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Asymptotics of Particle Trajectories in Infinite One Dimensional Systems with Collisions*

by

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Abstract

We investigate the asymptotic behavior of the trajectories of a tagged particle (tp) in an infinite one dimensional system of point particles. The particles move independently when not in contact: the only interactions between them are (generalized) elastic collisions which prevent crossings. This is achieved by relabeling the independent trajectories when they cross. When these trajectories are differentiable, as in particles with velocities undergoing Ornstein-Uhlenbeck processes, collisions correspond to exchange of velocities.

We prove very generally that the suitably scaled tp trajectory converges (weakly) to a simple Gaussian process. This extends the results of Spitzer for Newtonian particles to very general non-crossing processes. The proof is based on considering the simpler process which counts the crossings of the origin by the independent trajectories.

I. Introduction

We consider an infinite system of identical point particles uniformly distributed on the line. Suppose that the positions of the points change with time and that $x_i(t)$, $i \in \mathbb{Z}$, denotes the position of the i-th point at time t. Harris [1] introduced a generalized "elastic collision" process defined on the path space of this infinite particle system: Let $x_i(t)$, $i = 1,2$ be the independent motion of a pair of initially adjacent points with $x_1(0) < x_2(0)$. Then in the absence of the other particles the collision path of particle 1 is $y_1(t) = \min(x_1(t), x_2(t))$ and that of 2 is $y_2(t) = \max(x_1(t), x_2(t))$, i.e. the collisions do not change the trajectories of the system but preserve the order of particles.

In this paper we consider systems with quite general trajectories $x_i(t)$, $i \in \mathbb{Z}$, undergoing collisions of the Harris type, and prove central limit theorems and generalized invariance principles for the suitably normalized trajectory of a tagged particle. In fact we are able to establish asymptotically normal behavior for the tagged particle collision process (invariance principle) for quite general motions for which the free processes are i.i.d. and are independent of the initial positions $x_i(0)$, $i \in \mathbb{Z}$. Our work generalizes and strengthens the results of Harris [1] (colliding Brownian motions), Spitzer [2] (Newtonian particles) and Gisselquist [3] (colliding stable processes); it also covers the more recent work of Norio [6] (colliding fractional Brownian motions).

The key to our analysis is the process $n(t)$, the algebraic number of crossings of the origin up to time t. In section III we show that the tagged particle process $y(t)$ is asymptotically the same as $\rho^{-1} n(t)$, where ρ is the density of particles. Then, in sections IV and V, we analyze the asymptotics of $n(t)$, which in section VI are translated to the process $y(t)$ using the results of III. A partial summary of our results is given in section VII. Section VIII ends this paper with a collection of examples.

II. The model

The free motion

Let $(\underline{X}, \underline{F}_X, P_X)$ denote a Poisson point process on $\mathbb{R}$ of density ρ, with $X = \{x_i\}_{i \in n} \in \underline{X}$ representing the initial configuration of our system of particles. We let each particle move independently according to the same stochastic process $(\Xi, \underline{F}_\xi, P_\xi)$, where $\xi \in \Xi$ is a function $\xi : \mathbb{R}^+ \to \mathbb{R}$ with

$\xi(0) = 0$, concerning which we shall assume the following:

$$\mathbb{E}(|\xi(t)|) = \int |\xi(t)|\, dP_\xi < \infty \quad \text{for all} \quad t \in \mathbb{R}^+ \tag{2.1a}$$

$$\mathbb{E}(\xi(t)) = 0 \quad \text{for all} \quad t \in \mathbb{R}^+ \tag{2.1b}$$

$$\mathbb{E}(|\xi(t)|) \to \infty \qquad t \to \infty \tag{2.1c}$$

(We also require the technical assumption, automatically satisfied if the paths of ξ are in C or more generally in D (with $\underline{F}_\xi$ chosen suitably), that $t \to \mathbb{E}(|\xi(t)|)$ is measurable, used only to derive (4.7).) Thus the evolution of our system is given by the collection $(X, \{\xi_i\}_{i\in\mathbb{Z}})$, where the ξ_i are i.i.d. random variables also independent of X, each particle $x_i \in X$ undergoing the stochastic process

$$x_i(t) = x_i(0) + \xi_i(t), \qquad t \in \mathbb{R}^+ \tag{2.2}$$

with $\{x_i(0)\}_{i\in\mathbb{Z}}$ being the initial configuration X labeled so that

$$\ldots \le x_{-2}(0) \le x_{-1}(0) \le x_0(0) \le 0 < x_1(0) \le x_2(0) \ldots$$

We find it convenient to represent this evolution via a Poisson system $(\Omega, \underline{F}, \underline{P})$ built over $\mathbb{R} \times \Xi$ with intensity measure $d\mu = \rho\, dx\, dP_\xi$, where dx denotes the Lebesgue measure on $\mathbb{R}$. This means that for each $A \in \underline{F}$ with $\mu(A) < \infty$, $N_A(\omega)$, the number of points of ω in A, is a Poisson random variable with mean $\mu(A)$. $\mathbb{R} \times \Xi$ is naturally equipped with a family T_t, $t \in \mathbb{R}^+$, of mappings

$$T_t : \mathbb{R} \times \Xi \to \mathbb{R} \times \Xi$$

$$(x, \xi) \to (x + \xi(t), \theta_t \xi)$$

where $(\theta_t\xi)(s) = \xi(s+t)$,*

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* We assume that $\theta_t \Xi \subset \Xi$ for all $t \in \mathbb{R}^+$.

inducing a map $\underline{T}_t : \Omega \to \Omega$

$$\omega = \{(x,\xi)_i\}_{i\in Z} \to \{T_t[(x,\xi)_i]\}_{i\in\mathbb{Z}} \equiv \omega_t .$$

We identify each point $\omega \in \Omega$ with $(\{x_i(0)\}_{i\in Z}, \{\xi_i\}_{i\in Z})$ in the obvious way. Using this representation we immediately obtain that the distribution of points on $\mathbb{R}$, initially Poisson with density ρ, remains so for all times under the evolution (2.2) (see also [4]). In fact we obtain more. If $\xi(t)$ has stationary increments (so that $\theta(t)$ preserves P_ξ) then $\underline{T}_t$ preserves $\underline{P}$, since T_t preserves μ. More generally, $\underline{P} \cdot \underline{T}_t^{-1} \equiv \underline{P}_t$ is Poisson with intensity measure

$$d\mu^t = \rho\, dx\, dP_\xi^t , \quad \text{where} \quad P_\xi^t \equiv P_\xi \cdot \theta^{-1}(t) . \tag{2.3}$$

The collision process

Now we describe the evolution when "elastic collisions" are taken into account. We begin with an informal description. We may visualize the elastic collision motion by imagining that the particles at time t = 0 are all coloured differently, say the 0-th particle red, the -1-th particle blue and the +1-th particle green. Then at all later times the blue particle will be the left nearest neighbor and the green the right nearest neighbor of the red particle, i.e. the particle order is preserved. Although this description is not completely precise, it suggests the following definition.

Define (cf. (2.2))

$$N^+(t) = |\{i|\ i \le 0 ,\quad x_i(t) > 0\}| \tag{2.4a}$$

and

$$N^{-}(t) = |\{i|\ i > 0, \quad x_i(t) \leq 0\}| \tag{2.4b}$$

and define

$$n(t) = N^{+}(t) - N^{-}(t) . \tag{2.5}$$

We interpret n(t) as the algebraic number of crossings of the origin, which in fact it is if the particles have velocities, i.e. if $\xi(t)$ is P_ξ a.s. differentiable. Let x_i^t, $i \in \mathbb{Z}$, denote the positions of the particles at time t labeled in their natural order with respect to the origin, i.e.

$$\ldots \leq x_{-2}^t \leq x_{-1}^t \leq x_0^t \leq 0 < x_1^t \leq x_2^t \ldots$$

We define the position of the test particle (the particle which at time t = 0 is at $x_0(0)$) by

$$y(t) = x_{n(t)}^t \tag{2.6}$$

A moment of reflection will convince the reader of the appropriateness of this definition. See however [1,3].

