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PERCOLATION IN STRONGLY CORRELATED SYSTEMS

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We investigate the threshold percolation density, p_c, in strongly correlated lattice systems. The probability distribution of the system is the stationary state for a combined Kawasaki and Glauber dynamics; the voter model with flips. When the Glauber rate goes to zero, the pair correlation function of the system, at any density, decays in three dimensions as r^{-1} (and does not decay at all in two dimensions). Using Monte Carlo calculations and finite size scaling we obtain information about p_c as a function of the Glauber rate. Percolation in Gaussian fields on a lattice which have similar slow decay is also discussed.

1. Introduction

Percolation is a problem to which Piet Kasteleyn contributed some very significant ideas. In particular he showed how an interacting spin system, the q-component Potts model, reduces, in the limit $q \to 1$, to the percolation of independent (Bernoulli) bonds on the lattice[1]). In this note we shall consider site percolation for a strongly "interacting" particle system.

We put quotation marks around interacting because our system is not an equilibrium system with specified interaction U. Such systems have states or probability measures specified by a Gibbs ensemble $\mu \sim \exp[-\beta U]$ and strong interactions βU then give rise to strong correlations in μ. The probability measure of the system we shall consider here cannot be written in the Gibbs form for any reasonable U. It is characterized entirely by being stationary with respect to a certain type of stochastic time evolution.

Such systems go, in the probabilistic mathematical literature, under the name of interacting particle systems[2]). Unlike the usual Glauber or Kawasaki dynamics however the spin flip rates (we shall use spin and particle language

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interchangeably) we consider do not, in general, satisfy detailed balance with respect to the stationary probability measure. In fact the general global structure of the stationary measure is largely unknown. What is known however quite explicitly are the pair and other correlation functions which, for certain dynamics, corresponding to the voter model[2]), decay very slowly, i.e. like r^{d-2} in dimension $d \geqslant 3$ at any density p, $0<p<1$.

This fact prompted our curiousity about the value of the percolation threshold p_c in such a model – might it be arbitrarily small? Unfortunately we do not have a definite answer to this question. Numerical work, using finite size scaling ideas borrowed from equilibrium statistical mechanics, strongly suggests however that the answer is no, with $p_c \sim 0.16$ in $d=3$. This is rather close to the numerically estimated lower limit of $p_c(\beta)$ in the equilibrium Ising model with nearest neighbor ferromagnetic interactions (where however the correlations are known to decay exponentially) and other interacting systems[3]). Whether this is coincidental or deep (or even true) awaits some deeper insights than we are able to provide here. (Hopefully Kasteleyn will think about this now that he has more leisure time.)

Before describing the voter model and our analysis of its p_c we formulate the same question for a "simpler", equilibrium based, model. We believe, but again cannot prove, that the answers are essentially the same as for our model. Some analysis of this latter model is now in progress[4]).

2. Percolation for Gaussian fields on a lattice

Let $x \in \mathbb{Z}^d$ designate the sites of a d-dimensional simple cubic lattice and $\phi(x) \in \mathrm{R}$ be "displacements" of a Gaussian spin variable with n.n. interaction potential,

$$U = \tfrac{1}{2} J \sum_{\langle x,y \rangle} [\phi(x) - \phi(y)]^2 + \tfrac{1}{2} K \sum \phi^2(x)\,, \quad J>0,\ K \geqslant 0\,. \tag{2.1}$$

For $K>0$ the infinite volume Gibbs measure at reciprocal temperature β, μ_K is well defined – it is translation invariant and independent of boundary conditions. Define now the occupation variables $\rho_b(x)$,

$$\rho_b(x:\phi) = \begin{cases} 1\,, & \text{if } \phi(x) \geqslant b\,, \\ 0\,, & \text{if } \phi(x) < b\,, \end{cases} \tag{2.2}$$

and let $p = \langle \rho_b(x) \rangle$. We will call $\hat{\mu}_{K,p}$ the measure induced by μ_K on the variables $\{\rho_b(x)\}$. (By rescaling we set $\beta=1$, $J=1$.)

