ON THE SURFACE TENSION OF LATTICE SYSTEMS*

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There are various microscopic definitions of the surface tension $\beta^{-1}\tau$ in the literature, but it is far from obvious (or known), in general, that they are all equivalent.[1-4] A proof[5] that for the two-dimensional Ising model (on a square lattice with nearest-neighbor interactions), many different definitions give the same answer as that obtained explicitly by Onsager is therefore encouraging. Here, we use a "grand canonical"[2,4] definition of surface tension that seems natural to us. It is particularly simple when the two pure phases are related to each other by a symmetry of the Hamiltonian, as is the case for the Ising models we shall consider. To make things easy, we deal first with the simplest cases and leave all generalizations to the end.

We consider the d-dimensional ($d = 2, 3$) Ising model with nearest-neighbor ferromagnetic interactions on a simple cubic lattice. At each point, $i \in \mathbb{Z}^d$, there is a spin variable $\sigma_i = \pm 1$, and the Hamiltonian in a finite region $\Lambda \subset \mathbb{Z}^d$ is:

$$\mathscr{H}_{\Lambda,\,\mathrm{b.c.}} = -J\left(\sum_{\substack{\langle ij\rangle\\ i,j\in\Lambda}} \sigma_i\sigma_j + \sum_{\substack{\langle ij\rangle\\ i\in\Lambda\\ j\notin\Lambda}} \sigma_i\tilde{\sigma}_j\right), \tag{1}$$

with $\langle ij\rangle$ = nearest-neighbor pair. Here, $J > 0$, and $\tilde{\sigma}_j$ is some fixed value of the spin outside Λ, that is, some boundary condition (b.c.) on Λ. Let $\Lambda_{L,M} \subset \mathbb{Z}^d$ be a parallelepiped of height $2M$ and base $(2L+1)^{d-1}$, $i = (i_1, \ldots, i_d) \in \Lambda_{L,M}$ if $-M \le i_1 \le M-1$, $-L \le i_2, \ldots, i_d \le L$. We shall generally write Λ for $\Lambda_{L,M}$. We introduce three types of b.c.

(a) the + b.c. (respectively, − b.c.):
  $\tilde{\sigma}_j = +1 (\tilde{\sigma}_j = -1)$;
(b) the ± b.c.: $\tilde{\sigma}_j = +1$ if $j_1 \ge 0$,
  $\tilde{\sigma}_j = -1$ if $j_1 < 0$;
(c) the free b.c. where the second sum in Equation 1 is set equal to zero.

The Gibbs measure in Λ is, for a given b.c. and inverse temperature β,

$$\mu_{\Lambda,\,\mathrm{b.c.}} = \exp(-\beta\mathscr{H}_{\Lambda,\,\mathrm{b.c.}})/Z_{\Lambda,\,\mathrm{b.c.}} \tag{2}$$

with

$$Z_{\Lambda,\,\mathrm{b.c.}} = \sum_{\sigma_i=\pm 1} \exp(-\beta\mathscr{H}_{\Lambda,\,\mathrm{b.c.}}). \tag{3}$$

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We write $\langle\ \rangle_{\Lambda,+}$, $\langle\ \rangle_{\Lambda,\pm}$, $\langle\ \rangle_{\Lambda}$ for the expectation value with respect to the b.c. (a), (b), and (c). As L, $M \to \infty$, the Gibbs measures with these b.c. converge to infinite-volume Gibbs states denoted, respectively, ρ_+, (ρ_-), $\rho_\pm$, ρ; the states ρ_+, ρ_-, ρ are translation invariant in all directions,[6,7] and $\rho_\pm$ is translation invariant in the $i_2, \ldots, i_d$ directions.[8]

The surface tension $\beta^{-1}\tau$ is now defined[4] as the suitable thermodynamic limit of the excess free energy per unit cross section of the system with "mixed" $\pm$b.c. over the system with "pure" $+$b.c.; that is,

$$\tau(K; d) = \lim_{L\to\infty} \frac{1}{(2L+1)^{d-1}} \lim_{M\to\infty} \tau_\Lambda, \tag{4}$$

where $\tau_\Lambda = -\log(Z_{\Lambda,\pm}/Z_{\Lambda,+})$ and $K = \beta J$. This definition of τ is based on the following reasoning:

