}, \varepsilon u_0(x), \bar{T}; v)\right] + \varepsilon^2\bar{\rho}u_0^2(x)\right]\right\}$$

is determined by the conservation of energy, $\bar{\rho}$ and $\bar{T}$ being the initial density and temperature.

5.2. Fluid subject to external forces. The systems considered so far, approach, as $t \to \infty$, the state of global equilibrium with uniform ρ, u and T. We wish now to consider the more interesting situation in which the hydrodynamical flow is subject to an external force, i.e., when (2.1a) is replaced by

$$\bar{\rho}\partial_t u + \bar{\rho}u \cdot \nabla u = -\nabla p + \eta_N \Delta u + \bar{\rho}b(x), \tag{5.9a}$$

where $b(x)$ is smooth and periodic. (We consider for simplicity only the case $N = 2$, corresponding to the INS.) This corresponds to changing (2.4) to

$$\frac{\partial F^{\varepsilon}}{\partial t} + \varepsilon^{-1}v \cdot \nabla_x F^{\varepsilon} + \varepsilon b \cdot \nabla_v F^{\varepsilon} = \varepsilon^{-2}Q(F^{\varepsilon}, F^{\varepsilon}). \tag{5.9b}$$

Note that in the original Boltzmann equation (*) the force term would be $\varepsilon^3 b(\varepsilon r) \cdot \nabla_v f(r,v,t)$, i.e., the force varies on a macroscopic scale and is very small compared to the interparticle forces in the gas which give rise to the collision term. It is now straightforward to extend Theorem 1 to this case. Equations (3.2a) to (3.2b) are unchanged, while (3.6c), (3.1) and (3.8) are modified in obvious ways. The only difference is in the proof of the lemma (see Section 4) given in the appendix. We need to replace the free motion $(x, v) \to (x + vt, v)$ by the evolution in the force field

$$(x, v) \to \left(x + vt + \int_0^t ds \int_0^s ds' \, b(x(s')), v + \int_0^t ds \, b(x(s)\right),$$

which of course changes also the various constants in the bounds.

There are two simple special cases which are worth considering:

(a) *The conservative case.* We take $b = -\nabla_x U(x)$. In this case the energy of a gas particle $\frac{1}{2}v^2 + \varepsilon^2 U(x)$ is conserved and the equilibrium state of the gas is $M(\rho_\varepsilon(x), 0, \bar{T}; v)$ with the density $\rho_\varepsilon(x)$ given by

$$\rho_\varepsilon(x) = \bar{\rho} \exp\{-\varepsilon^2 U(x)/2\bar{T}\}. \tag{5.10}$$

We could therefore replace $\bar{\rho}$ by $\rho_\varepsilon(x)$ in (2.5) and state the theorem in terms of this local Maxwellian.

(b) *The uniform case.* We take $b = \bar{b}$ independent of x and consider an initial distribution $F^\varepsilon(x, 0; v) = M(\bar{\rho}, \varepsilon\bar{u}, \bar{T}; v)$, where $\bar{u}$ is a constant (which as already noted can be set equal to zero by a Galileian transformation). The solutions of the INS and of the BE (5.8) and (5.9) are then

$$u(x, t) = \bar{u} + \bar{b}t, \tag{5.11a}$$

$$F^\varepsilon(x, t; v) = M(\bar{\rho}, \varepsilon[\bar{u} + \bar{b}t], \bar{T}; v), \tag{5.11b}$$

which are seen to satisfy (2.5) with $C = 0$.

The interesting case is of course when $b \neq -\nabla U$ but $\int_\Pi b \, dx = 0$. We expect then that (5.8) will have a stationary solution $u_s(x)$ which will be approached asymptotically as $t \to \infty$ starting with any u_0. The gas distribution $F^\varepsilon(x, t; v)$ on the other hand does not approach a stationary state in this case. The reason for this is that the external force continues to do work on the system even in the stationary state which, since the system is finite for $\varepsilon \neq 0$, leads to an ever increasing temperature. This however does not appear at the level of the INS (5.8a) which only describes the organized flow field.