We ask now for the probability $W(K,p)$ that the origin, $x = \boldsymbol{0}$, belongs to an infinite cluster of occupied sites. It is easy to show that the measure $\hat{\mu}$ satisfies the Fortuin, Kasteleyn, Ginibre (FKG)[5]) inequalities from which it follows directly that $W(K, p)$ is a monotone non-decreasing function of p. We call $p_c(K)$ the percolation threshold: $p_c(K) = \min p$ such that $W(K, p) > 0$ for all $p > p_c(K)$. The question now is: how does $p_c(K)$ behave as $K \to 0$?

The answer will certainly depend on d. For $d = 1$ general arguments give $p_c(K) = 1$ all K. For $d = 2$ the fluctuations in the ϕ variables become unbounded as $K \to 0$. This corresponds to $\hat{\mu}_{K,p} \to p\delta_1 + (1-p)\delta_0$, as $K \to 0$. Here δ_1 (δ_0) are the measures concentrated on all sites being occupied (empty). We can argue however that for any $K > 0$, $p_c(K) \geq \frac{1}{2}$ for $d = 2$ since by symmetry if $p_c(K) = \frac{1}{2} - \delta$, then there will be percolation of "holes" for $p < \frac{1}{2} + \delta$ and we expect that there cannot be simultaneously infinite clusters of both particles and holes in $d = 2$[3]). Hence $\delta < 0$ for any $K > 0$ but we expect that $p_c(K) \to \frac{1}{2}$ as $K \to 0$. The intriguing question is: What happens for $d \geq 3$? Here the state $\hat{\mu}_{K,p}$ approaches as $K \to 0$ a well defined limit with correlations between sites x and y decaying as $|x-y|^{2-d}$ [4,6]). It is possible that $p_c(K) \to 0$ as $K \to 0$? The answer to this question requires information about the geometrical structure of the level sets in the harmonic system with $K = 0$: Do the sets $\{x : \phi(x) > b\}$ form a percolating cluster for b very large?

An argument for the possibility of such behavior comes from the slow decay of the truncated pair correlation, $\langle \rho_b(0)\rho_b(x)\rangle - p^2 \sim |x|^{-(d-2)}$, $\langle \rho_b(x)\rangle = p$. This implies that if the origin is occupied then the expected number of occupied sites in a sphere of radius R centered on the origin grows like R^2 independent of p as $p \to 0$. The occupied sites, i.e. those for which $\phi(x) \geq b$, thus form a "two dimensional set" in $\mathbb{Z}^d$ even in the limit of zero density. What is unclear however is the connectivity of this set, $\{\phi(x) \geq b\}$ in a typical configuration – does it form a "ramified" percolating cluster or does it consist of compact islands which have a tendency to cluster because of the general elevation of the $\phi(x)$ in a given area. It is here where the fact that the state $\hat{\mu}_{K,p}$ is not Gibbsian with any finite range interaction is crucial. The islands of occupied sites separated by empty sites are not independent as they would be in a Gibbs state for an Ising system with short range spin interactions.

3. The voter model with independent flips

The voter model is well known (and beloved) in the mathematics literature devoted to stochastic evolutions of infinite particle systems on a lattice – for a comprehensive review of the whole subject see the book by Liggett[2]). In words the model is described by saying that each lattice site is occupied by a voter who has to decide between voting yes or no ($\rho(x) = 1$ or 0) on a certain issue.

He does this by looking (at random times) at one of his $2d$ neighbors (chosen at random) and adopting this neighbor's position on the issue. Using spin language $\boldsymbol{\sigma} = \{\sigma(x)\}$, $\sigma(x) = \pm 1$, $x \in \mathbb{Z}^d$, the time evolution of this voter model is specified by giving the rate $C_v(x, \boldsymbol{\sigma})$ for a spin flip at the site x when the spin configuration is $\boldsymbol{\sigma}$,

$$C_v(x, \boldsymbol{\sigma}) = \frac{1}{\tau}\left[1 - \frac{1}{2d}\,\sigma(x) \sum_{|y-x|=1} \sigma(y)\right]. \tag{3.1}$$

Here τ merely sets the unit of time so we can take it equal to one.

