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# MECHANICAL SYSTEM WITH STOCHASTIC BOUNDARIES

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## ABSTRACT

We consider a model of a fluid confined in a box $\Lambda$ whose sides are maintained at different temperatures through contact with thermal reservoirs. We prove that the fluid has a unique steady state, to which any initial state will converge.

The model is as follows: Let $\Lambda \subset \mathbf{R}^3$ be compact and convex with smooth boundary $\partial\Lambda$. Let the boundary temperature function $T(\vec{q})$, $\vec{q}$ $\partial\Lambda$, be smooth and nowhere vanishing. We represent our fluid by $n$ classical particles evolving according to Hamilton's equations, with Hamiltonian

$$H = \text{kinetic energy} + V(q_1, \ldots, q_n),$$

where

$$V = \sum_{i \neq j} u(|\vec{q}_i - \vec{q}_j|).$$

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It is assumed that $u(r)$, $r \geqslant 0$, is smooth and repulsive: $u(r)$ is $C^\infty$, $u'(0) = 0$, $u'(r) < 0$ for $r \neq 0$. When a particle hits $\partial\Lambda$ we stipulate that it be instantly thermalized. This means that its momentum immediately following collision with $\partial\Lambda$ at $\vec{q}$ is a random variable with a "Maxwellian" distribution at temperature $T(\vec{q})$: the distribution is proportional to

$$\vec{p} \cdot \vec{n}(\vec{q})\, e^{-\frac{\vec{p}^{\,2}}{2mkT(\vec{q})}}\, d\vec{p},$$

where $\vec{n}(\vec{q})$ is the inward directed unit normal vector to $\partial\Lambda$ at $\vec{q}$ (on the set $\vec{p} \cdot \vec{n}(\vec{q}) \geqslant 0$). If two or more particles collide with $\partial\Lambda$ at the same time, their outgoing momenta are required to be independent. A Markov process is thus defined.

We prove the following: Let $\Omega = (\Lambda \times \mathbf{R}^3)^n$. Then there exists a subset $\hat{\Omega} \subset \Omega$ whose complement has vanishing Lebesgue measure, such that the Markov process

$$X(t) = (\mathbf{q}(t), \mathbf{p}(t)),$$

$$\mathbf{q} = (\vec{q}_1, \ldots, \vec{q}_n), \quad \mathbf{p} = (\vec{p}_1, \ldots, \vec{p}_n) \quad \text{with } X(0) \epsilon \hat{\Omega},$$

(1) is well defined for all $t \geqslant 0$, and

(2) lives on $\hat{\Omega}$ for all $t \geqslant 0$:

For $x \epsilon \hat{\Omega}$,

$$\text{Prob}\left\{ X(t) \epsilon\, \Omega - \hat{\Omega}, \text{ for some } t > 0 \mid X(0) = x \right\} = 0.$$

Moreover,



(3) the $\hat{\Omega}$-valued process $X(t)$ has a unique stationary probability measure $\mu_s$,

$$\int \mu_s(dx)\, P^t(x, dy) = \mu_s(dy),$$

where $P^t(x,dy) = \text{Prob}\left\{ X(t) \epsilon\, dy \mid X(0) = x \right\}$, and $\mu_s$ is equivalent to Lebesgue measure.

(4) For any probability measure $\nu$ on $\hat{\Omega}$

$$\nu P^t \equiv \int \nu(dx) P^t(x,dy) \rightarrow \mu_s \text{ in variation norm.}$$

The convexity of $\Lambda$ and the repulsiveness of $u$ are needed to establish the existence of a nontrivial stationary probability measure. In case $T(\vec{q})$ is constant, so that the canonical ensemble is stationary, (3) and (4) may be established without these assumptions.

## 1. INTRODUCTION

Consider a gas in a box $\Lambda \subset \mathbf{R}^3$ consisting of $n$ (identical) interacting particles, moving according to Newton's equations of motion. What happens when a particle hits $\partial\Lambda$, the boundary of $\Lambda$? If the system is closed, so that there is no exchange of energy with the outside, then it is natural to assume that the particles undergo elastic collisions with the boundary – the sign of the normal component of the momentum is reversed. This assumption gives rise to a deterministic dynamical system with evolution $S^t$.