Remark: Though in this paper we are interested only in the motion of the test particle, once this is defined we easily obtain a precise definition for the elastic collision motion of the entire system, since the order of particles is preserved and the unlabeled system evolves as described earlier. This prescription agrees with that of Harris [1]. $\triangleright$

Note that n(t) is a simple random variable, since $N^{+}(t)$ and $N^{-}(t)$ in (2.4a,b) are i.i.d. Poisson random variables: By definition

$$N^{+}(t) = N_{\underline{A}^{+}}(t)$$

and

$$N^{-}(t) = N_{\underline{A}^{-}}(t)$$

where

$$\underline{A}^+(t) = \{\omega = (x,\xi) \mid x \leq 0 \text{ and } \xi(t) + x > 0\} ,$$

and

$$\underline{A}^-(t) = \{\omega = (x,\xi) \mid x > 0 \text{ and } \xi(t) + x \leq 0\} .$$

Therefore

$$\mathbb{E}(N^-(t)) = \mathbb{E}(N^+(t)) = \mu(\underline{A}^-(t)) = \mu(\underline{A}^+(t)) =$$

$$= \int_{-\infty}^{0} \rho \, dx \, dP_\xi \, \mathbb{1}(\xi(t) > -x) = \rho \int (\xi(t) \vee 0) dP_\xi =$$

$$= \frac{\rho}{2} \mathbb{E}(|\xi(t)|) \tag{2.7}$$

by (2.1a) and (2.1b); $\mathbb{1}(\cdot)$ is the indicator function of the set $(\cdot)$. Since $A^-(t) \cap A^+(t) = \phi$, $N^+(t)$ and $N^-(t)$ are i.i.d..

Therefore

$$\mathbb{E}(n(t)^2) = \rho \, \mathbb{E}(|\xi(t)|) . \tag{2.8}$$

Since at time t the particles are Poisson distributed, we expect by the law of large numbers that $x_n^t/n \sim \rho^{-1}$ and hence by (2.6)

$$y(t) \sim \rho^{-1} n(t) \tag{2.9}$$

for n(t) large. By (2.1c) we expect n(t) to be large for t large, so that by (2.8) the asymptotics of $(y(t))_{t\in[0,\infty)}$ should be given by the asymptotics of the process $(\rho^{-1}n(t))_{t\in[0,\infty)}$. We now give a precise formulation of (2.9).

III. Approximation of tp motion by the crossing process

(3.1) Proposition:

(i) Suppose that for $t \in [0,T]$, $T < \infty$, there exists a function $\phi(A)$ such that $\phi(A) \to \infty$ as $A \to \infty$ and

$$\lim_{x\to\infty} \overline{\lim_{A\to\infty}} \mathbb{P}\left(x \le \frac{|n(At)|}{\sqrt{\phi(A)}}\right) = 0 \quad . \tag{3.2}$$

Then for each $t > 0$

$$\frac{y(At)}{\sqrt{\phi(A)}} - \rho^{-1} \frac{n(At)}{\sqrt{\phi(A)}} \xrightarrow[A\to\infty]{} 0 \tag{3.3}$$

in probability.

(ii) Suppose there exists a function $\phi(A)$ and $\beta > 0$ such that

$$\lim_{A\to\infty} \frac{A^{\beta}}{\phi(A)} = 0 \tag{3.4}$$

$$\lim_{x\to\infty} \overline{\lim_{A\to\infty}} \mathbb{P}\left\{x \le \sup_{t\in[0,T]} \left|\frac{n(At)}{\sqrt{\phi(A)}}\right|\right\} = 0 \tag{3.5}$$

for any $T < \infty$, and

$$\sup_{t\in \mathbb{R}^+} \mathbb{E}\left(\sup_{s\in[t,t+1]} |\xi(s) - \xi(t)|\right) < \infty \quad . \tag{3.6}$$

Then the rescaled difference process

$$\frac{y(At) - \rho^{-1} n(At)}{\sqrt{\phi(A)}} \xrightarrow[A\to\infty]{} 0$$

in "distribution", i.e.

$$\lim_{A\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left|\frac{y(At)}{\sqrt{\phi(A)}} - \rho^{-1} \frac{n(At)}{\sqrt{\phi(A)}}\right| > \delta\right) = 0 \quad . \qquad \triangleright \tag{3.7}$$

Remark: If the paths of ξ are highly irregular, then $\sup |\xi(t) - \xi(s)|$ and $\sup |n(t)|$ need not be measurable. In cases of dubious measurability.

"measure" should be understood as "outer measure" and "expectation" as "outer expectation"

$$\overline{\mathbb{E}}(f) = \inf_{\substack{g \geq f \\ g \text{ measurable}}} \mathbb{E}(g) .$$

Note that if ξ has paths in D and (3.6) is satisfied, then $\sup |n(t)|$ is measurable (see Remarks at the end of section VI).

Proof of Proposition (3.1): We first prove (i). The key is the following

(3.8) Lemma:

Given any $\varepsilon > 0$ and $\eta > 0$ there exists $n_0 > 0$ such that for any $t > 0$

$$\mathbb{P}(|y(t) - \rho^{-1} n(t)| < n_\varepsilon(t)) > 1 - \eta , \tag{3.9}$$

where

$$n_\varepsilon(t) = \begin{cases} \varepsilon|n(t)| & \text{for } |n(t)| \geq n_0 \\ 2n_0 & \text{for } |n(t)| < n_0 . \end{cases} \tag{3.10}$$

▷

Proof: We assume for notational convenience that $\rho = 1$. Let

$$G^{\varepsilon,n_0} = \{\omega \mid |x_n(0) - n| < \varepsilon|n| \quad \text{for all } |n| \geq n_0\} . \tag{3.11}$$

By the law of large numbers

$$\frac{x_n(0)}{n} \to 1 \quad \underline{P} \text{ a.s.} , \tag{3.12}$$

i.e.

$$\lim_{n_0 \to \infty} \underline{P}(G^{\varepsilon,n_0}) = 1 . \tag{3.13}$$

Therefore, by the invariance of the Poisson measure on $\mathbb{R}$, given $\eta > 0$ there exists $n_0 > 0$ such that

$$\underline{P}(\{\omega_t \in G_t^{\varepsilon,n_0}\}) > 1 - \eta \quad .$$

But for $\omega_t \in G_t^{\varepsilon,n_0}$

$$|x^t_{n(t)} - n(t)| < \varepsilon|n(t)| \qquad \text{for } |n(t)| \geq n_0$$

while

$$|x^t_{n(t)} - n(t)| < 2\,n_0 \qquad \text{for } |n(t)| \leq n_0 \;,$$

since for $|n| \leq n_0$

$$|x^t_n| \leq |x^t_{-n_0}| \vee |x^t_{n_0}| \leq 2n_0 \quad . \qquad \triangleright$$

(3.3) now easily follows from (3.2) using the observation that

$$\left\{|y(At) - \rho^{-1}n(At)| < n_\varepsilon(At)\right\} \cap \left\{\left|\frac{y(At)-\rho^{-1}n(At)}{\sqrt{\phi(A)}}\right| > \delta\right\} \subset \left\{\frac{n_\varepsilon(At)}{\sqrt{\phi(A)}} > \delta\right\} =$$

$$= \left\{\frac{|n(At)|}{\sqrt{\phi(A)}} > \delta/\varepsilon\right\}$$

for A sufficiently large. $\triangleright$

We now turn to the proof of (ii). For $\gamma > 0$ and $k > 0$ define

$$n_0(t) = t^\gamma + k \tag{3.14}$$

and let

$$n_\varepsilon(t) = \begin{cases} \varepsilon|n(t)| & \text{for } |n(t)| \geq n_0(t) \\ 2n_0(t) & \text{for } |n(t)| < n_0(t) \end{cases} \tag{3.15}$$

(3.16) Lemma: Given any $\varepsilon > 0$, $\eta > 0$ and $\gamma > 0$ there exists a $k > 0$ such that

$$\mathbb{P}(|y(t) - \rho^{-1} n(t)| < n_\varepsilon(t) \quad \text{for all } t \geq 0) > 1 - \eta, \tag{3.17}$$

where $n_\varepsilon(t)$ is given by (3.14) and (3.15). $\triangleright$

Proof: We again assume that $\rho = 1$. Suppose first that $\xi(t)$ has stationary increments, so that the full Poisson system is invariant under $\underline{T}_t$.