It is clear from the description of the model as well as from eq. (3.1) that if we consider this system in a finite region of the lattice, $\Lambda \subset \mathbb{Z}^d$, with periodic boundary conditions (so that each site has the same number of neighbors and (3.1) does not have to be changed) then the only stationary states are that of a consensus $\sigma(x) = 1$ or $\sigma(x) = -1$ all $x \in \Lambda$, everybody up or everybody down. This is a simple consequence of the fact that these configurations are trapping, once reached there is no way out, and in a finite system they will always be reached, whatever the initial state, via some fluctuations. Thus starting from any initial state $\mu_\Lambda^{(v)}(\boldsymbol{\sigma}, 0)$,

$$\lim_{t\to\infty} \mu_\Lambda^{(v)}(\boldsymbol{\sigma}, t) = p\delta_1^{(\Lambda)} + (1-p)\delta_{-1}^{(\Lambda)}, \tag{3.2}$$

where p is the expected fraction of sites with spins up at $t = 0$,

$$p = \tfrac{1}{2}\left(1 + \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \langle\sigma(x)\rangle\;(t=0)\right). \tag{3.3}$$

This follows from the fact, easy to verify from (3.1) that

$$\begin{aligned}\frac{\mathrm{d}\langle\sigma(x)\rangle(t)}{\mathrm{d}t} &= -2\left\{\langle\sigma(x)\rangle - \frac{1}{2d}\sum_{|y-x|=1}\langle\sigma(y)\rangle\right\}\\ &\equiv \Delta_x\langle\sigma\rangle\end{aligned} \tag{3.4}$$

so that $\sum_{x\in\Lambda}\langle\sigma(x)\rangle$ is independent of t.

This leaves open the question concerning the stationary state of the voter model on the infinite lattice. We clearly still have that δ_1 and δ_{-1} are stationary states (and so is any linear combination of them) but are there any other "extremal" stationary states? (i.e. states which are not linear combinations of other states or equivalently states in which there is some decay of correlations). The answer[2]) is that for $d = 1, 2$ there are only consensus states while for $d \geqslant 3$ there are unique extremal states for every density of up spins $p = \frac{1}{2}$ (1+

$\langle \sigma(x) \rangle$). The expectations $\langle \quad \rangle$ here is in the stationary state, which are all translation invariant. The correlations in these states decay like $(\text{distance})^{2-d}$, i.e. putting $\rho(x) = \frac{1}{2}(1 + \sigma(x))$, $\langle \rho(x) \rangle = p$,

$$\langle \rho(x)\rho(y) \rangle - p^2 = p(1-p)G_d(x-y) \tag{3.5}$$

where $G_d(x)$ is the probability for a random walker, starting at $x \in \mathbb{Z}^d$, to hit the origin (before disappearing to infinity). As is well known $G_d(x) \sim |x|^{d-2}$, $d \geqslant 3$.

Our question concerning p_c in the voter model in $d \geqslant 3$ is now well defined and the analogy with the harmonic crystal for $K = 0$ is clear. In order to study it numerically we first introduce a modification of the model which makes the states non-trivial also for finite Λ. We do this by considering voters who occasionally change their opinions spontaneously, i.e. independently of what their neighbors are doing. In terms of flip rates we set

$$C(x, \boldsymbol{\sigma}) = (1-\lambda)C_v(x, \boldsymbol{\sigma}) + \lambda[1 + (1-2p)\sigma(x)]\,, \tag{3.6}$$