Statistical mechanics says that the state of the gas in thermodynamic equilibrium should be represented by the probability measure $\mu_E$, the microcanonical ensemble at the appropriate energy, on the energy surface $\Gamma_E$. $\mu_E$ is stationary under $S^t$, as it must be if it is to represent equilibrium. A strong theoretical indication of the appropriateness of $\mu_E$ would be provided if it could be shown that $\mu_E$ is in some sense the only stationary probability distribution on $\Gamma_E$. A form of this uniqueness would be provided by the ergodicity of $(S^t, \mu_E)$. This would guarantee that $\mu_E$ is the unique absolutely continuous stationary probability distribution on $\Gamma_E$. Unfortunately ergodicity is extremely difficult to establish; it has been established for only a few mechanical systems [1]. Moreover, we should demand more: Systems do not start out in equilibrium, but they end up there; thus we should demand that any initial probability distribution should be driven toward equilibrium ($\mu_E$) by $S^t$. A version of this is equivalent to the mixing property for $(S^t, \mu_E)$. Mixing would give the above convergence for any absolutely continuous initial distribution. Moreover, not much more can be expected: if $\nu = \delta_x$, then $S^t\nu = \delta_{S^t x}$, still a $\delta$ measure.

Now consider an open system [2],[3]. Our gas is in contact at $\partial\Lambda$ with a thermal reservoir at temperature $T$. Statistical mechanics says that the state of thermodynamic equilibrium (with the reservoir) should now be represented by the canonical ensemble $\mu_T \sim e^{-\frac{H}{kT}}\, d\mathbf{q}d\mathbf{p}$, where $H$ is the energy. Suppose we assume immediate thermalization upon contact with the boundary. Then we should require that after colliding with $\partial\Lambda$ the momentum of the outgoing particle be a random variable with a "Maxwellian distribution" at temperature $T$. This is the simplest way to model the effect of the boundary, and it gives rise to a Markov process $P^t$. It is easy to check that $\mu_T$ is stationary for $P^t$. Furthermore, we show that $\mu_T$ is the unique probability distribution stationary for $P^t$, and that an initial state described by any probability distribution (including $\delta$ measures) converges to $\mu_T$ in a very strong sense.

Finally, consider the case where the temperature varies along the boundary. The stochastic dynamics are as previously described, except that now the "Maxwellian" giving the velocity of a particle leaving $\partial\Lambda$ at $\vec{q} \in \partial\Lambda$ will be at $T(\vec{q})$, the temperature at that point. Here, though there is no obvious candidate for a stationary probability distribution, it seems clear physically that the system should settle down to a steady state of heat flow.



We further expect that this should be represented by a stationary probability distribution $\mu_s$ (the steady state) which represents this state. We show that this is indeed the case, that $\mu_s$ is equivalent to the Lebesgue measure, that it is unique, and that any other initial state converges to $\mu_s$ in a very strong sense.

We also hope that this steady state can be taken to the thermodynamic limit in such a way that a mathematical object (analogous to the Gibbs state for equilibrium phenomena) appropriate for the description of steady state non-equilibrium phenomena can be extracted and that this object will be reasonably independent of boundary conditions – i.e., the details of the stochastic mechanism on the boundary. For example, for a system confined between two walls, separated by a distance $L$, at temperatures $T_1$ and $T_2$ respectively, we might expect that the steady state $\mu_s^L$ should be given by

$$\mu_s^L = \mu_e + \left(\frac{T_2 - T_1}{L}\right) \mu_\infty + o\left(\frac{1}{L}\right),$$

where $\mu_e$ is the appropriate equilibrium (Gibbs) state and $\mu_\infty$ represents the thermodynamic limit of the steady state, e.g., it can be used to describe linear transport processes.

## 2. MAIN RESULTS

We now describe the main results more precisely and in greater detail. We first indicate exactly what happens at $\partial\Lambda$. When one particle collides with $\partial\Lambda$ the only effect is that the momentum distribution (on the set $\vec{p} \cdot \vec{n}(\vec{q}) \geqslant 0$) of the outgoing particle becomes proportional to

$$\vec{p} \cdot \vec{n}(\vec{q})\, e^{-\frac{\vec{p}^{\,2}}{2mkT(\vec{q})}}\, d\vec{p},$$

where $\vec{n}(\vec{q})$ is the inward directed unit normal vector to $\partial\Lambda$ at $\vec{q}$. All other coordinates are unaffected. If two or more particles collide with $\partial\Lambda$ simultaneously, we further require that their outgoing momenta be independent.