(a) For $\Lambda \nearrow \mathbb{Z}^d$, the state of the system with $+$b.c., ρ_+, always corresponds to a pure phase (translation-invariant extremal Gibbs state). For $T \geq T_c$, the critical temperature for the onset of the spontaneous magnetization (which, as is well known, equals $\rho^+(\sigma_0)$), this infinite-volume Gibbs state is the same as that obtained with any b.c. since the system can exist only in one phase (Gibbs state). For $T < T_c$, however, ρ_+ is different from the state obtained with pure $-$b.c., ρ_-. Moreover, for these systems, it is known that there are only two pure phases at low temperatures[6] (all $T < T_c$ for $d = 2$, almost all for $d \geq 3$[9]).

(b) When $M \to \infty$, the state of the system in the infinite cylinder B_L with base area $(2L+1)^2$ resembles very closely, as $i_1 \to \infty$ (or $-\infty$), the state of a system in B_L with pure $+$ (or pure $-$) b.c.[10] This can be interpreted to mean that the system with $\pm$b.c. in B_L is spatially segregated into a $+$ and $-$ phase (vapor and liquid phase in the lattice-gas language). τ then measures the excess free energy due to the interface thus created. Although this interface may fluctuate wildly as $L \to \infty$, depending on d and T,[10-12] it is present somewhere (with, presumably, a finite thickness at all $T < T_c$; this will be discussed elsewhere).

As mentioned earlier, Equation 4 can be shown[5] to give, for $d = 2$, the value, found by Onsager,[1]

$$\tau(K; 2) = \begin{cases} 2K + \log[\tanh K], & \text{for } T < T_c \\ 0, & \text{for } T \geq T_c \end{cases}. \tag{5}$$

Although there is unfortunately no explicit formula for τ in other systems, it will be shown that Equation 5 is a lower bound to τ for $d \geq 3$. This is based on the monotonicity property of τ (Theorem 1): when the strength of ferromagnetic interactions is increased, τ increases too. The consequent monotonicity of τ in the temperature is certainly what we would expect of the physical surface tension. We would also expect the surface tension to vanish whenever there is only one phase present ($T \geq T_c$). This is the content of Theorem 2.

We remark here that it has been shown that for a large class of ferromagnetic spin systems:

(a) the limit τ in Equation 4 exists[4] and, for a subset of these spin systems,

(b) $\tau > 0$ for sufficiently low temperatures.[13]

These results apply to spin systems in which the different low-temperature phases (there can be more than two) are related by some symmetry of the Hamiltonian (as in our case where $Z_{\Lambda,+} = Z_{\Lambda,-}$). It is an interesting problem to extend these results to systems without symmetry of the kind considered by Pirogov and Sinai.[14]

RESULTS

Inequalities

Theorem 1. $\tau(K; d)$ is monotone increasing in K and d.

Proof. Consider a general ferromagnetic Ising spin system with (free b.c.) Hamiltonian

$$\mathscr{H}_\Lambda = \sum_{A \subset \Lambda} J_A \sigma_A, \qquad J_A \geq 0.$$

Then, by well-known arguments,[7]

$$\frac{\partial}{\partial J_B} \ln \left[\frac{Z_{\Lambda,+}}{Z_{\Lambda,\pm}}\right] \equiv \langle\sigma_B\rangle_{\Lambda,+} - \langle\sigma_B\rangle_{\Lambda,\pm} \geq 0.$$

The inequality survives in the limit $L \to \infty$ in Equation 4, and so we have monotonicity of $\tau(K; d)$ in K. The monotonicity in d follows from the observation that the d-dimensional system can be obtained from the $(d+1)$-dimensional one by "cutting" the bonds between (hyper-) planes. (In particular, $\tau(K; 3) > 0$ for $T < T_c$ of the $d = 2$ system.)

Remark. The monotonicity of τ is analogous to the well-known monotonicity of the spontaneous magnetization.

We shall now prove that the surface tension vanishes above T_c, the critical temperature for the spontaneous magnetization.