where $0 \leqslant p$, $\lambda \leqslant 1$, and C_v is given in (3.1). Here p sets the desired final density and λ plays a role very similar to the mass term K ($\lambda \sim K/(1+K)$) in the harmonic case. For $\lambda = 0$ we have the regular voter model while for $\lambda = 1$ we are just in the independent site model. Our numerical studies of $p_c(\lambda)$ for this model are described in the next section. They are based on finite size scaling ideas which appear to work well for equilibrium systems, i.e. those whose measures are of the form $\mu_\Lambda \sim \exp[-\beta U_\Lambda(\boldsymbol{\sigma})]$ with U short range. These measures are, for different Λ, simply related to each other – since U_Λ varies in a simple way with Λ. This is no longer true for the measures we are considering; the "effective" potential, defined by taking the logarithm of the measure, will be very strongly dependent on Λ. Nevertheless the procedure seems to work yielding "critical exponents" in good agreement with those obtained for equilibrium. This is encouraging but not clearly understood at the present time.

4. Numerical study of the voter model with independent flips

We now present our numerical results for the percolation properties of the model (3.6). The method we used in our calculation is a simple generalization of what has been done in ref. 7 for the random site percolation problem. Using a Monte Carlo procedure we first generate – according to the dynamics (3.6) – configurations of squares or cubes of linear size N ($|\Lambda| = N^d$) with periodic boundary conditions. Expectation values in the stationary state are then obtained by taking time averages after the system has settled down. For each value of λ and p we estimate

$$\Gamma_N(\lambda, p) = \sum_s s^2 n_s \,, \tag{4.1}$$

where n_s is the mean number of clusters of s sites with spin up in the system. In the case $\lambda = 1$ which corresponds to random percolation, the finite size scaling theory[8]) shows that Γ_N has the following form for N large and $(p - p_c) \ll 1$:

$$\Gamma_N \sim N^{(\gamma/\nu + d)} F(N^{1/\nu}(p - p_c)) \,. \tag{4.2}$$

In this expresion γ and ν are critical exponents of the random percolation problem[9]). They can be defined in different ways. For example ν gives the divergence of the correlation length near the threshold p_c by

$$\frac{\sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s} \sim |p - p_c|^{-2\nu} \,. \tag{4.3}$$

(In this formula, R_s^2 is the mean squared distance between two points belonging to a cluster of size s.) In a similar way one has

$$\sum_s s^2 n_s \sim |p - p_c|^{-\gamma} \,. \tag{4.4}$$

These exponents can also be related to the geometrical properties of the clusters[9]). Once γ and ν are known, all the other exponents follow by the scaling relations.

In our calculation we have assumed that Γ_N takes a form similar to (4.2) whenever $\lambda < 1$. The threshold is now a funcion of λ, $p_c(\lambda)$. The exponents can a priori depend on λ although universality considerations suggest that they may not – at least one can hope.

For a fixed value of λ, one can obtain estimates of the critical properties by comparing the quantities Γ_N for different sizes. For example, using (4.2) the ratios[7])

$$R_N = \frac{\Gamma_{2N}}{\Gamma_N} \tag{4.5}$$

should be independent of N (for N large) if $p = p_c$. We have used this to get $p_c(\lambda)$ by drawing the ratio R_N versus p for different sizes N and looked for the points where the curves cross. Once $p_c(\lambda)$ is obtained, the slopes of the curves give $\nu(\lambda)$ while the values of the ratios at the threshold give $[\gamma/\nu](\lambda)$.

Let us first consider the two dimensional case. We give some details for a typical value $\lambda = 0.40$. A similar analysis has been carried out for several values of λ between 0 and 1. The case $\lambda = 1$ where one recovers the random site percolation has already been studied in detail in ref. 7. Configurations of squares with sizes $N \leq 32$ have been generated with the dynamics (3.6) and the random number generator already used in ref. 7. For configurations separated by a time interval T (in Monte Carlo step per site units) we have calculated the sizes of all the clusters in the sample using a known algorithm[7]). For $\lambda = 0.40$ a value of T around $T = 30$ has given quasi independent configurations. The quantities have been averaged over 10^4 samples for $N \leq 16$ and 5×10^3 samples for $N = 32$.