Using the notation $\bar{\cdot}$ for complements, we have that

$$\underline{P}(\overline{G}^{\varepsilon,m}) = \underline{P}(|x_n^t - n| > |n|\varepsilon \quad \text{for some } |n| \geq m)$$

$$\leq \sum_{|n|\geq m} \mathbb{P}\left(\left|\sum_{i=1}^{n} (d_i - 1)\right| > |n|\varepsilon\right)$$

where $(d_i)_{i\in\mathbb{N}}$ are i.i.d. exponential variables with mean 1. Therefore for any $\ell > 1$

$$\underline{P}(\overline{G}^{\varepsilon,m}) \leq \text{const. } m^{-\ell} . \tag{3.18}$$

Let

$$\underline{A}^{\varepsilon} = \{(x,\xi) \in \mathbb{R} \times \Xi \mid \sup_{0\leq t\leq 1} |\xi(t)| > \varepsilon|x|\} \tag{3.19}$$

and note that

$$\mu(\underline{A}^{\varepsilon}) = \rho\, \mathbb{E}(\varepsilon^{-1} \sup_{0\leq t\leq 1} |\xi(t)|) \equiv \rho\, \varepsilon^{-1} q . \tag{3.20}$$

Thus (by Markov's inequality)

$$\mathbb{P}(N_{\underline{A}^\varepsilon} > \Gamma) \leq e^{-\Gamma} e^{\rho\varepsilon^{-1} q(e-1)} . \tag{3.21}$$

Define

$$\underline{G}^{\varepsilon,m} = G^{\varepsilon,m} \bigcap \{ N_{\underline{A}^{\varepsilon}} \leq \varepsilon m \} \tag{3.22}$$

and observe that by (3.18) with $\ell > \gamma^{-1}$ and (3.21),

$$\alpha(k) \equiv \sum_{t \in \mathbb{Z}^+} \underline{P}(\overline{\underline{G}}^{\varepsilon, n_o(t)/2}) < \infty$$

and therefore

$$\lim_{k\to\infty} \alpha(k) = 0 .$$

Thus, for any $\eta > 0$,

$$\underline{P}(\{\omega_t \in \underline{G}^{\varepsilon, n_o(t)/2} \text{ for all } t \in \mathbb{Z}^+\}) > 1 - \eta$$

for k sufficiently large.

(3.16) now follows from the observation that if $\omega \in \underline{G}^{\varepsilon,m}$ then $\omega_t \in G^{4\varepsilon,2m}$ for all $0 \leq t \leq 1$:

Suppose that $\omega \in \underline{G}^{\varepsilon,m}$ and $\omega_t \notin G^{4\varepsilon,2m}$ for some $t \in [0,1]$. Then there exists n, with $|n| \geq 2m$, such that $|x_n^t - n| > 4\varepsilon|n|$. Suppose $n > 0$ and $x_n^t > n + 4\varepsilon n$; the other cases are similar. Then at time t there are fewer than n particles between the origin and $n + 4\varepsilon n$, while at time $t = 0$ there are at least $n + [\varepsilon n]$ particles between the origin and $n + 3\varepsilon n$, since $x_{n+[\varepsilon n]}(0) \leq n + [\varepsilon n] + \varepsilon(n+[\varepsilon n]) < n + 3\varepsilon n$. Therefore at least $[\varepsilon n]$ particles which at time $t = 0$ are in $[0, n+3\varepsilon n]$ must at time t be in $(-\infty,0] \bigcup [n+4\varepsilon n, \infty)$. But because $\omega \in \underline{G}^{\varepsilon,m}$, there can be at most εm such particles.

More or less the same argument covers the case where $\xi(t)$ may not have stationary increments, the main difference being that in (3.20) and (3.21) q should be replaced by (3.6). $\triangleright$

ii) now follows from the lemma with $\gamma < \beta/2$ in the same way as for i). ▷

(3.23) Remark:

(i) If $\frac{n(At)}{\sqrt{\phi(A)}}$ converges in distribution, as $A \to \infty$, to a random variable $Z(t)$, (3.2) holds and, by (3.3), $\frac{y(At)}{\sqrt{\phi(A)}}$ converges in distribution $\rho^{-1} Z(t)$.

(ii) If $\left(\frac{n(At)}{\sqrt{\phi(A)}}\right)_{t\in[0,T]}$ converges weakly (i.e. in distribution, for the Skorohod topology on D) to a process $(Z(t))_{t\in[0,T]}$, then (3.5) holds and, by (3.7), $\left(\frac{y(At)}{\sqrt{\phi(A)}}\right)_{t\in[0,T]}$ converges weakly to $\rho^{-1}(Z(t))_{t\in[C,T]}$ if (3.4) and (3.6) hold . (Technically speaking the latter conclusion requires that y has paths in D (cf. remarks at the end of section VI).) ▷

(3.3) can be sharpened to L^2 convergence.

(3.24) Lemma:

If

$$\lim_{A\to\infty} \frac{\mathbb{E}|\xi(At)|}{\phi(A)} < \infty$$

for some function $\phi(A)$ and some $t \geq 0$, then

$$\lim_{A\to\infty} \frac{1}{\phi(A)} \mathbb{E}((y(At) - \rho^{-1} n(At))^2) = 0 \quad .$$ ▷

Proof: We compute, again setting $\rho = 1$,

$$\mathbb{E}((y(At)-n(At))^2) = \sum_{n=-\infty}^{\infty} \mathbb{E}((x_n^{At} - n)^2 \mathbb{1}(n(At)=n))$$

$$\leq \sum_{n=-\infty}^{\infty} \mathbb{E}((x_n(0)-n)^{2p})^{1/p}\, \mathbb{P}(n(At)=n)^{1/q}$$

$$\leq C(p) \sum_{n=-\infty}^{\infty} |n|\, \mathbb{P}(n(At)=n)^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $C(p)$ is a suitable constant, and the last inequality follows from the fact that $x_n(0)-n$ is a sum over centered i.i.d. exponential variables.