Let $H$ = kinetic energy + $V(\vec{q}_1, \ldots, \vec{q}_n)$, where

$$V(\vec{q}_1, \ldots, \vec{q}_n) = \sum_{i<j} u(|\vec{q}_i - \vec{q}_j|).$$

Suppose $u(r)$, $r \geqslant 0$, is smooth and repulsive: $u(r)$ is $C^\infty$, $u'(0) = 0$, and $u'(r) < 0$ for $r \neq 0$. (The conditions on $u$ guarantee, in particular, that $V$ gives rise to bounded forces, a fact which plays an important role in our argument.)

Suppose that $\Lambda$ is compact and convex with smooth boundary $\partial\Lambda$, and suppose the boundary temperature $T(\vec{q})$, $\vec{q} \in \partial\Lambda$, is smooth and nowhere vanishing. Let $\Omega = (\Lambda \mathrm{x} \mathbf{R}^3)^n$. Then there exists a subset $\hat{\Omega} \subset \Omega$ whose complement has vanishing Lebesgue measure, such that the Markov process $X(t) = (\mathbf{q}(t), \mathbf{p}(t))$, $\mathbf{q} = (\vec{q}_1, \ldots, \vec{q}_n)$, $\mathbf{p} = (\vec{p}_1, \ldots, \vec{p}_n)$, induced by $H$ and the stochastic boundary effect corresponding to $T(\vec{q})$, with $X(0) \epsilon \hat{\Omega}$,

(1) is well defined for all $t \geqslant 0$, and

(2) lives on $\hat{\Omega}$ for all $t \geqslant 0$:

For $x \in \hat{\Omega}$,

$$\mathrm{Prob}\left\{ X(t) \in \Omega - \hat{\Omega}, \text{ for some } t > 0 \mid X(0) = x \right\} = 0.$$

Moreover,

(3) the $\hat{\Omega}$-valued process $X(t)$ has a unique stationary probability measure, the "steady state" $\mu_s$, and $\mu_s$ is equivalent to the Lebesgue measure $m$:



Let $P^t(x,dy) = \mathrm{Prob}\{X(t) \epsilon dy \mid X(0) = x\}$. Then a measure $\nu$ is stationary if $\nu P^t = \nu$ for all $t \geqslant 0$, i.e.,

$$\int \nu(dx) P^t(x,dy) = \nu(dy) \quad \text{for all} \quad t \geqslant 0.$$

Furthermore,

(4) For $x \epsilon \hat{\Omega}$,

$$\delta_x P^t = P^t(x,dy) \longrightarrow \mu_s(dy) \quad (t \to \infty) \text{ in variation norm.}$$

Consequently, for any probability measure $\nu$ on $\hat{\Omega}$ we have

$$\nu P^t \equiv \int \nu(dx) P^t(x,dy) \to \mu_s \text{ in variation norm.}$$

In particular, for $\nu$ a stationary probability measure on $\hat{\Omega}$

$$\nu = \nu P^t \to \mu_s,$$

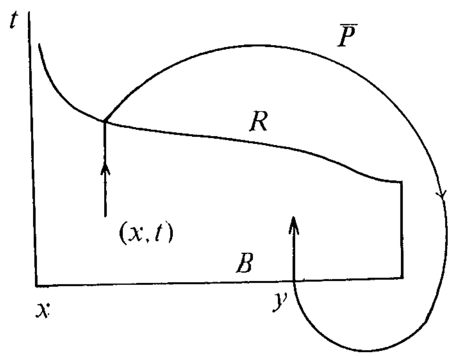
i.e., $\nu = \mu_s$, giving the uniqueness described in (3). ($\nu P^t \to \mu_s$ in variation norm means that $\sup_A |\nu P^t(A) - \mu_s(A)| \to 0$ $(t \to \infty)$.

## 3. SKETCH OF THE PROOF

(a) *Existence.* As far as the existence of the Markov process is concerned, note that since the deterministic part of the evolution is well defined, as is the stochastic part, the only way the process could break down would be by the occurrence of an infinite number of collisions with $\partial\Lambda$ in a finite amount of time. For this to occur at least one of the particles would have to undergo an infinite number of collisions in that time. But since the forces are bounded, if the particle leaves $\partial\Lambda$ in a direction bounded away from the tangential with a speed bounded away from 0 and $\infty$, its return time to $\partial\Lambda$ will be bounded away from 0 ($> \alpha$), uniformly in the rest of the coordinates. Thus in any infinite sequence of returns to $\partial\Lambda$, an infinite number of the return times will be greater than $\alpha$.