To do this, we introduce a modified Hamiltonian, $\mathscr{H}^s_{\Lambda,\text{b.c.}}$, defined as in Equation 1 but with a coupling sJ instead of J for those $\langle ij\rangle$ with $i_1 = 0$ and $j_1 = -1$. $\tau_\Lambda(K; d, s)$ is defined as in Equation 4 but with $\mathscr{H}^s_{\Lambda,+}$, $\mathscr{H}^s_{\Lambda,\pm}$ (thus, $\tau_\Lambda(K; d, 1) = \tau_\Lambda(K; d)$), and $\langle\ \rangle^s_{\Lambda,\text{b.c.}}$, ρ^s_+, ρ^s_-, $\rho^{\pm}_s$, denote the corresponding expectation value and Gibbs states. ρ^s_+ and ρ^s_- are now translation invariant only in the $i_2, \ldots, i_d$ directions.

For $s = 0$, there is no coupling between the top and bottom parts of Λ, and the system splits into two uncoupled systems with free b.c. on the spins with $i_1 = 0$ and $i_1 = -1$. In the thermodynamic limit, we obtain two uncoupled "semi-infinite" lattices. We use the subscript s.i. for Gibbs states of this semi-infinite system. The b.c. refer, then, to the b.c. put on the $(2d - 1)$ other sides of Λ.

Theorem 2. For any K and d,

(i) $\tau(K; d) = K \int_0^1 [\rho^s_+(\sigma_0\sigma_{-1}) - \rho^s_\pm(\sigma_0\sigma_{-1})]\, ds,$

where $-1 = (-1, 0, \ldots, 0)$.

(ii) $\tau(K; d) \leq 2K(\rho_+(\sigma_0))^2$.

(iii) $\tau(K; d) \leq 2K\rho_{+,\text{s.i.}}(\sigma_0)$.

(iv) $\tau(K; d) \geq 2K \int_0^1 \rho^s_\pm(\sigma_0)\rho^s_+(\sigma_0)\, ds.$

Proof. (i) We first remark that $\tau_\Lambda(K; d, 0) = 0$. Therefore,

$$\begin{aligned}\tau_\Lambda(K; d, 1) &= \int_0^1 \frac{d}{ds}\tau_\Lambda(K; d, s)\, ds \\ &= K \int_0^1 \sum_{\langle ij\rangle \subset \Lambda}' [\langle\sigma_i\sigma_j\rangle^s_{\Lambda,+} - \langle\sigma_i\sigma_j\rangle^s_{\Lambda,\pm}]\, ds,\end{aligned}$$

where the prime indicates $i_1 = 0, j_1 = -1$. We claim that

$$\lim_{L\to\infty} \frac{1}{(2L+1)^{d-1}} \lim_{M\to\infty} \sum_{\langle ij\rangle \subset \Lambda}{}' \langle\sigma_i\sigma_j\rangle^s_{\Lambda;+} = \rho^s_+(\sigma_0\sigma_{-1})$$

and similarly for $\langle\sigma_i\sigma_j\rangle_{\Lambda,\pm}$. Indeed, we know, from monotonicity in Λ, the existence of the limit and its translation invariance (in the $i_2, \ldots, i_d$ directions) and that, for any $\varepsilon > 0$, there exists Λ_0 such that for every $\Lambda \supset \Lambda_0 + i$, $i_1 = 0$, $|\langle\sigma_i\sigma_j\rangle^s_{\Lambda,+} - \rho^s_+(\sigma_0\sigma_{-1})| < \varepsilon$, $j = (-1, i_2, \ldots, i_d)$. It is clear that

$$\lim_{L, M\to\infty} \frac{1}{(2L+1)^{d-1}} (\#\{i \in \Lambda, i_1 = 0 | \Lambda_0 + i \not\subset \Lambda\}) = 0$$

for any given Λ_0. To conclude the proof of (i), we may simply use the Dominated Convergence Theorem.

(ii) From Lebowitz' inequalities on duplicate variables,[15] one concludes that

$$\rho^s_+(\sigma_0\sigma_{-1}) - [\rho^s_+(\sigma_0)]^2 \leq \rho^s_\pm(\sigma_0\sigma_{-1}) - \rho^s_\pm(\sigma_0)\rho^s_\pm(\sigma_{-1}).$$

On the other hand, by Griffiths' inequalities,[7]

$$\pm\, \rho^s_\pm(\sigma_0) \leq \rho^s_+(\sigma_0) \leq \rho_+(\sigma_0);$$

and this, together with (i), shows (ii).