For $q < 2$ we proceed by writing the last sum as

$$\begin{aligned} C(p) \sum_{n=-\infty}^{\infty} |n| \, \mathbb{P}(n(At)=n)^{\frac{1}{2}} \, \mathbb{P}(n(At)=n)^{\frac{1}{q}-\frac{1}{2}} & \\ \leq C(p) \, \mathbb{E}(n(At)^2)^{\frac{1}{2}} \left(\sum_{n=-\infty}^{\infty} \mathbb{P}(|n(At)| \geq |n|)^{\frac{2}{q}-1} \right)^{\frac{1}{2}} & \\ \leq C(p) \, \mathbb{E}(n(At)^2)^{\frac{1}{2}} \, \mathbb{E}(n(At)^2)^{\frac{1}{q}-\frac{1}{2}} \left(\sum_{n=-\infty}^{\infty} \frac{1}{|n|^{\frac{4}{q}-2}} \right)^{\frac{1}{2}} & \\ = \tilde{C}(p) \, \mathbb{E}(n(At)^2)^{\frac{1}{q}} , & \end{aligned}$$

where for $q < \frac{4}{3}$, $\tilde{C}(p) < \infty$. But

$$\lim_{A\to\infty} \frac{1}{\phi(A)} \mathbb{E}(n(At)^2)^{\frac{1}{q}} = \rho^{\frac{1}{q}} \lim_{A\to\infty} \frac{(\mathbb{E}|\xi(At)|)^{\frac{1}{q}}}{\phi(A)} = 0$$

for $q > 1$. $\triangleright$

<u>Remark</u>: In previous works [1,2] a collision process $\hat{y}(t)$ with $\hat{y}(0) = 0$ was considered, i.e. at time $t = 0$ particles are distributed according to the Palm-distribution,which may be obtained from the Poisson distribution by adding to each configuration of the Poisson point process an extra point at the origin. Let $\{\hat{x}_i^t\}_{i\in\mathbb{Z}}$ denote the positions of the particles at time t in this new process,again labeled in their natural order with respect to the origin. Since under $\underline{T}_t$ the particles move independently

of each other and of their initial positions, it is easily seen e.g. that for $n > 0$

$$x_{n-1}^t \leq \hat{x}_n^t \leq x_n^t \quad .$$

For the collision path $\hat{y}(t)$ we have of course that $\hat{y}(t) = \hat{x}^t_{\hat{n}(t)}$, where $\hat{n}(t)$ is the number of "signed" crossings of the origin in the $\hat{}$-process. Thus

$$x^t_{n(t)-1} \leq \hat{y}(t) \leq x^t_{n(t)+1}$$

Since Proposition (3.1), formulated in terms of the crossing process $n(t)$, applies as well to any process $m(t)$ which satisfies the hypothesis for $n(t)$, and in particular to $m(t) = n(t) \pm 1$, it follows that

$$x^t_{n(t)\pm 1} \sim \rho^{-1}(n(t) \pm 1)$$

and therefore

$$\hat{y}(t) \sim \rho^{-1}\, n(t) \quad .$$

IV. Asymptotics of $n(t)$

(4.1) Proposition:

$$\frac{n(t)}{\sqrt{\rho \mathbb{E}|\xi(t)|}} \xrightarrow[t\to\infty]{L} N(0,1)$$

(" $\xrightarrow{L}$ " denotes convergence in distribution.) $\triangleright$

Proof: Since $\dfrac{N^+(t)}{\sqrt{\rho \mathbb{E}|\xi(t)|}}$ is Poisson with mean $\frac{1}{2}$, it follows from the Poisson convergence to the normal that

$$\frac{N^+(t) - E(N^+(t))}{\sqrt{\rho \mathbb{E}(|\xi(t)|)}} \xrightarrow{L} N(0,\tfrac{1}{2}) \ .$$

Since $N^+(t)$ and $N^-(t)$ are i.i.d., the result follows. $\triangleright$

We wish to know more than the behavior of $n(t)$, $t\to\infty$; we wish to consider the asymptotic behavior of the paths as a whole, i.e. that of the process

$$\left(\frac{n(At)}{\sqrt{\phi(A)}}\right)_{t\in[0,\infty)} \quad \text{as } A\to\infty, \text{ where } \phi(A) \text{ is a suitable normalization.} \tag{4.1}$$

suggests, and we shall prove below, that in order for such a process to have a nontrivial limit the following condition should be satisfied by ξ: For all $t \geq 0$

$$\lim_{A\to\infty} \frac{\mathbb{E}(|\xi(At)|)}{\mathbb{E}(|\xi(A)|)} = c_0(t) < \infty \quad . \tag{4.2}$$

(4.1) and (4.2) obviously imply that for each fixed $t \geq 0$, $\frac{n(At)}{\sqrt{\Phi(A)}}$ is well behaved as $A\to\infty$, where

$$\Phi(A) = \mathbb{E}(|\xi(A)|) \quad . \tag{4.3}$$

We wish to allow normalizations $\phi(A)$ more general than $\Phi(A)$, since $\Phi(A)$ is frequently not as simple as we would like. Again by (4.1) any normalization $\phi(A)$ must satisfy

$$\lim_{A\to\infty} \frac{\mathbb{E}(|\xi(At)|)}{\phi(A)} = c(t) < \infty \tag{4.4}$$

for all $t \geq 0$.

4.5 Remark:

Suppose $\phi(A)$ satisfies (4.4). Then

$$c(st) = \lim_{A\to\infty} \frac{\mathbb{E}(|\xi(Ast)|)}{\phi(A)} = \lim_{A\to\infty} \frac{\mathbb{E}(|\xi(As)|)}{\phi(A)} \frac{\mathbb{E}(|\xi(Ast)|)}{\mathbb{E}(|\xi(As)|)}$$

$$= c(s)\, c_o(t) \quad . \tag{4.6}$$

Thus either $c(t) = 0$ for all t (and $\frac{n_A(t)}{\sqrt{\phi(A)}}$ has a trivial limit) or

$c_o(t) < \infty$ for all t (i.e. (4.2) is satisfied). Note also that

$$\lim_{A\to\infty} \frac{\Phi(A)}{\phi(A)} = c(1) .$$

Suppose (4.2) is satisfied. Then since $c_o(t)$ is measurable and

$$c_o(st) = \lim_{A\to\infty} \frac{\Phi(Ast)}{\Phi(A)} = \lim_{A\to\infty} \frac{\Phi(As)}{\Phi(A)} \frac{\Phi(Ast)}{\Phi(As)} = c_o(s)\, c_o(t) ,$$

it follows that

$$c_o(t) = \begin{cases} t^\alpha & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases} \tag{4.7}$$

where $\alpha \geq 0$, since $c_o(1) = 1$ and $\xi(0) = 0$. That in fact $\alpha \geq 0$ follows from (2.1c).

Remark: If ξ has asymptotically stationary increments then $\alpha \leq 1$, since in this case $c_o(t+s) \leq c_o(t) + c_o(s)$.

Remark: $\alpha = 0$ may occur. Consider for example $\xi(t) = v \log(1+t)$ for some centered random variable v.

Remark: Note that $\Phi(A) = A^\alpha h(A)$ with $h(A)$ a slowly varying function. Thus (3.4) with $\phi = \Phi$ can fail only if $\alpha = 0$.

From now on we fix a normalization $\phi(A)$ satisfying (4.4) and define the process n_A by

$$n_A \equiv (n_A(t))_{t\geq 0} \equiv \left(\frac{n(At)}{\sqrt{\phi(A)}}\right)_{t\geq 0} . \tag{4.8}$$

Note that by (4.6) and (4.7), $c(t) = c(1)\, t^\alpha$ for some $\alpha \geq 0$.

<u>(4.9) Proposition</u>:

Suppose that there exists a function $c(s,t)) < \infty$ such that for all $s,t \in [0,\infty)$

$$\lim_{A\to\infty} \frac{\mathbb{E}(|\xi(As) - \xi(At)|}{\phi(A)} = c(s,t) . \tag{4.10}$$

Let Z be the centered Gaussian process with covariance

$$\sigma(s,t) = \frac{\rho}{2}(c(t) + c(s) - c(t,s)) .$$

Then the finite dimensional distributions of n_A converges to those of Z as $A \to \infty$.

<u>Proof</u>: By the Poisson convergence to the normal and (2.8), for all $t \geq 0$

$$n_A(t) \underset{A\to\infty}{\overset{L}{\Longrightarrow}} N(0, \rho c(t))$$

Similarly, for $t > s$, by (4.10) and (2.3)

$$n_A(t) - n_A(s) \underset{A\to\infty}{\overset{L}{\Longrightarrow}} N(0, \rho c(s,t)) ;$$

since

$$n(t) - n(s) = (N^+(t) - N^+(s)) - (N^-(t) - N^-(s))$$

and

$$N^+(t) - N^+(s) = N_s^+(t-s) ,$$

where

$$N_s^+(t-s) = |\{i \mid x_i(s) \leq 0 \quad \text{and} \quad x_i(t) > 0\}|$$

and similarly for $N^-(t) - N^-(s)$.