(b) *Special Flow Representation.* Markov chains (discrete time) are frequently easier to handle directly than continuous time processes. There is a way of extracting from our process a Markov chain in which all the stochastic behavior is nicely isolated from the deterministic motion. Using this Markov chain we construct a special "flow" representation of our process [4]:

Let us define the base $B$ of our system to be the set of phase points $x$ which have one or more particles on $\partial\Lambda$. Let $R(x)$, $x \epsilon B$, be the return time to $B$. Then, modulo a set of Lebesgue measure zero, $\Omega$ may be identified with the portion of $B \times \mathbf{R}^+$ ($\mathbf{R}^+ = \{\ t \epsilon \mathbf{R} \mid t \geqslant 0\}$) lying below the graph of $R$. A point $(x,t)$ in this set is identified with the phase point $S^t x$ – the phase point which was last in $B$ $t$ units ago at $x$. Our process assumes the form: flow directly "upward" with unit speed until the graph of $R$ is rached; then return to the base at a random point given by a stochastic kernel $\bar{P}$: $B \to B$.




The probability distribution of the "return" point $y$ is $\bar{P}(x,dy)$. Moreover $\bar{P} = UK$. Here $U$ is deterministic: for $x \epsilon B$, $Ux = S^{R(x)}x$, the point of deterministic first return to $B$, and $K$ is stochastic: it thermalizes the outgoing momenta of particles on $\partial\Lambda$.

We remark that there is a simple relationship between stationary

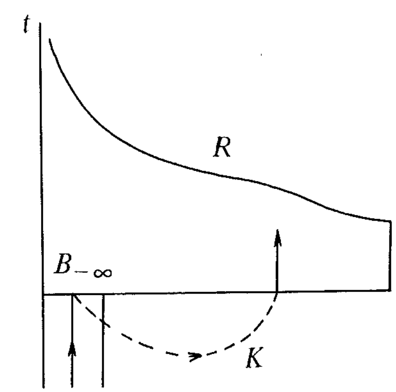


probability measures for the Markov chain on $B$ induced by $\bar{P}$ and stationary probability measures on $\Omega$ for $P^t$. If $\bar{\mu}$ (on $B$) is stationary for $\bar{P}$, then $d\bar{\mu} \times dt$ is stationary for $P^t$. Moreover, all measures stationary for $P^t$ are of this form. (Thus all questions concerning the existence and uniqueness of stationary probability measures for $P^t$ can be settled by considering the base process induced by $\bar{P}$. It turns out, however, that it is easier to establish the existence of nontrivial stationary probability measures directly for the full $P^t$ process.)

The last statement is not completely correct because the identification we have proposed for $\Omega$ is only a mod 0 identification. We have neglected points $x \epsilon \Omega$ for which either $S^t x \epsilon B$ for no $t$ or for no $t \leqslant 0$. For $x \epsilon B$, let $R^-(x)$ be the smallest $t > 0$ for which $S^{-t}x \epsilon B$. (If $S^{-t}x \notin B$ for any $t > 0$, we let $R^-(x) = \infty$. Similarly, if $S^t x \notin B$ for any $t > 0$, $R(x) = \infty$.) Let $B_\infty = \{x \epsilon B \mid \ R(x) = \infty\}$, $B_{-\infty} = \{\ x \epsilon B \mid R^-(x) = \infty\}$. Then $\Omega$ may be identified with

$$C \cup \left\{ (x,t) \in B \times \mathbf{R} \mid \begin{array}{ll} -\infty < t < R(x) & \text{for } x \in B_{-\infty} \\ 0 \leqslant t < R(x) & \text{otherwise} \end{array} \right\},$$

where $C = \{\ x \in \Omega \mid \ S^t x \in B \text{ for no } t \in \mathbf{R}\}$ is the set of phase points which never reach $B$. The structure of our process $P^t$ on the special representation of $\Omega - C$ is as before, except that when $B$ is reached from below, a random change governed by $K$ occurs.



Now it is easy to see that any extremal stationary measure $\mu$ either comes from a stationary measure $\bar{\mu}$ on $B$ for $\bar{P}$ or is supported by $C$. It is also easy to see that if $\mu$ is finite, $\bar{\mu}(B_{-\infty} \cup B_{+\infty}) = 0$.