(iii) We use another modification of the Hamiltonian Equation 1 by putting $s = 1$ but adding an external field $h \geq 0$ on all sites with $i_1 = 0$ or $i_1 = -1$. Then, by Theorem 1, $\frac{d}{dh}\tau_\Lambda(h) \geq 0$, and

$$\tau_\Lambda(K; d) = \tau_\Lambda(K; d, h = 0) \leq \lim_{h\to\infty} \tau_\Lambda(K; d, h) = \log\left(\frac{Z^{++}_{\Lambda'}}{Z^{-+}_{\Lambda'}}\right),$$

where $\Lambda' = \{i \in \Lambda | i_1 > 0\}$ and the superscript $+\,+$ refers to $+$b.c. on Λ' and $-\,+$ to $+$ b.c. on the line $i_1 = 0$ and $-$ everywhere else. Introducing now a factor λ that multiplies the coupling between the spins (equal to $+1$) on $i_1 = 0$ and on $i_1 = 1$, one sees that our last expression equals, after taking the limits of Equation 4,

$$K \int_0^1 [\rho_{+,\lambda}(\sigma_1) - \rho_{-,\lambda}(\sigma_1)]\, d\lambda,$$

$1 = (1, 0, \ldots, 0)$. The subscript λ refers here to the coupling, that is, to the strength of the external field imposed on the spins with $i_1 = 1$. By use of the duplicate variables as in (ii), one sees that the derivative with respect to λ of $\rho_{+,\lambda}(\sigma_1) - \rho_{-,\lambda}(\sigma_1)$ is negative; that is, the integral is bounded from above by its value at $\lambda = 0$. But for $\lambda = 0$, the integrand is just $2\rho_{+,s.i.}(\sigma_0)$, and this shows (iii).

(iv) We now use (i) and the inequalities of Lebowitz,[9] which show that

$$\rho^s_+(\sigma_0\sigma_{-1}) - \rho^s_\pm(\sigma_0\sigma_{-1}) \geq |\rho^s_+(\sigma_0)\rho^s_\pm(\sigma_{-1}) - \rho^s_+(\sigma_{-1})\rho^s_\pm(\sigma_0)|.$$

By symmetry, $\rho^s_\pm(\sigma_{-1}) = -\rho^s_\pm(\sigma_0)$, which concludes the proof.

Remarks. (1) Although we have used Lebowitz' inequalities to prove (ii) and (iii), we remark that F.K.G. inequalities and (i) give $\tau(K; d) \leq 2K\rho_+(\sigma_0)$ for Ising spins. In fact, we can prove that $\tau(K; d) = 0$ above T_c for systems with ferromagnetic two-body interactions and even a priori measure with compact support on the real line. However, the stronger results (ii) and (iii) hold only for some measures, for example, the uniform measure on $[-1, +1]$ or $\exp(-\lambda\sigma_i^4 + h\sigma_i^2)\, d\sigma_i$ that satisfy

Lebowitz' inequalities. The restriction to nearest-neighbor interactions is not important. Analogous results hold for general ferromagnetic two-body interactions.

(2) (iii) shows that $\tau(K; d)$ vanishes if there is no spontaneous magnetization in the semi-infinite system. However, the critical temperatures for the infinite and the semi-infinite systems are expected to coincide, as they do in two dimensions.[16] On the other hand, if we could show that, whenever $\rho_{+,\mathrm{s.i.}}(\sigma_0) \neq 0$, there exists an $s \neq 0$ such that $\rho^s_{\pm}(\sigma_0) \neq 0$, it would follow from (iii) and (iv) that $\tau(K; d) \neq 0$ if, and only if, $\rho_{+,\mathrm{s.i.}}(\sigma_0) \neq 0$. Indeed, Lebowitz' inequalities in the form used by Messager and Miracle-Sole[8] show that $\rho^s_{\pm}(\sigma_0)$ is monotone decreasing with s and, of course, $\rho^s_{+}(\sigma_0) \geq \rho^s_{\pm}(\sigma_0)$.

Low Temperatures

Let $z = \exp(-K)$.