Thus it is clear that if the finite dimensional distributions of n_A converge to those of any process Z' as $A \to \infty$, or even along a subsequence

$A_n \to \infty$, then

$$Z'(t) = N(0, \rho c(t))$$

and

$$Z'(t) - Z'(s) = N(0, \rho c(s,t)) .$$

Thus the process Z' must have covariance

$$\mathbb{E}(Z'(s)\, Z'(t)) = \tfrac{1}{2} (\mathbb{E}(Z'(t)^2) + \mathbb{E}(Z'(s)^2) - \mathbb{E}((Z'(t) - Z'(s))^2))$$

$$= \sigma(s,t)$$

Thus we are done once we show that for every $0 \le t_1 < t_2 < \ldots < t_n < \infty$

(i) $(Z'(t_1), \ldots, Z'(t_n))$ must be jointly Gaussian

and

(ii) $(n_A(t_1), \ldots, n_A(t_n))$ is (asymptotically, $A \to \infty$) tight.

We now fix $0 \le t_1 < t_2 \ldots < t_n < \infty$. For

$$\alpha = (\alpha_1,\ldots,\alpha_n) \in \{>, \le\}^n$$

define

$$N_\alpha^+ = |\{i \mid i \le 0,\ x_i(t_1)\ \alpha_1 0, \ldots, x_i(t_n)\ \alpha_n\ 0\}|$$

and

$$N_\alpha^- = |\{i \mid i > 0,\ x_i(t_1)\ \alpha_1 0, \ldots, x_i(t_n)\ \alpha_n\ 0\}|$$

Similarly define $N^+_{A,\alpha}$ and $N^-_{A,\alpha}$, replacing $(t_1,\ldots,t_n)$ by $(At_1,\ldots,At_n)$. For each fixed A

$$\{N^+_{A,\alpha}\}_{\alpha\in\{>,\le\}^n} \cup \{N^-_{A,\alpha}\}_{\alpha\in\{>,\le\}^n}$$

is an independent family of Poisson random variables.

Let

$$\overline{N}^{\pm}_{A,\alpha} = \frac{N^{\pm}_{A,\alpha} - \mathbb{E}(N^{\pm}_{A,\alpha})}{\sqrt{\phi(A)}}$$

and observe that for any $1 \leq i \leq n$

$$n_A(t_i) = \sum_{\substack{\alpha \\ \alpha_i = >}} \overline{N}^{+}_{A,\alpha} - \sum_{\substack{\alpha \\ \alpha_i = \leq}} \overline{N}^{-}_{A,\alpha} . \tag{4.11}$$

Thus

$$\sum_{\substack{\alpha \\ \alpha_i = >}} \mathbb{E}(\overline{N}^{+\,2}_{A,\alpha}) + \sum_{\substack{\alpha \\ \alpha_i = \leq}} \mathbb{E}(\overline{N}^{-\,2}_{A,\alpha}) = \mathbb{E}(n_A(t_i)^2) = \frac{\rho\,\mathbb{E}(|\xi(At_i)|)}{\phi(A)}$$

$$\xrightarrow[A\to\infty]{} \rho c(t_i) \quad ,$$

from which follows not only (ii) but also the (asymptotic) tightness of the $\overline{N}^{\pm}_{A,\alpha}$.

Thus every sequence $A_n \to \infty$ along which the finite dimensional distributions of n_A converge has a subsequence along which the family $\{\overline{N}^{\pm}_{A,\alpha}\}_{\alpha\in\{>,\leq\}^n}$ jointly converges in distribution to an independent Gaussian family, again by the Poisson convergence to the normal and the fact that $\phi(A) \xrightarrow[A\to\infty]{} \infty$.
It follows from (4.11) that all limit points of $(n_A(t_1),\ldots,n_A(t_n))$ as $A \to \infty$ are Gaussian. $\triangleright$

Remark: If ξ has (asymptotically) stationary increments, (4.10) follows from (4.4) and $c(s,t) = c(t-s)$ for $t > s$. Thus, in this case,

$$\sigma(s,t) = \tfrac{1}{2}\rho c(1)(t^{\alpha} + s^{\alpha} - (t-s)^{\alpha}) \tag{4.12}$$

with $0 \leq \alpha \leq 1$, where for $\alpha = 0$, $0^{\alpha} = 0$, and $Z(t)$ has stationary increments, a fact which is also directly obvious without considering the form of $\sigma(s,t)$. If $\alpha > 0$, Z has continuous paths, since $E(Z(t)^2) = \rho c(1)\, t^\alpha$.

Remark: For $\lambda > 0$ (cf. Remark (4.5))

$$c(\lambda s, \lambda t) = \lim_{A\to\infty} \frac{\mathbb{E}(|\xi(A\lambda t) - \xi(A\lambda s)|}{\phi(A)}$$

$$= \lim_{A\to\infty} \frac{\phi(A\lambda)}{\phi(A)} \quad \frac{\mathbb{E}(|\xi(A\lambda t) - \xi(A\lambda s)|)}{\phi(A\lambda)}$$

$$= c_o(\lambda)\; c(s,t) ,$$

provided $c(1) > 0$, while if $c(1) = 0$, $c(s,t) \equiv 0$. Thus

$$c(\lambda s, \lambda t) = \lambda^\alpha\, c(s, t)$$

and Z is a self similar Gaussian process with parameter $\alpha/2$.

Remark: If ξ is self similar with parameter $\beta > 0$, i.e.

$$(\xi(\lambda t))_{t\geq 0} \overset{L}{=} (\lambda^\beta\, \xi(t))_{t\geq 0} ,$$

then (4.10) is satisfied with $\phi(A) = A^\beta$ and Z is a self similar Gaussian process with parameter $\beta/2$.

Remark: The self similar Gaussian processes with stationary increments have covariance (4.12) with $0 \leq \alpha \leq 2$ (where for $\alpha = 0$ (4.12) should be regarded as allowing for many possibilities - which we shall not describe here). For $0 < \alpha < 2$ these processes are called fractional Brownian motions [7]. For the case $\alpha = 2$, see Example (8.1).

Remark: In view of the previous remark, all the Gaussian processes Z with covariance (4.12) with $0 < \alpha \leq 1$ arise as a limit of $n_A(t)$ as $A \to \infty$. The case $\alpha = 0$ is special. The process with covariance (4.12) with $\alpha = 0$ is the following:

$$Z(t) = \begin{cases} X(t) + \tilde{Z} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

where $(X(t),\tilde{Z})_{t\in[0,\infty)}$ is a family of i.i.d. Gaussian variables. We do not know, however, whether this process can arise as a limit of $n_A(t)$ as $A \to \infty$. (The reader should consider, however, the 2nd remark following (4.7).)

V. Tightness of the scaled crossing processes

We now consider the process $(n_A(t))_{t\in[0,T]}$, for some fixed $0 \leq T < \infty$, denoted again by n_A.

The paths of n_A can be very irregular, and in particular fail to be in D, even if ξ has continuous paths. Consider for example $\xi(t) = W(t)$, a standard Wiener process. Let $H \subset \mathbb{R}^{[0,T]}$ be such that for all A the paths of n_A are in H almost surely. We may assume that $D = D([0,T]) \subset H$. Since the paths of n_A in general need not be in D we cannot expect $f(n_A)$ to be measurable, for f a bounded function on H continuous in the sup-norm or even Skorohod topology. We shall see however that we nevertheless obtain for every bounded function f continuous in the sup-norm topology that

$$\mathbb{E}(f(n_A)) \xrightarrow[A\to\infty]{} \mathbb{E}(f(Z)) \tag{5.1}$$

in a natural sense.