To evaluate the error bars on our estimation of Γ_N we first divided our samples into 10 subsets. We then calculated the averages for each subset and estimated the variance assuming that the distribution of the subset averages was Gaussian[7]). The R_N calculated in this way are represented versus p in fig. 1.

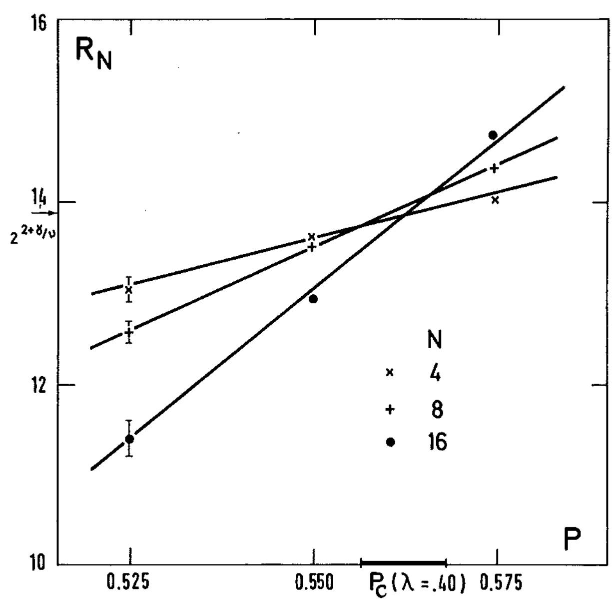

Fig. 1. Plot of the ratio R_N (4.5) versus p for $\lambda = 0.40$ and $d = 2$. (The error bars are indicated on the left points.) The intersections of the curves give $p_c(\lambda = 0.40) = 0.562 \pm 0.006$. The values of R_N at p_c and the slopes fo the curves give estimates of the exponents γ and ν which are in good agreement with the values of the random percolation problem (4.8), (4.9).

The finite size scaling form (4.2) tells us that these curves should intersect at $p_c(\lambda = 0.40)$. Several intersections are in fact observed due to statistical errors and to corrections to scaling[7]). From this figure we deduce

$$p_c(\lambda = 0.40) = 0.562 \pm 0.006 . \tag{4.6}$$

At this value of the threshold one can estimate ν by

$$\nu^{-1}(\lambda = 0.40) = \frac{\log \dfrac{dR_{2N}}{dp} \Big/ \dfrac{dR_N}{dp}}{\log 2} . \tag{4.7}$$

For $N = 8$ this gives

$$\nu(\lambda = 0.40) = 1.2 \pm 0.3 . \tag{4.8}$$

This estimate is not very precise but it is in good agreement with the value $\frac{4}{3}$ [9]) of the random percolation problem. From (4.5) the ratio $R_N(p_c)$ should be equal to $2^{(2+\gamma/\nu)}$ for N large enough. Fig. 1 gives

$$\frac{\gamma}{\nu}(\lambda = 0.40) = 1.80 \pm 0.05 . \tag{4.9}$$

This is more precise than (4.8) and in good agreement with the value $\frac{43}{24} = 1.792$ [9]) for random percolation.

A similar analysis was carried out for several values of λ. We obtained in this way the results of fig. 2. Of particular interest is the limiting value $\lim_{\lambda \to 0^+} p_c(\lambda)$. When λ goes to zero, the simulation becomes more difficult because the correlation time between configurations increases rapidly and also because the results seem to converge more slowly with increasing N. We have thus been limited to $\lambda \geq 0.1$. The dotted line of fig. 2, which is essentially an eye's guide, suggests

$$\lim_{\lambda \to 0^+} p_c(\lambda) = \tfrac{1}{2} , \tag{4.10}$$

a result which agrees with the expectations of section 2.