Theorem 3 (proven by Bricmont *et al.*,[17] Part III). There exists an $r > 0$ such that $\tau(K; 3) - 2K = f(z)$ is analytic in z for $|z| < r$.

Remark. For $d = 2$, it follows from Formula 5 that $\tau(K; 2) - 2K$ is analytic in z for $|z| < \exp(-K_c)$, $K_c = \beta_c J$, and has an analytic continuation for all K. However, even if we did not have Formula 5, we could prove analyticity of $\tau(K; 2)$ for large K by use of the results of Gallavotti.[11] This proof is slightly more difficult than that of Theorem 3 and proceeds in two steps. We define

$$\bar{\tau} = -\lim_{L\to\infty} \frac{1}{(2L+1)} \lim_{M\to\infty} \log\left(\frac{\sum_{i\in Z} Z^i_{\Lambda,\pm}}{Z_{\Lambda,+}}\right),$$

where $Z^i_{\Lambda,+}$ is defined with $\pm$ b.c. but where on one side the separation between $+$ and $-$ is put at the height i. One shows that $\bar{\tau}$ is analytic in z and then that for K, real $\tau = \bar{\tau}$. Both results are contained implicitly in Gallavotti.[11]

Let $e(i) = -\frac{1}{2}K\sigma_i \sum'_j \sigma_j$, where the sum $\sum'$ is over the $2d$ nearest-neighbor sites, be the energy of site i (times β) in a given configuration.

Theorem 4 (proven by Bricmont *et al.*,[17] Part II). There exists a $K_0 < \infty$ such that for $K > K_0$,

$$\begin{aligned}\frac{d}{dK}\tau(K; 3) &= \sum_{i_1=-\infty}^{+\infty} K^{-1}\{\rho_{\pm}[e(i_1, 0, 0)] - \rho_{+}[e(i_1, 0, 0)]\}(K) \\ &= -\frac{1}{2}\sum_{i_1=-\infty}^{+\infty}\sum_j{}' [\rho_{\pm}(\sigma_i\sigma_j) - \rho_{+}(\sigma_i\sigma_j)](K),\ i = (i_1, 0, 0). \qquad (6)\end{aligned}$$

Remarks. (1) Clearly, for fixed Λ, we always have:

$$\begin{aligned}(2L+1)^{-(d-1)}\frac{d}{dK}\log\left(\frac{Z_{\Lambda,\pm}}{Z_{\Lambda,+}}\right) &= \frac{[\langle H_{\Lambda,\pm}\rangle_{\Lambda,\pm} - \langle H_{\Lambda,+}\rangle_{\Lambda,+}]}{(2L+1)^{d-1}} \\ &= \frac{1}{2}\sum_{i\in\Lambda}\sum_j{}' \frac{[\langle\sigma_i\sigma_j\rangle_{\Lambda,\pm} - \langle\sigma_i\sigma_j\rangle_{\Lambda,+}]}{(2L+1)^{d-1}}, \qquad (6')\end{aligned}$$

where in the sum $\sum'$ the σ_j for $j \notin \Lambda$ are fixed $= \pm 1$ by the b.c.

What has to be proven to get Equation 6 is the validity of the interchange in Equation 6′ of the limits $L \to \infty$, $M \to \infty$, with the summation. Whereas the first interchange, $M \to \infty$, is valid also in two dimensions, the second one, $L \to \infty$, is certainly not; due to the fluctuations of the interface as $L \to \infty$, the Gibbs state ρ_+ is a superposition of the pure-phase Gibbs states ρ_+ and ρ_-;[8,11,18] that is,

$$\rho_{\pm}(\sigma_A) = \tfrac{1}{2}[\rho_{+}(\sigma_A) + \rho_{-}(\sigma_A)]. \qquad (7)$$

In particular, $\rho_{\pm}(\sigma_i\sigma_j) = \rho_{+}(\sigma_i\sigma_j)$, so the right-hand side of Equation 6 would be equal to zero for $d = 2$. In fact, this happens also presumably in three dimensions for temperatures above the roughening temperature (see part 2 in the discussion below).