(c) *Existence of a "nontrivial" stationary probability measure for $P^t$.* If $\Omega$ were compact, and if $P^t$ were continuous for all $t$ (in the sense that $P^t(x,\cdot)$ depends continuously on $x$, i.e.,

$$(P^t f)(x) \equiv \int P^t(x,dy)\, f(y)$$

is a continuous function of $x$ for every bounded continuous $f$), then we could use a fairly standard argument to obtain a stationary probability measure. Take any probability measure $\nu$ on $\Omega$ and let

$$\mu_\tau = \frac{1}{\tau} \int_0^\tau \nu P^t dt.$$

By compactness, $\mu_{\tau_j} \to \mu$, which by the continuity of $P^t$ must be stationary.

Unfortunately $\Omega$ is not compact and $P^t$ is not continuous. And, in general, if we attempt to discard the points of discontinuity of $P^t$ (assuming that they are few), what remains of $\Omega$ will be even less compact. $\Omega$ is not compact because it allows for arbitrarily large momenta. $P^t$ is not continuous because the stochastic boundary effect does not occur gradually: For $x \in \Omega$ , $P^t$ may fail to be continuous at $x$ if

(i) $R(x) = t$ ($R(x)$ is the hitting time for $B$),

or

(ii) the collision with $\partial\Lambda$ at time $R(x)$ is tangential and $t > R(x)$,

or

(iii) $x \in B$.

(i) presents no real problem. If we alter the topology on $\Omega$ by ignoring the momenta of particles $\in \partial\Lambda$, this case no longer gives rise to a



discontinuity. Case (ii) is more serious; however, our assumptions of convexity of $\Lambda$ and repulsiveness of the potential preclude tangential collisions. Case (iii), which is also serious, is handled using the long range repulsive forces. We omit the details here.

Even if the difficulties just described did not arise, the argument for the existence of a stationary probability measure would be seriously flawed. This is because the stationary measure whose existence it establishes could be trivial. For example, a phase point $x$ all of whose particles are at rest at the very same position is a fixed point for $S^t$ and $\delta_x$ is stationary. In general, there may be other stationary measures concentrated on $C$. We call such measures trivial. One consequence of the assumed repulsiveness is that $C \subset \{\vec{p}_i = 0, \text{all } i = 1, \ldots, n\} \equiv \hat{C}$. Thus, such triviality will be precluded if we can find a stationary probability measure on $\Omega - \hat{C}$, which is even less compact than $\Omega$. To push the argument through we isolate $\hat{C}$ and the point at $\infty$ in a controllable, uniform way. By this we mean that the measures $\mu_\tau$ should form a tight family of measures on $\Omega - \hat{C}$, i.e., we have to prove that for all $\varepsilon > 0$ there exists a compact subset $\Omega_\varepsilon$ of $\Omega - \hat{C}$ such that

$$\mu_\tau(\Omega_\varepsilon) > 1 - \varepsilon$$

for all $\tau$. If this can be shown, then $\{\mu_\tau\}$ is a compact family of measures and we can extract a subsequence converging to a stationary probability measure $\mu_s$ on $\Omega - \hat{C}$. In particular, $\mu_S(C) = 0$ and therefore for $\mu_S$ a.e. $x \in \Omega$, $S^t x \in B$ for some $t > 0$ (since $\bar{\mu}_s(B_{-\infty} \cup B_{+\infty}) = 0$).

The main ingredients in establishing tightness are the following:

(i) Since the forces are bounded, a particle with a large momentum will tend to move freely. Such particles will tend to rapidly hit $\partial\Lambda$ and give up most of their momentum. This enables us to control the large momenta.

(ii) The density for the probability distribution of the thermalized outgoing momentum of a particle of $x\epsilon B$ leaving $\partial\Lambda$ is bounded uniformly in $x$.

(iii) Since our potential is translation invariant, the total momentum $\vec{P} = \vec{p}_1 + \ldots + \vec{p}_n$ of our system will be conserved until $B$ is reached. It follows that $R(x) \leqslant \frac{b}{|\vec{P}|}$, where $b$ is a constant, independent of $x$. Thus, using (ii)

$$\int_{|\vec{P}|<\varepsilon} \bar{P}(x,dy)\, R(y) \leqslant b \int_{|\vec{P}|<\varepsilon} \frac{\bar{P}(x,dy)}{|\vec{P}(y)|}$$

is small, uniformly in $x$. Also, $\{ |\vec{P}| < \varepsilon \}$ is a neighborhood of $\hat{C}$.