The main ingredient in the proof of this is

5.2 Definition:

For $h \in H$ let

$$w_h(\delta) = \sup_{\substack{s,t\in[0,T] \\ |t-s|<\delta}} |h(t) - h(s)| .$$

A sequence of measures $(P_n)_{n\in\mathbb{N}}$ on H is C-tight if for each positive η there exists an a such that

$$P_n(\{h \mid |h(0)| > a\}) \leq \eta \quad \text{for all } n \geq 1 \tag{5.3}$$

and for each positive ε and η there exist $\delta > 0$ and n_o such that

$$P_n(\{h \mid w_h(\delta) \geq \varepsilon\}) \leq \eta \quad \text{for all } n \geq n_o . \tag{5.4}$$

A sequence $\{X_n\}$ of processes (i.e. random elements of H) is C-tight if the induced measures $\{P_n\}$ form a C-tight sequence. ▷

5.5 Remark: If H = C then C-tightness amounts to the usual tightness for measures on C. If H = D then C-tightness implies the usual tightness for measures on D, and if furthermore P_n converges to P in finite dimensional distribution then P_n converges weakly to P and P(C) = 1. [5] ▷

We now make (5.1) precise. As before we denote by $\overline{E}(\cdot)$ the outer expectation and by $\underline{E}(\cdot)$ the inner expectation, i.e.

$$\underline{E}(f) = \sup_{\substack{g\leq f \\ g \text{ measurable}}} \mathbb{E}(g).$$

(5.6) Lemma: Let $\{X_n\}$ be a C-tight sequence of random elements of H, and suppose that the finite dimensional distributions of X_n converge to those of X. Then X has (a version also denoted by X with) continuous paths, and for every bounded function $f:H \to \mathbb{R}$ continuous in the sup norm topology

$$\overline{\mathbb{E}}(f(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(X))$$

and

$$\underline{\mathbb{E}}(f(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(X)) \,. \qquad \triangleright$$

Proof: We first define (for n sufficiently large) a regularization $\hat{X}_n$ of X_n with continuous paths. Let $\eta_m = \varepsilon_m = 2^{-m}$, $m \geq 1$. Then by C-tightness there exist δ_m and $n_o(m)$ such that for all m

$$\mathbb{P}(w_{X_n}(\delta_m) > \varepsilon_m) < \eta_m \qquad \text{for all} \quad n \geq n_o(m) \,.$$

Let $m(n) = \sup_{n_o(m)\leq n} m$. If $m(n) = \infty$ then (a version of) X_n has continuous paths, so we set $\hat{X}_n = X_n$ in this case. For $-\infty < m(n) < \infty$, let $\hat{X}_n$ be the linear interpolation of $\{X_n(t)\}_{t\in\delta_{m(n)}\mathbb{N}}$.

Since $\lim_{n\to\infty} m(n) = \infty$, $\{\hat{X}_n\}$ is also C-tight and its finite dimensional distributions also converge to those of X. Therefore X has a version with continuous paths and $\hat{X}_n \overset{L}{\Longrightarrow} X$. Moreover, for every $\varepsilon > 0$, there exists a set $K \subset C$, compact in the sup-norm topology, such that for all n

$$\mathbb{P}(\hat{X}_n \in K) \geq 1 - \varepsilon$$

(provided $\hat{X}_n$ is defined). Any bounded function f on H continuous in the sup norm topology is, of course, uniformly continuous on K; moreover, by the proof of uniform continuity, for any $\varepsilon' > 0$ there exists a $\delta > 0$ such that

$$|f(h_1) - f(h_2)| \leq \varepsilon' \,.$$

provided $h_1 \in K$ and $\|h_1 - h_2\| \leq \delta$. It follows that for any $\varepsilon > 0$

$$\mathbb{E}(f(\hat{X}_n)) - \varepsilon \leq \underline{\mathbb{E}}(f(X_n)) \leq \overline{\mathbb{E}}(f(X_n)) \leq \mathbb{E}(f(\hat{X}_n)) + \varepsilon$$

for n sufficiently large. $\triangleright$

We use the following proposition to establish the C-tightness of $\{n_A\}$ (i.e. of $\{n_{A_n}\}$ for every sequence $A_n \to \infty$).

<u>5.7 Proposition:</u>

Suppose $\{X_n\}$ is a sequence of random elements of H satisfying:

(i) $\{X_n(0)\}$ is tight.

(ii) For all n there exists $\delta_n > 0$ such that

(a) there exist $\beta > 0$, $\sigma > 1$, and $C > 0$ such that for all n (sufficiently large)

$$\mathbb{E}(|X_n(t) - X_n(s)|^\beta) \leq C|t-s|^\sigma \tag{5.8}$$

for all $s,t \in [0,T]$ with $|t-s| > \delta_n$,

and

(b) for every $\varepsilon > 0$ and $\eta > 0$ there exists an n_o such that

$$\mathbb{P}(w_{X_n}(\delta_n) > \varepsilon) < \eta \qquad \text{for all } n \geq n_o . \tag{5.9}$$

Then $\{X_n\}$ is C-tight. $\triangleright$

<u>Proof</u>: We may assume without loss of generality that $T = 1$ and $\delta_n = 2^{-k}$ for some $k \geq 0$ depending on n. Given n we have that

$$|h(t)-h(s)| \leq |h(t)-h(t')| + |h(s)-h(s')| + |h(t')-h(s')|$$

where $s',t' \in \delta_n\mathbb{N}$ and $0 \leq t-t', s-s' \leq \delta_n$. Thus for any $\delta > 0$

$$w_h(\delta) \leq 2w_h(\delta_n) + \sup_{\substack{t,s\in\delta_n\mathbb{N} \\ |t-s|\leq\delta+\delta_n \\ t,s\in[0,1]}} |h(t)-h(s)| .$$

Therefore, by (5.9), we need only show that for any $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$ such that for all n sufficiently large

$$\mathbb{P}(w_{X_n}^{(n)}(\delta) > \varepsilon) < \eta \tag{5.10}$$

where

$$w_h^{(n)}(\delta) = \sup_{\substack{t,s\in\delta_n\mathbb{N} \\ |t-s|<\delta \\ t,s\in[0,1]}} |h(t) - h(s)| .$$

But this follows from (5.8) by the usual "dyadic rational" argument:

Choose a λ such that $2^{-(\sigma-1)} < \lambda^{\beta} < 1$. Given n for which (5.8) is satisfied, let

$$G_k = \{|X_n(\tfrac{i}{2^k}) - X_n(\tfrac{i+1}{2^k})| \leq \lambda^k \quad \text{for } i = 0,1,\ldots,2^k-1\} .$$

By (5.8)

$$\mathbb{P}(G_k) \geq 1 - c\gamma^k$$

where

$$\gamma = \lambda^{-\beta} 2^{(1-\sigma)} < 1 .$$

Given $\varepsilon > 0$ and $\eta > 0$, choose k_o such that

$$c \sum_{k\geq k_o} \gamma^k < \eta$$

and

$$2 \sum_{k\geq k_o} \lambda^k < \varepsilon .$$

Since every interval in [0,1] can be expressed as a union of elementary dyadic intervals involving at most 2 intervals of length 2^k, it follows that (5.10) is satisfied with $\delta = 2^{-k_o}$. $\triangleright$

We now apply Proposition (5.7) to the family $\{n_A\}$ with $\delta_A = A^{-1}$. We shall see that (5.9) follows easily from (3.6); while if $\xi(t)$ has stationary increments, then if (4.4) holds uniformly in t, (5.8) is satisfied. In fact, a uniform version of a condition substantially weaker than (4.4) is sufficient, even if $\xi(t)$ does not have stationary increments.