(2) Adapting Proposition 4.2 of Bricmont *et al.*[17] (Part II) to the Ising model, one shows that $\tau - 2K$ is composed of two terms, both exponentially small as $K \to \infty$. The first term is the difference in the free energy (per unit interface, in the limit $L \to \infty$) between a system in B_L with $+$ b.c. and one in B_L with $\pm$b.c. and the additional constraint that all the spins $\sigma_i = +1$ for $i_1 = 0$ and $= -1$ for $i_1 = -1$. This separates B_L into two semi-infinite cylinders with all $+$ or all $-$b.c. The second term results from the fluctuation of the interface separating the $+$ and the $-$ phases in B_L, under $\pm$b.c.

Duality

For a large class of ferromagnetic Ising models, it is possible to construct dual models.[19] Writing the Hamiltonian of the system with "free" b.c. in the form (see Equation 1)

$$-\beta\mathscr{H}_\Lambda = K \sum_{B \subset \Lambda} \sigma_B, \tag{8}$$

where B is a lattice bond and $\sigma_B = \prod_{i \in B} \sigma_i$, the dual model is constructed on a dual lattice with dual bonds B^*, such that

$$-\beta^*\mathscr{H}^*_{\Lambda^*} = K^* \sum_{B^* \subset \Lambda^*} \sigma_{B^*},$$

the coupling K^* is given by

$$K^* = -\tfrac{1}{2} \log \tanh K. \tag{9}$$

In Equation 8 we have free b.c., but the dual of a model with free b.c. has to be taken with $+$b.c. and vice versa.

Our interest in this duality comes from the observation that by "flipping the spins" in the bottom half of Λ for a system with $\pm$b.c., we obtain, with $\mu_B = exp(-2K\sigma_B)$,

$$\begin{aligned} Z_{\Lambda,\pm}/Z_{\Lambda,+} &= \Big\langle \prod_{\substack{\langle ij\rangle \subset \Lambda \\ i_1 = 0,\, j_1 = -1}} \mu_{\langle ij\rangle} \Big\rangle_{\Lambda,+} \\ &= \Big\langle \prod_{\langle ij\rangle^* \subset \Lambda^*} \sigma_{\langle ij\rangle^*} \Big\rangle^*_{\Lambda^*}. \end{aligned} \tag{10}$$

The first product here is over all bonds crossing the plane ($d = 3$) or line ($d = 2$) $i_1 = -\frac{1}{2}$, whereas the second product is over the dual bonds. Thus, the surface tension of an Ising system at reciprocal temperature β is directly related to the asymptotic behavior of certain spin correlations in the dual model at β^*. This leads to useful relations, as we shall now see.

We note first that for Ising ferromagnets with free boundary conditions,

$$-\lim_{k \to \infty} \frac{1}{k} \rho(\sigma_0\sigma_k) = m. \tag{11}$$

In Equation 11, 0 and k are the lattice sites i with $i_2 = \cdots = i_d = 0$ and $i_1 = 0$ or $i_1 = k$, m is the mass gap, or inverse correlation length. The existence of the limit in

Equation 11 can be shown in the same way as the existence of τ in Equation 4.[4] It is known[19] that, for $d = 2$, our Ising model is self-dual; the dual square lattice $(\mathbb{Z}^2)^*$ has its vertices at the centers of the squares of $\mathbb{Z}^2$, and $\langle ij \rangle^*$ is the bond crossing $\langle ij \rangle$. For $d = 3$, the dual of our model is the Ising gauge model, which is constructed as follows.[20] We take as lattice $\mathcal{L}'$, instead of $\mathbb{Z}^3$, the set of centers of the faces of the unit cubes in $\mathbb{Z}^3$. For each $i \in \mathcal{L}'$, we have a spin $\sigma_i = \pm 1$. Instead of the *n.n.* pairs, we take as bonds, B^*, the four-point sets given by the centers of all the faces to which a given bond of the *n.n.* $\mathbb{Z}^3$ Ising model (i.e., an edge of an elementary cube) belongs.

A distinctive feature of the gauge model is that $\mathcal{H}$ is invariant under symmetry transformations that involve only the flipping of finitely many spins; for example, changing the sign of σ_i, for all $i \in \mathcal{L}'$, which belong to the faces of a given unit cube, does not change the energy. However, this symmetry has no role here (see point 1 in the discussion). In the gauge model, one introduces the "Wilson loop"[20,21]: $\sigma_c = \prod_{i \in c} \sigma_i$, where c is a square of size $L \times L$ in $\mathcal{L}'$ for and the coefficient of the (area-law) decay of this loop is

$$-\lim_{L\to\infty} \frac{1}{L^2} \log \rho(\sigma_c) = \alpha. \tag{12}$$

This limit exists and is nonnegative.