We now turn to the more interesting case of three dimensions. When $\lambda = 1$ one recovers the 3d random site percolation. We shall first present some results for this case which has not been studied in ref. 7. With 5×10^3 samples for cubes of size $N \leq 12$ and 10^3 samples for $N = 24$ we have obtained the results given in fig. 3 where the ratio R_N is plotted versus p. One can see that the results converge more slowly with increasing N than in the 2d case (compare

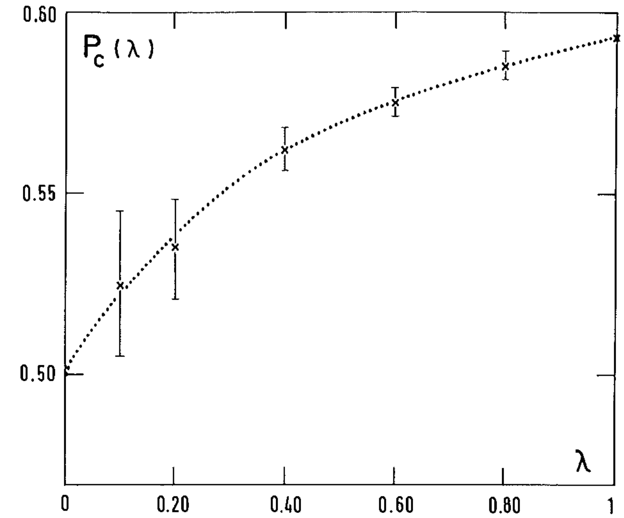

Fig. 2. Values of the threshold $p_c(\lambda)$ for several values of λ between 0 and 1 in the case $d=2$. The limiting value $\lim_{\lambda\to 0^+} p_c(\lambda)$ is compatible with $\frac{1}{2}$.

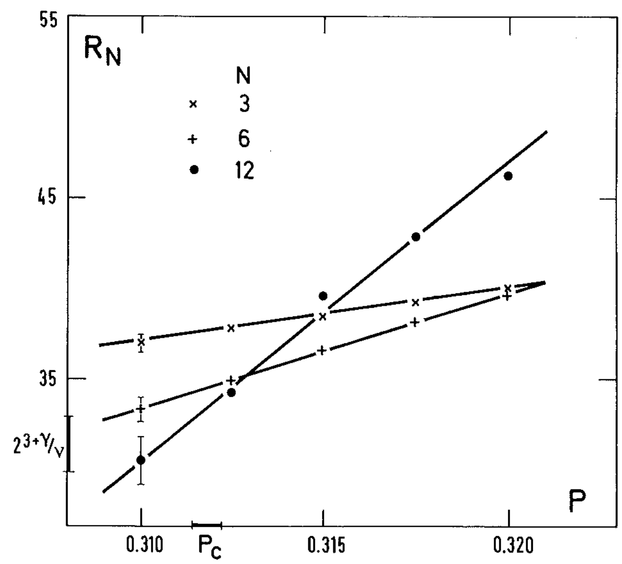

Fig. 3. Plot of R_N versus p in the case $d=3$, $\lambda=1$ (random site percolation). The indicated estimate of p_c is the threshold evaluated in several other works[7,10]). The intersections of our curves go rapidly to this value when N increases. The values of R_N at p_c suggest $2^{3+\gamma/\nu} \cong 30$ i.e. $\gamma/\nu \cong 1.90$. This agrees with ref. 7 but seems slightly too low when compared to ref. 10.

with fig. 1A, 1B of ref. 7). The intersection 6/12 is in good agreement with the value $p_c = 0.3118 \pm 0.0004$ given in refs. 7, 10. The slopes of the curves at p_c give, as in the 2d case, estimates of the exponent ν. By comparing the slopes for $N = 6$ and $N = 12$ one gets

$$\nu = 0.80 \pm 0.2\,, \tag{4.11}$$

a value not very precise but compatible with the result of more complete calculations $\nu = 0.90 \pm 0.02$ [9,10]). The value of R_N at p_c should be equal to $2^{3+\gamma/\nu}$ for N large enough. From fig. 3 we obtain the estimate

$$\gamma/\nu = 1.90 \pm 0.05\,. \tag{4.12}$$

This agrees with ref. 7 but seems slightly too low when compared to other works[10]).