(iv) For $\delta$ small $\hat{C}_\delta \equiv \{ |\vec{p}_i| < \delta, i = 1, \ldots, n \} \subset \{ |\vec{P}| < \varepsilon \}$ .

Moreover, starting from $x \epsilon B$, the hitting time of $\hat{C}_\delta$ can be uniformly bounded away from 0 for a large (with respect to the thermalized momentum distribution) set of values of the momentum of the particle leaving $\partial\Lambda$, uniformly in $x$. (iii) and (iv) allow us to control the small neighborhoods $\hat{C}_\delta$ of $\hat{C}$.

(d) *The Harris Condition.* Having established the existence of a nontrivial stationary probability distribution $\mu_s$, we use the following condition very strongly in establishing most of the remaining results. For "any" $x$

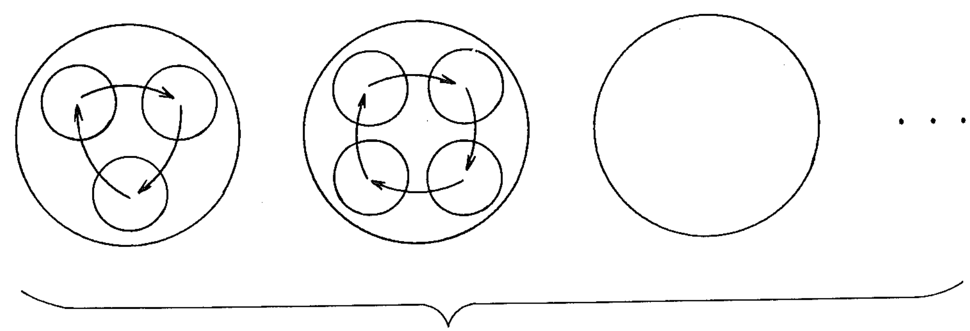
(*) there exists a $t > 0$ such that $P^t(x,\cdot)_{\text{a.c.}} \neq 0$.

Here $P^t(x,\cdot)_{\text{a.c.}}$ denotes the absolutely continuous component of the measure $P^t(x,.)$. Very roughly, (*) is proven as follows: each collision with $\partial\Lambda$ produces spreading in 3-dimensions. Thus after $2n$ such collisions we



should have spreading in all the $6n$-dimensions. of $\Omega$ and thus have an absolutely continuous component. Part of the difficulty in making this rigorous arises because the argument assumes that the spreading produced by collision with $\partial\Lambda$ will be independent of the spreading which has already occurred. Since we can't solve the equations of motion exactly, it is difficult to keep track of exactly what spreading occurs. One of the main ingredients in carrying the argument through is the fact that particles with large momenta will move almost freely. Thus, by always having the particle at $\partial\Lambda$ leave with an appropriately large momentum, we gain sufficient control of the sequence of collisions with $\partial\Lambda$ , and the effects there of, which ensues.

Let us indicate in a bit more detail what (*) is good for. Let $P = P^1.P$ is the transition kernel for a Markov chain on $\Omega$, with stationary probability measure $\mu_s$. For such a Markov chain, with $\mu_s$ absolutely continuous, the satisfaction of (*) by $\mu_s$ a.e. $x$ (with $t \epsilon \mathbf{Z}$) is equivalent to the "Harris condition". In such a case we have a Harris process [5], for which we have the following picture:



at most a countable number of ergodic (irreducible) components

Moreover, if the chain is ergodic (only one component) and aperiodic (no cycles), there exists a set $\Omega_0$, with $\mu_s(\Omega_0) = 1$, such that for $x \epsilon \Omega_0$,

$P^n(x,\cdot) \to \mu_s(\cdot)$ in variation norm.

Note that since our Harris process is embedded in a continuous time process $P^t$, it must be aperiodic.

Now if $P^t(x,\cdot)$ is to grow an absolutely continuous component it is clear that $S^t x$ must eventually hit $B$; this is assured for $x$ with total momentum $\vec{P} \neq 0$. But this alone is obviously not sufficient, since if $x$ contains two or more *coincident* particles (i.e., having the same position and momentum), these coincident particles cannot be parted until they themselves reach $\partial\Lambda$. Let us denote the submanifold formed by the phase points $x$ containing coincident particles by $\Omega_c$(the coincidence submanifold). For $x \in \{R < \infty\} - \Omega_c$, it may be shown that (*) is indeed satisfied. Since the Lebesgue measure

$$m(\{\vec{P} = 0\} \cup \Omega_c) = 0,$$

it follows that (*) is satisfied for $m$ a.e. $x$. Thus the Harris condition will be proven once the absolute continuity of $\mu_s$ has been established.