We now state the result of this section.

Theorem 5. (1) For the two-dimensional Ising model,

$$\tau(K; 2) = m(K^*). \tag{13}$$

(2) For the three-dimensional Ising model and for the gauge model,

$$\tau(K; 3) = \alpha(K^*), \tag{14}$$

with K^* given by Equation 9.

Proof. We start with Equation 13. By use of Equations 4, 10, and 11 and the self-duality of the model, we see that we have only to show:

$$\lim_{L\to\infty} \frac{1}{(2L+1)} \lim_{M\to\infty} \log\langle \sigma_{-L}\sigma_L \rangle_\Lambda = \lim_{L\to\infty} \frac{1}{(2L+1)} \log \rho(\sigma_{-L}\sigma_L); \tag{15}$$

that is, the decay of the long-long range order $\langle \sigma_{-L}\sigma_L \rangle_\Lambda$ equals the decay of the short-long range order $\rho(\sigma_{-L}\sigma_L)$.[22]

It is clear that the left-hand side is less than or equal to the right-hand side by Griffiths' inequalities:[7] for fixed A, $\langle \sigma_A \rangle_\Lambda$ is monotone increasing in Λ. To show the converse inequality, we take $L = nk$, for n, k integers and k fixed (since we know that the limits exist in Equation 13, we may use subsequences). Then, since $\sigma_i^2 = 1$, $\sigma_{-L}\sigma_L = \prod_{i=-n}^{n-1} \sigma_{ik}\sigma_{(i+1)k}$ and, by Griffiths' inequalities,[7]

$$\log\langle \sigma_{-L/2}\sigma_{L/2} \rangle_\Lambda \geq \sum_{i=-n}^{n-1} \log\langle \sigma_{ik}\sigma_{(i+1)k} \rangle_\Lambda .$$

We claim that

$$\lim_{M, L\to\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} \log\langle \sigma_{ik}\sigma_{(i+1)k} \rangle_\Lambda = \rho(\sigma_0\sigma_k), \tag{16}$$

by the same argument as the one used in the proof of Theorem 2. Since $(2nk + 1)/2n \to k$; we see that the left-hand side of Equation 15 is larger, for any k, than $1/k \log \rho(\sigma_0\sigma_k)$. Letting $k \to \infty$ finishes the proof.

The proof of Equation 14 is similar. With Equation 9, we obtain

$$\frac{Z_{\Lambda,\pm}}{Z_{\Lambda,+}} = \left\langle \prod_{\substack{i_1=-1/2 \\ i\in\partial\Lambda}} \sigma_i \right\rangle^*_\Lambda,$$

where the expectation is in the dual (gauge) model, and the product runs over all the spins in Λ adjacent to the exterior of Λ, which are in the plane $i_1 = -\frac{1}{2}$ crossing the bonds in Equation 10. Thus, this product is over a square of size $L \times L$, and we write this square as the union of n^2 squares of size $k \times k$, and the rest of the proof goes through.

Discussion

(1) Although we have restricted ourselves to isotropic nearest-neighbor Ising models on $\mathbb{Z}^d$ and their duals, some of our results extend immediately to other systems. Thus, $\tau(K)$ for the triangular lattice with *n.n.* interactions is equal to the mass gap $m(K^*)$ of the honeycomb lattice and vice versa, since these two models are duals.[23] Similarly, our argument, combined with the results of Fontaine and Gruber,[13] shows that some models in three dimensions exhibit the area-law decay of the Wilson loop ($\alpha \neq 0$ in Equation 12). These are all systems obtained through low-temperature–high-temperature duality from a model for which the surface tension is nonzero at low temperature. In fact, the proof in Fontaine and Gruber[13] that $\tau > 0$ at low temperatures for some systems also shows that $\alpha > 0$ at high temperatures for the dual models. In particular, this area law holds also when there is no gauge symmetry. For example, we can take a low-temperature–high-temperature dual of the Ising model where we have "plaquettes," as in the gauge model (4-points bonds) only in planes perpendicular to some axis and *n.n.* two-body bonds between the plaquettes in the different planes. This model has no gauge symmetry but still has a Wilson loop that decays proportionally to the area at high temperatures for the same reason as in the gauge model. This last example was suggested to us by Gruber. Of course, $\alpha = 0$ at low temperatures for all these models because $\tau = 0$ at high temperatures.[13,24]