(5.11) Proposition: Suppose that (3.4) (i.e. $\alpha > 0$) and (3.6) are satisfied, and that there exist a $\kappa > 0$ and a $K > 0$ such that (for all A sufficiently large)

$$\frac{\mathbb{E}(|\xi(At) - \xi(As)|)}{\phi(A)} \leq K|t-s|^{\kappa} \tag{5.12}$$

for all $s,t \in [0,T]$. Then the family $\{n_A\}$ is C-tight. $\triangleright$

Proof: By (2.1c), $\mathbb{E}(|\xi(t)|) > 0$ for some t, and without loss of generality we may assume that $\mathbb{E}(|\xi(1)|) > 0$. Moreover, just as in the proof of Proposition (4.9), we consider only ξ having stationary increments.

We establish the hypotheses of Proposition (5.7) with $\delta_A = A^{-1}$. Choose an even number $\nu > \frac{2}{\kappa}$. Then

$$\begin{aligned}\mathbb{E}(|n(t)|^{\nu}) &\leq 2^{\nu}\,\mathbb{E}((N^{+}(t) - \mathbb{E}(N^{+}(t)))^{\nu}) \\ &\leq \text{const.}((\mathbb{E}(|\xi(t)|))^{\nu/2} + \mathbb{E}(|\xi(t)|)),\end{aligned}$$

since $N^{+}(t)$ is a Poisson random variable with mean $\frac{\rho}{2}\mathbb{E}(|\xi(t)|)$. It follows that for $t \geq \delta_A$,

$$\mathbb{E}(|n_A(t)|^{\nu}) \leq \text{const.}\left(\left(\frac{\mathbb{E}(|\xi(At)|)}{\phi(A)}\right)^{\nu/2} \vee \left(\frac{\mathbb{E}(|\xi(1)|)}{\phi(A)}\right)^{\nu/2}\right) \leq \text{const.}\ t^{\kappa\nu/2}$$

by (5.12) (for A sufficiently large), which establishes (5.8).

To establish (5.9) we observe that for any $\varepsilon > 0$,

$$\{w_{n_A}(\delta_A) > \varepsilon\} = \left\{ \sup_{\substack{s,t\in[0,AT] \\ |s-t|\leq 1}} |n(t) - n(s)| > \varepsilon \sqrt{\phi(A)} \right\}$$

$$\subset \bigcup_{s=0,\ldots,[AT]} \left\{ \sup_{\substack{t\in[s,s+1) \\ t\leq AT}} |n(t) - n(s)| > \frac{\varepsilon}{3} \sqrt{\phi(A)} \right\}$$

$$\subset \bigcup_{s=0,\ldots,[AT]} \{\omega \mid \omega_s \in \{N_{\underline{A}^1} > \frac{\varepsilon}{3}\sqrt{\phi(A)}\}\} ,$$

where $\underline{A}^1$ is defined in (3.19). Thus (cf. (3.20) and (3.21)),

$$\mathbb{P}(\{w_{n_A}(\delta) > \varepsilon\}) \leq AT\, e^{-\frac{\varepsilon}{3}\sqrt{\phi(A)}} \exp\{\rho\mathbb{E}(\sup_{0\leq t\leq 1} |\xi(t)|)(e-1)\}$$

and (5.9) thus follows from (3.4). $\triangleright$

If ξ has stationary increments and $\phi(A) = A^{\alpha}$, $\alpha > 0$, then (5.12) can be replaced by a condition which is easier to check.

<u>(5.13) Proposition</u>: Suppose that ξ has stationary increments, that (3.6) is satisfied, and that $\phi(A) = A^{\alpha}$, $\alpha > 0$. Suppose further that there exists a $\tau > 0$, a $\gamma > 0$, and a $C > 0$ such that

$$\mathbb{E}(|\xi(t)|) < C\, t^{\gamma} \quad \text{for all} \quad t < \tau . \tag{5.14}$$

Then the family $\{n_A\}$ is C-tight. $\triangleright$

<u>Proof</u>: We may assume without loss of generality that $\gamma \leq \alpha$. By (4.4)

$$\mathbb{E}(|\xi(t)|) \leq \text{const. } t^{\alpha}$$

for t sufficiently large. Note that because ξ has stationary increments, it follows from (5.14) that $\mathbb{E}(|\xi(t)|)$ is bounded on compact subsets of $[0, \infty)$. Thus for all $t \geq 0$

$$\mathbb{E}(|\xi(t)|) \leq \text{const. } t^{\gamma} \vee t^{\alpha} .$$

Therefore

$$\frac{\mathbb{E}(|\xi(At)|)}{\phi(A)} \leq \text{const.}\left(\left(\frac{t^{\gamma}}{A^{\alpha-\gamma}}\right) \vee t^{\alpha}\right) \leq \text{const. } t^{\gamma}$$

for $t < T$ (and $A > 1$). ▷

VI. The rescaled collision process

Let $\phi(A)$ satisfy (4.4). Then we define the rescaled collision process y_A by

$$y_A(t) = \frac{y(At)}{\sqrt{\phi(A)}} , \qquad t \in [0,T]$$

By virtue of Proposition (3.1), $\{y_A\}$ has the same asymptotic behavior as $\{n_A\}$; this would follow immediately from Remark (3.23) if n_A had paths in D, but,as we have already noted,this will frequently not be the case.

(6.1) Proposition: (i) If ξ satisfies (4.10), then the finite dimensional distributions of y_A converge to those of $\hat{Z}$, where $\hat{Z} = \rho^{-1} Z$ is the Gaussian process with covariance $\hat{\sigma}(s,t) = \frac{\rho^{-1}}{2}(c(t) + c(s) - c(s,t))$.

(ii) If ξ satisfies (3.4), (3.6) and (5.12), then $\{y_A\}$ is C-tight. ▷

Proof: (i) is an immediate consequence of (3.1)(i) and Proposition (4.9). Moreover, the C-tightness of $\{n_A\}$ implies (3.5), and hence, by Proposition (3.1)(ii), the C-tightness of $\{y_A\}$. ▷

Remark: Our tightness condition is essentially (3.6). If this is not satisfied, the paths of y_A are presumably highly irregular and, in particular, not in D. Thus (3.6) appears to be the minimal condition for the existence of a decent collision process.

Remark: If for all $0 \leq T < \infty$

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\xi(t)| \right) < \infty ,$$

then it is easy to see that if the paths of ξ are in C (respectively D) then the paths of y_A are in C (respectively D). Note also that if the paths of ξ are in D then, $\underline{P}$ almost surely,

$$\sup_{0 \leq t \leq T} |n_A(t)| = \sup_{\substack{0 \leq t \leq T \\ t \text{ rational}}} |n_A(t)|$$

is measurable. Similarly $w_{n_A}(\delta)$ is measurable in this case.

VII. Partial summary

(7.1) Theorem:

Suppose ξ has stationary increments and satisfies (2.1).

Then the following are true:

(i) (Central Limit Theorem)

$$\frac{y(t)}{\sqrt{\mathbb{E}(|\xi(t)|}} \xrightarrow[t \to \infty]{L} N(0, \rho^{-1})$$

(ii) (Convergence of finite dimensional distributions)

If for all $t \geq 0$

$$\lim_{A\to\infty} \frac{\mathbb{E}(|\xi(At)|)}{\phi(A)} = c(t) < \infty ,$$

then $c(t) = c(1)\,t^{\alpha}$, $\alpha \geq 0$,

and

$$y_A(t) \equiv \frac{y(At)}{\sqrt{\phi(A)}} \underset{A\to\infty}{\Longrightarrow} \hat{Z}(t)$$

in finite dimensional distribution, where $\hat{Z}$ is the Gaussian process with covariance $\hat{\sigma}(s,t) = \frac{1}{2}\rho^{-1} c(1)(t^{\alpha} + s^{\alpha} - |t-s|^{\alpha})$.