(e) *Absolute Continuity.* Let us write $\mu$ for $\mu_s$. Then $\mu_{\text{a.c.}} + \mu_{\text{sing}} = \mu = \mu P^t = \mu_{\text{a.c.}} P^t + \mu_{\text{sing}} P^t$. The first term on the right is absolutely continuous and the second term will have an absolutely continuous component for sufficiently large $t$ unless $\mu_{\text{sing}}$ is concentrated on the set of $x$ for which (*) fails. Note that the above equation implies that $\mu_{\text{a.c.}} = \mu_{\text{a.c.}} P^t$ and $\mu_{\text{sing}} = \mu_{\text{sing}} P^t$. We conclude that unless $\mu$ is singular, $\mu_{\text{a.c.}}$ provides us with a stationary absolutely continuous probability measure, and that the singular part of $\mu$ must be concentrated on $\Omega_c$, since we have shown that $\mu_s(R = \infty) = 0$. Now $\Omega_c$ may be regarded as the phase space of a system of $n-1$ particles similar to the one we are investigating, except that one of the particles will have twice the mass of the others, and the pair potential involving this heavy particle will be $2u$ rather than $u$. Moreover, the stochastic evolution on $\Omega_c$ viewed this way will agree with the stochastic evolution on $\Omega_c$ regarded as a subset of $\Omega$, until the heavy particle hits $\partial\Lambda$. We may decompose $\Omega$ into invariant sets $\Omega_c^{(i)}$, $i = 1, \ldots, n-1$ according to which particle is heavy. All the arguments and results for the stochastic evolution on $\Omega$ will be valid for the $n-1$ particle stochastic evolution on $\Omega_c^{(i)}$. In particular, we will soon show that if $\mu_s$ is absolutely continuous, then it is equivalent to the Lebesgue measure and the Markov process starting from $\mu_s$ is ergodic. This would imply that every particle must eventually hit $\partial\Lambda$ (with probability 1). Similarly, if the $n-1$ particle process on $\Omega_c^{(i)}$ has an absolutely continuous (with respect to the Lebesgue measure on $\Omega_c^{(i)}$) stationary probability measure, every particle, in particular the heavy one, must hit $\partial\Lambda$. But when this happens, the phase point will leave $\Omega_c$ under the $n$-particle stochastic evolution, for which this probability measure will therefore not bestationary. Thus $\mu_{\text{sing}}$ must be concentrated on $(\Omega_c)_c$, the coincidence submanifold of $\Omega_c$. Iterating the above argument, we see that $\mu_{\text{sing}}$ must be concentrated on the minimal coincidence submanifold in which all particles are at the same position and have the same momentum. But this is impossible. Thus $\mu_s$ is absolutely continuous.

(f) *Ergodicity and Equivalence to Lebesgue Measure.* We give a sketch of an argument, which though not completely correct as formulated here, captures the flavor of the correct argument. By an analysis similar to the one which we use to establish (*), we may show that if we discard $\{\vec{P} = 0\} \cup \Omega_c$, all sets invariant for our Markov process are open. Since $\Omega - (\{\vec{P} = 0\} \cup \Omega_c)$ is connected, the results follow.

## 4. CONCLUDING REMARKS

We have stated our results for particles moving in a 3 dimensional space, but everything should apply verbatim in any dimension greater than one. For $\Lambda \subset \mathbf{R}$, however, our arguments break down in several places, though we believe that our results are valid in this case, too.

As far as our other assumptions are concerned, we wish to emphasize that most of these are unnecessary when the temperature is constant on $\partial\Lambda$. We also believe that our results will remain true even if $\Lambda$ is not convex (but just connected), and our potential is not repulsive. In the case $u = 0$, i.e. the ideal gas, our arguments do not all apply directly. However in this case all our results follow from the observation that the $n$-particle process may be factored into $n$ independent single particle processes, for which our arguments easily apply.

Finally, we remark that as a consequence of the result (4), we obtain that the time shift on trajectories of our process, starting in the steady state, defines a Bernoulli flow [6].

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