(2) There is some numerical evidence[25] that, in the three-dimensional Ising model, the nontranslation-invariant Gibbs states that exist at low temperatures do not persist above a roughening temperature T_R strictly less than T_c. It has been questioned[26] whether the surface tension exhibits nonanalytic behavior at T_R.

On the other hand, combining Theorem 5 and Theorem 3, one sees that $\alpha + \log \tanh(\beta J)$ is the restriction to real temperatures of a function of β, analytic around $\beta = 0$. One may ask: What is the interpretation of the corresponding, possibly nonanalytic, behavior of α at T_R^* or, rather, how do the nontranslation-invariant Gibbs states reflect themselves in the gauge model at high temperatures? We remark first that in two dimensions, taking some fixed *n.n.* pair $\langle ij \rangle$, the difference

$$\langle \mu_{\langle ij\rangle} \rangle_{\Lambda,\pm} - \langle \mu_{\langle ij\rangle} \rangle_{\Lambda,+} \tag{17}$$

goes to zero as $\Lambda \to \infty$ (see Equation 7)[8,11,18] for any temperature (this is equivalent in Lebowitz[9] to saying that the state $\langle\ \rangle_{\Lambda,\pm}$ is translation invariant in the thermodynamic limit).

By duality, this means, as pointed out to us by D. Merlini,

$$\frac{\langle \sigma_{-L/2}\sigma_i\sigma_j\sigma_{L/2}\rangle^*_\Lambda}{\langle \sigma_{-L/2}\sigma_{L/2}\rangle^*_\Lambda} - \langle \sigma_i\sigma_j\rangle^*_\Lambda \xrightarrow[\Lambda\to\infty]{} 0 \tag{18}$$

(we denote by $\langle ij \rangle$ also the pair dual to $\langle ij \rangle$). At high temperatures, this is a nontrivial cluster property of the high-temperature state. At low temperatures, Equation 18 still holds but is less striking since then both the numerator and denominator of the first term in Equation 18 tend to a nonzero value.

In three dimensions, duality applied to Equation 17 gives

$$\frac{\langle \sigma_B \sigma_C \rangle^*_\Lambda}{\langle \sigma_C \rangle^*_\Lambda} - \langle \sigma_B \rangle_\Lambda, \tag{19}$$

where B is the bond that crosses $\langle ij \rangle$, and $C = \{i \mid i_1 = -\frac{1}{2}, i \text{ adjacent to the exterior of } \Lambda\}$. Therefore, we know that, as soon as $\langle\ \rangle_{\Lambda,\pm}$ is nontranslation invariant in the thermodynamic limit [and this holds at least up to T_c (two dimensions)[27]], Equation 19 does not go to zero at the dual temperature [i.e., for temperatures higher than T_c (two dimensions) since T_c (two dimensions) $= T_c^*$ (two dimensions) (with $J = 1$)]. So, if there is a $T_R < T_c$ (three dimensions), Equation 19 does not go to zero above the dual temperature T_R^* and goes to zero below. This is another kind of transition in the gauge model.

Summary

We collect here some exact results concerning the surface tension $\beta^{-1}\tau$ of two- and three-dimensional Ising ferromagnets. Some of these results are new: the monotonicity of τ in the coupling constants, the fact that $\tau = 0$ above the critical temperature, the analyticity in $z = \exp(-\beta J)$ of $\tau - 2\beta J$ at low temperatures for the three-dimensional system with nearest-neighbor interaction J, and an expression for τ in terms of correlation functions. Other results are already known, for example, the equality of τ in the two-dimensional square lattice to the mass gap m (inverse correlation length) at the dual temperature β^*; this is proven here simply by means of inequalities. The proof of $\tau(\beta) = m(\beta^*)$ then extends immediately to other lattices, for example, the triangular-honeycomb dual lattices.

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