(iii) (Functional Central Limit Theorem)

Suppose that for some $\alpha > 0$

$$\frac{\mathbb{E}(|\xi(A)|)}{A^{\alpha}} \underset{A\to\infty}{\longrightarrow} \sigma < \infty \tag{7.2}$$

and that

$$\mathbb{E}(\sup_{0\leq t\leq 1} |\xi(t)|) < \infty . \tag{7.3}$$

Suppose further that there exist $\tau > 0$, $\gamma > 0$ and $C > 0$ such that

$$\mathbb{E}(|\xi(t)|) \leq Ct^{\gamma} \quad \text{for all } t \leq \tau . \tag{7.4}$$

Then as $A \to \infty$

$$\left(\frac{y(At)}{A^{\alpha/2}}\right)_{t\geq 0} \quad \text{converges weakly to } (\hat{Z}(t))_{t\geq 0} ,$$

where $\hat{Z}$ is the Gaussian process with covariance $\hat{\sigma}(s,t) = \frac{1}{2}\rho_0^{-1}\sigma(t^{\alpha} + s^{\alpha} - |t-s|^{\alpha})$. in the sense of Lemma (5.6). ▷

(7.5) Remark: By self similarity and the fact that $\hat{Z}$ is Gaussian

$$\mathbb{E}(\hat{Z}(xs) \mid \hat{Z}(s)) = C(x)\, \hat{Z}(s)$$

and hence

$$\hat{\sigma}(s,t) = s^{\alpha}\, C(t/s)\ .$$

If $\hat{Z}(t)$ is Markovian, we furthermore obtain that for $x,y \geq 1$

$$C(xy) = C(x)\, C(y), \quad \text{so that}$$

so that

$$C(x) = x^{\beta} \quad \text{or} \quad c(x) \equiv 0\ .$$

Thus $\hat{\sigma}(s,t) = \text{const.}\ s^{\alpha-\beta}\, t^{\beta}$ or $\hat{\sigma}(s,t) \equiv 0$ in this case, and hence (a nondegenerate) $\hat{Z}$ is Markovian only for $\alpha = 1$ and $\alpha = 2$ (see also [7]).

VIII. Examples

(8.1) The ideal gas [1,2]:

Let $\xi(t) = vt$, where v is a centered random variable with $\mathbb{E}(|v|) < \infty$. Then (7.2) is satisfied with $\alpha = 1$, $\sigma = \mathbb{E}(|v|)$, and (7.4) is satisfied with $C = \mathbb{E}(|v|)$ and $\gamma = 1$. Thus by (7.1)iii, $\left(\frac{y(At)}{\sqrt{A}}\right)_{t\geq 0}$ converges weakly to the Wiener process with diffusion constant $\rho^{-1}\, \mathbb{E}(|v|)$.

(8.2) Colliding Brownian particles [2]:

Let $\xi(t) = W(t)$, a standard Wiener process. Then (7.2) is satisfied with $\alpha = \frac{1}{2}$, $\sigma = \sqrt{\frac{2}{\pi}}$. Furthermore

$\mathbb{E}(\sup_{0\leq t\leq 1} |\xi(t)|) = 2\mathbb{E}(|\xi(1)|)$ and (7.4) holds with $C = \sqrt{\frac{2}{\pi}}$ and $\gamma = \frac{1}{2}$.

Thus by (7.1)(iii), $\left(\frac{y(At)}{A^{1/4}}\right)_{t\geq 0}$ converges weakly to the Gaussian process with covariance $\hat{\sigma}(s,t) = \frac{1}{2}\, \rho^{-1} \sqrt{\frac{2}{\pi}}\, (t^{\frac{1}{2}} + s^{\frac{1}{2}} - |t-s|^{\frac{1}{2}})$.

(8.3) Colliding Ornstein Uhlenbeck particles:

Let $\xi(t) = x(t)$, where

$$dx(t) = v(t)\,dt$$

$$dv(t) = -v(t)\,dt + dW(t),$$

with the stationary initial distribution for v.

By the "Central Limit Theorem" for the Ornstein Uhlenbeck process

$\frac{\xi(t)}{\sqrt{t}} \to N(0,D)$ with variance $D = \lim_{t\to\infty} \frac{E(\xi(t)^2)}{t} = 1$.

Hence

$$\frac{\mathbb{E}(|\xi(A)|)}{\sqrt{A}} \xrightarrow[A\to\infty]{} \mathbb{E}(|W(1)|) = \sqrt{\frac{2}{\pi}}\ .$$

Moreover

$$\mathbb{E}(|\xi(t)|) \leq \mathbb{E}(|v|)\,t\ ,$$

and since

$$|v(t)| \leq \int_0^t |v(s)|\,ds + C \quad \text{for all} \quad 0 \leq t \leq 1\ ,$$

where

$$C = |v(0)| + \sup_{0\leq t\leq 1} |W(t)|\ ,$$

we obtain by Gronwall's inequality that

$$|v(t)| \leq Ce^t \quad \text{for all} \quad 0 \leq t \leq 1\ .$$

Therefore

$$\mathbb{E}(\sup_{0\leq t\leq 1} |\xi(t)|) \leq \mathbb{E}(\sup_{0\leq t\leq 1} |v(t)|) \leq \mathbb{E}(|v(0)|) + \mathbb{E}(\sup_{0\leq t\leq 1} |W(t)|)e\ .$$

Hence, by (7.1)iii, $\left(\frac{y(At)}{A^{1/4}}\right)_{t\geq 0}$ converges weakly to the same limit as in Example (8.2). $\triangleright$

Examples (8.1) and (8.2) are special cases of

(8.4): Let ξ be the Gaussian process with covariance

$$\sigma(s,t) = \frac{1}{2}(t^{\nu} + s^{\nu} - |t-s|^{\nu}), \qquad 0 < \nu \leq 2 .$$

Since $\mathbb{E}(|\xi(t)|) = \sqrt{\frac{2}{\pi}}\, t^{\nu/2}$, (7.2) and (7.4) are satisfied with $\alpha = \gamma = \frac{\nu}{2}$ and $\sigma = C = \sqrt{\frac{2}{\pi}}$.

Since there exist $\beta > 1$ and $\sigma > 1$ for which

$$\mathbb{E}(|\xi(t)| - \xi(s)|^{\beta}) \leq \bar{C}\,|t-s|^{\sigma},$$

we obtain by the proof of (5.6), replacing λ^k by $a\lambda^k$, that

$$\mathbb{P}(\sup_{0\leq t\leq 1} |\xi(t)| \geq x) \leq \frac{\tilde{C}}{x^{\beta}}$$

where $\tilde{C} < \infty$. Thus (7.3) is satisfied. Therefore $\left(\frac{y(At)}{A^{\nu/2}}\right)_{t\geq 0}$ converges weakly to the Gaussian process with covariance

$$\hat{\sigma}(s,t) = \frac{1}{2}\rho^{-1}\sqrt{\frac{2}{\pi}}\,(t^{\nu/2} + s^{\nu/2} - |t-s|^{\nu/2}).$$

(8.5): Let ξ be the symmetric stable process of order $0 < \nu < 1$, so that $\mathbb{E}(|\xi(t)|) = t^{1/\nu}\,\mathbb{E}(|\xi(1)|)$ [3,4]. Thus (7.2) and (7.4) are satisfied with $\alpha = \gamma = \nu^{-1}$ and $\sigma = C = \mathbb{E}(|\xi(1)|)$.

Since ξ is a martingale [4]

$$\mathbb{P}(\sup_{0\leq t\leq 1} |\xi(t)| > x) \leq \frac{1}{x^{\beta}}\,\mathbb{E}(|\xi(1)|^{\beta}) < \infty,$$

provided $\beta > 1$ is sufficiently small, and (7.3) follows. Therefore $\left(\frac{y(At)}{A^{1/2\nu}}\right)_{t\geq 0}$ converges weakly to the Gaussian process with covariance

$$\hat{\sigma}(s,t) = \frac{1}{2}\rho^{-1}\,\mathbb{E}(|\xi(1)|)(t^{1/2\nu} + s^{1/2\nu} - |t-s|^{1/2\nu}).$$

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