To see this explicitly we consider the case where $u_0(x) = u_s(x)$. Multiplying (5.8a) by $u_s(x)$, integrating over the torus Π using (5.8b), we get

$$\bar{\rho} \int_\Pi (u_s \cdot b) \, dx = \eta \int_\Pi \sum_{\alpha=1}^{d} \left(\frac{\partial u_{s\alpha}}{\partial x_\alpha}\right)^2 dx. \tag{5.12}$$

On the other hand, if we multiply (5.9) by $\frac{1}{2}v^2$ and integrate over both v and x we find

(5.13)
$$\frac{d}{dt}\int dx \int dv\, \tfrac{1}{2}v^2 F^\varepsilon(x,t;v) \equiv \frac{d}{dt}\left[\rho\langle v\rangle^2 + \tfrac{1}{2}d\rho T\right] = \varepsilon\int_\Pi \rho\langle v\rangle \cdot b\, dx,$$

so that for $\langle v\rangle = \varepsilon u_s$ the rate of heat production in the system is proportional to ε^2 times the right-hand side of (5.12).

It appears therefore that in order to get a stationary distribution for the positions and velocities of the gas particle in the presence of a stationary macroscopic flow one needs either (i) provide contact with external heat reservoirs, e.g. through contact with walls at fixed temperatures, which will absorb the energy or (ii) consider an infinite system in which any finite heat generation can "dissipate to infinity". An example of case (i) is the study by Bardos, Caflisch and Nicolaenko [20] of the stationary distribution of a gas between two plates kept at different fixed temperatures. An example of the case (ii) is the study by Asano and Ukai [21] of the flow past an obstacle. They prove the existence of stationary distributions which are actually attracting for a certain class of initial distributions.

5.3 Linear Boltzmann and incompressible Stokes equations. Many of the questions raised in the last sections become easier to answer if instead of starting with (*) one starts with the linearized BE for f,

$$\partial_t f + v\cdot\nabla_x f = Lf, \tag{5.14}$$

where

$$Lf = 2Q[m_0, f], \tag{5.15}$$

$m_0 = M(\bar\rho, 0, \bar T; v)$ being the global Maxwellian with density $\bar\rho$ and temperature $\bar T$. Scaling variables like in Section 2 and assuming $N = 2$ we get, instead of (2.4), the equation

$$\frac{\partial F^\varepsilon}{\partial t} + \varepsilon^{-1}v\cdot\nabla_x F^\varepsilon = \varepsilon^{-2}LF^\varepsilon. \tag{5.16}$$

In this case, using the arguments of Section 3, it is quite easy to prove the following

THEOREM 3. *Given a divergence-free field $u_0 \in H_s(\Pi)$, $s \geqq 2$, $\bar\rho > 0$, $\bar T > 0$, equation* (5.16) *has a unique solution $F^\varepsilon(x,t;v) \in L_2[m_0^{-1}(v)\,dv\,dx]$ satisfying, for any finite t,*

$$\|F^\varepsilon(x,t;v) - M(\bar\rho, \varepsilon u(x,t), \bar T; v)\| < C\varepsilon^2, \tag{5.17}$$

whenever $u(x,t)$ *is a solution in* $\mathscr{H}_s(\Pi)$ *of the Stokes equation*:

$$\bar{\rho}\partial_t u = -\nabla p + \eta\Delta u, \tag{5.18a}$$

$$\operatorname{div} u = 0, \tag{5.18b}$$

$$u(x,0) = u_0(x), \tag{5.18c}$$

and η *is determined as in Theorem* 1.