We then studied with the same method several values of λ between 0 and 1. For example when $\lambda = 0.60$ we obtained the results of fig. 4. In this case the quantities have been averaged over 10^3 samples for $N \leqslant 12$ and 5×10^2 samples

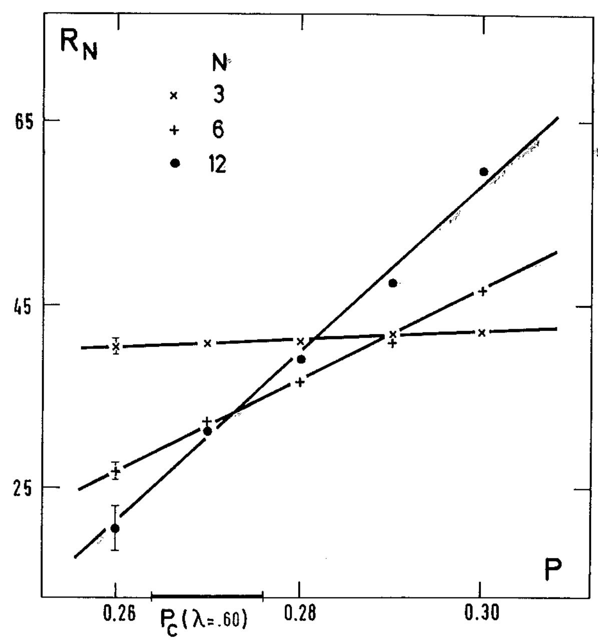

Fig. 4. The same as in fig. 3 for $\lambda = 0.60$. The curves are similar to those of fig. 3, the threshold being lowered to $p_c(\lambda = 0.60) = 0.27 \pm 0.006$.

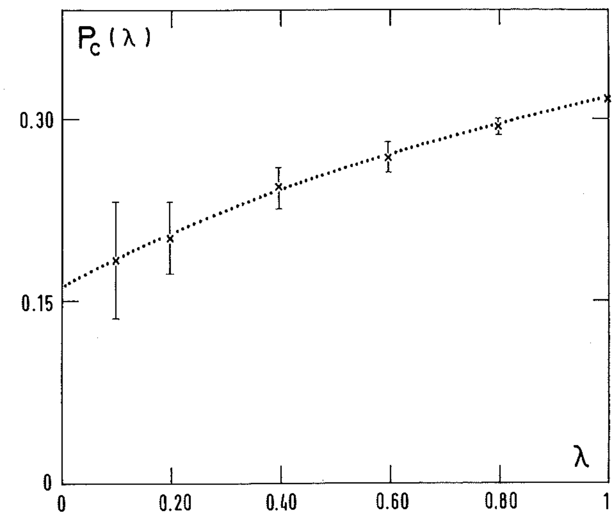

Fig. 5. Values of the threshold $p_c(\lambda)$ for several values of λ between 0 and 1 in the case $d=3$. From these values one deduces $\lim_{\lambda\to 0^+} p_c(\lambda) \cong 0.16$.

for $N=24$. Independent configurations were separated by a value of T around 15 Monte Carlo steps per site. From fig. 4 we estimate

$$p_c(\lambda=0.60)=0.27\pm 0.006\,. \tag{4.13}$$

The slopes and the values of R_N at this estimate of the threshold are in good agreement with (4.11), (4.12).

The complete curve $p_c(\lambda)$ obtained in this way is given in fig. 5. It suggests that in the three dimensional case $p_c(\lambda)$ has the limiting value

$$\lim_{\lambda\to 0^+} p_c(\lambda)\cong 0.16\,. \tag{4.14}$$

5. Concluding remarks

These calculations indicate that the finite size scaling methods which work well for random percolation[7]) can be extended to the study of the voter model with independent flips.

Although the region $\lambda\to 0$ is difficult to explore numerically, our results suggest a non-zero limiting value of the threshold $p_c(\lambda)$ in the three dimensional case. Our results also give estimates of the critical exponents in agreement with the random percolation values, suggesting thus "universality" for this case also.

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