To prove Theorem 3 it is enough to realize that following the lines of Section 2, equations (3.6) are modified as follows:

$$\varepsilon^{-1}: \quad v \cdot \nabla m_0 = 0, \tag{5.19a}$$

$$\varepsilon^{0}: \quad \partial_t m_0 + v \cdot \nabla \varphi_1 = L(f_2 + \varphi_2), \tag{5.19b}$$

$$\varepsilon^{1}: \quad \partial_t \varphi_1 + v \cdot \nabla (f_2 + \varphi_2) = L(f_3 + \varphi_3). \tag{5.19c}$$

From (5.19b) we get that $f_2 = \bar{f}_2 - \varphi_2$, where $\bar{f}_2$ is equal to the right-hand side of (3.27). Therefore, (5.19c) becomes

$$\partial_t \varphi_1 + v \cdot \nabla \bar{f}_2 = L(f_3 + \varphi_3). \tag{5.20}$$

From (5.20), equations (5.18) easily follow. The equation for the remainder is linear and this allows us to control it by the nonpositivity of the operator L on $L_2[m_0^{-1}(v)\, dv\, dx]$.

Appendix

We use the following estimates proved in [2], [7] and [9]:

$$\|\mathscr{K}_\varepsilon f\|_{js} \leqq c\|f\|_{j-1s}, \quad \|\mathscr{K}^* f\|_{js} \leqq c\|f\|_{j-1s} \quad \text{for} \quad j \geqq 1,\ s \geqq 0, \tag{A.1}$$

$$\|\mathscr{K}_\varepsilon f\|_{00} \leqq c|||f|||, \quad \|\mathscr{K}^* f\|_{00} \leqq c|||f|||, \tag{A.2}$$

$$0 < \nu_0 \leqq \nu_\varepsilon(x,v,t) \leqq \nu_1(I + |v|)^\beta, 0 < \nu_0 \leqq \nu^*(v) \leqq \nu_1(1+|v|)^\beta, \quad 0 < \beta \leqq 1, \tag{A.3}$$

$$\left\| \frac{1}{\nu_\varepsilon} \widetilde{\mathscr{L}}_\varepsilon f \right\|_{js} \leqq c\|f\|_{js}, \tag{A.4}$$

$$|\nabla^s \nu_\varepsilon| \leqq c\nu_\varepsilon, \quad \left\| \frac{1}{\nu_\varepsilon} (\nabla^s \mathscr{K}_\varepsilon) f \right\|_{j0} \leqq \varepsilon^s \|f\|_{j+s}. \tag{A.5}$$

In (A.2) the $|||\cdot|||$ denotes L_2-norm in velocities and positions. Estimate (A.4) follows from the bounds (3.43) and (A.5) follow from a direct computation using the fact that the temperature is constant.

We denote by $x + vt$ the translation on the torus Π with velocity v in time t. Of course the Lebesgue measure is invariant under this translation. We write (4.15)–(4.17) in integral form:

$$h(x,v,t) = \int_0^t ds \left\{ \exp\left\{ -\int_s^t ds' \nu_\varepsilon(s') \varepsilon^{-2} \right\} \cdot \left[-\mu\sigma g + \varepsilon^2 \bar{\chi} \mathscr{K}^* h + \widetilde{\mathscr{L}}(\sigma g + h) + \varepsilon^2 b \right] \right\}^{\#}, \tag{A.6}$$

$$g(x,v,t) = \int_0^t ds \left\{ \exp\left\{ -\int_s^t ds' \nu_\varepsilon(s') \varepsilon^{-2} \right\} \cdot \left[\varepsilon^{-2} \mathscr{K}_\varepsilon g + \varepsilon^{-2} \chi \sigma^{-1} \mathscr{K}^* h \right] \right\}^{\#}, \tag{A.7}$$

where the symbol $\{\ \}^{\#}$ means that the functions in braces have to be computed at $(x - (t-s)v\varepsilon^{-1}, v, s)$. Since we compute integral norms in x, the presence of ε^{-1} makes no difference for the invariance of the Lebesgue measure on the torus. Perform the time integral which gives a factor $\varepsilon^2/\nu_\varepsilon$. Using (A.1) (A.4) and choosing γ large enough we get, by (A.6),

$$\|h\|_{j0} \leqq \varepsilon^2 \|g\|_{00} + \varepsilon^4 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{j0}. \tag{A.8}$$

Differentiating (4.16) and writing it in integral form we have

$$\nabla h(x,v,t) = \int_0^t ds \left\{ \exp\left\{ -\int_s^t ds' \nu_\varepsilon(s') \varepsilon^{-2} \right\} \cdot \left[-\nabla(\mu\sigma g) + \varepsilon^{-2}(\nabla \nu_\varepsilon) h + \varepsilon^{-2} \bar{\chi} \nabla(\mathscr{K}^* h) + \nabla\left(\widetilde{\mathscr{L}}_\varepsilon(\sigma g + h) \right) + \varepsilon^2 \nabla b \right] \right\}^{\#}. \tag{A.9}$$

Using (A.8) and (A.1), (A.2), (A.4), (A.5) we get, for γ large enough,

$$\|\nabla h\|_{j0} \leqq \varepsilon^2 \|g\|_{01} + \varepsilon^4 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{j1}. \tag{A.10}$$

By the same argument higher derivatives may be estimated and we get

$$\|h\|_{js} \leqq \varepsilon^2 \|g\|_{0s} + \varepsilon^4 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{js}. \tag{A.11}$$

Now we estimate g. Following [2] we take advantage of the bound (A.1) to reduce the estimate of $\|g\|_{js}$ to an L_2 estimate which follows by the nonpositivity of the operator $\mathscr{L}_\varepsilon$. Namely, denoting by $C(\gamma)$ a constant diverging with γ, we have

$$\begin{aligned} \|g\|_{j0} &\leqq \|\mathscr{K}_\varepsilon g\|_{j0} + \|\chi\sigma^{-1}\mathscr{K}^* h\|_{j0} \leqq \|\mathscr{K}_\varepsilon g\|_{j0} + C(\gamma)\|\mathscr{K}^* h\|_{j0} \\ &\leqq \|g\|_{j-10} + C(\gamma)\|h\|_{j-10} \leqq \|g\|_{j-10} + C(\gamma)\left(\varepsilon^2 \|g\|_{00} + \varepsilon^4 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{j-10}\right) \\ &\leqq \|g\|_{j-10} + \varepsilon^{7/2} \left\| \frac{1}{\nu_\varepsilon} b \right\|_{j-10}, \end{aligned} \tag{A.12}$$

for ε small enough. In the same way, using (A.2) we get

$$\|g\|_{00} \leqq |||g||| + \varepsilon^{7/2} \left\| \frac{1}{\nu_\varepsilon} b \right\|_{00}. \tag{A.13}$$

A recursive argument (Grad hierarchy) then gives

$$\|g\|_{j0} \leqq |||g||| + \varepsilon^{7/2} \left\| \frac{1}{\nu_\varepsilon} b \right\|_{j-10}. \tag{A.14}$$

The L_2 estimate is obtained using integration by part to cancel the streaming operator $\varepsilon^{-1} v \cdot \nabla$, and the nonpositivity of $\mathscr{L}_\varepsilon$ to get

$$\begin{aligned} \partial_t |||g||| &\leqq \varepsilon^{-2} |||\chi\sigma^{-1}\mathscr{K}^* h||| \leqq C(\gamma)\varepsilon^{-2} |||\mathscr{K}^* h||| \leqq C(\gamma)\varepsilon^{-2} |||h||| \leqq C(\gamma)\varepsilon^{-2} \|h\|_{20} \\ &\leqq C(\gamma)\left[\|g\|_{00} + \varepsilon^2 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{20}\right] \leqq C(\gamma)\left[|||g||| + \varepsilon^2 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{20}\right]. \end{aligned} \tag{A.15}$$

Therefore, by the Gronwall inequality we have

$$|||g||| \leqq \exp\{C(\gamma) t_0\} \varepsilon^2 \left\| \frac{1}{\nu_\varepsilon} b \right\|_{20} \leqq \varepsilon^{3/2} \left\| \frac{1}{\nu_\varepsilon} b \right\|_{20}, \tag{A.16}$$

if ε is small enough. Therefore,

$$\|g\|_{j0} \leqq \varepsilon^{3/2}\left\|\frac{1}{\nu_\varepsilon}b\right\|_{j-10}, \qquad j \geqq 3. \tag{A.17}$$

The equation for the derivatives of g can be studied in the same way. Higher norms of b appear in consequence of (A.5).

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