# OPTIMAL INVESTMENT POLICY FOR A REGULATED FIRM

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## ABSTRACT

We investigate the optimal capital investment policy, when disinvestment is prohibited, of a model regulated firm, constrained to meet a given (inelastic) demand. The firm seeks to minimize the present worth of total costs. We show that demand in the future enters the optimal policy primarily in determining whether or not the firm is to undertake new investment; once the decision to invest is made, the optimal amount, when the retirement table is exponential, can be determined myopically, i.e., on the basis of only the current demand. We also show that inflation induces the firm to increase its current investment, so as to offset the effect of higher prices to come. The argument uses only simple variational methods, having a natural economic interpretation.

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## 1.  Introduction

In this paper, we investigate the optimal capital invest-
ment policy of a model regulated firm.  Our firm differs from
a nonregulated firm in two ways:  (a) it is constrained by its
franchise to meet a certain price-inelastic demand for its
services or output at each time t, and (b) the revenues R(t)
it collects from its customers are regulated.  The regulated
revenues consist of a certain return on the firm's invested
capital, depreciation of that capital, taxes, and the cost of
labor.  With its production function given, the firm can meet
demand by employing a variety of combinations of labor, con-
tracted for period by period, and capital equipment; the firm's
stock of capital equipment at any time depends on the firm's
prior investment policy.  We express the latter by choice of

a (cumulative) construction budget $C(t)$. We also assume irreversibility of investment, i.e., the firm cannot divest itself of capital equipment. Thus $C(t)$ is a monotone non-decreasing function of time. (This is a point of much significance for the mathematical analysis which follows.) We do, however, permit discontinuous increases in capital.

If the firm views itself as being owned by its customers, a reasonable objective for it is to choose a construction budget $C(t)$ which will minimize the present worth of the revenues it has to collect.

We investigate this minimization problem, considering especially informational requirements and the effects of inflation. We owe our sense of the importance of the former to Arrow (1968), who investigated a problem of similar type in a formulation which placed somewhat greater restriction on the mathematical form of the firm's objective function. Our present treatment, which incorporates arbitrary capital inflation and depreciation, is also more elementary, in relying on simple variational arguments rather than on the Pontryagin principle.

Our principal conclusions are, firstly, that the course of future demand determines whether or not the firm undertakes new investment; once the decision to invest has been made, then, with an exponential retirement schedule, the optimal amount can be determined myopically, i.e., on the basis of only the current demand. In sum then, we can perhaps describe this as a "world enough and time" model: if the

firm has enough of a future, it can ignore it, but not otherwise. Secondly, we show that inflation in the price of capital goods and wages induces the firm to increase its current investment so as to offset, to some extent, the effect of the higher prices to come; this phenomenon has no connection with the Averch-Johnson effect (1962).

## 2. Formulation of Problem

Our firm is assumed to provide products or services so essential that the demand for them, $S(t)$, is completely inelastic. Thus we take $S(t)$ as given once and for all. The regulator adjusts prices to produce the revenues $R(t)$ required by the firm. These required revenues are determined so as to cover: an allowed rate of return $\rho$ on a rate base $X$, income taxes $T$, book depreciation $D$, and labor costs $wL$, where $w$ is the wage rate and $L$ is the amount of labor employed. (We have suppressed in our notation the time dependence of these quantities.) Our model firm thus collects from its customers, per unit time, the revenues (Linhart, 1970)

$$R = \rho X + D + T + wL. \tag{2.1}$$

Taxes are given by

$$T = \theta(R - wL - D_T - i\nu X),$$

where $\theta$ is the income tax rate

$D_T$ is the tax depreciation

i is the interest rate

$\nu$ is the debt ratio;

that is, wages, depreciation, and interest payments are tax deductible. If, for simplicity, we take book and tax depreciation to be identical, $D_T = D$, then it follows from (2.1) and (2.2) that

$$R = aX + D + wL, \qquad (2.2)$$

where $a \equiv (\rho - \theta i \nu)/(1-\theta)$ is the pre-tax rate of return.

The objective of the firm at t=0 is to minimize the present worth of $R(t)$, subject to the constraint that the given demand $S(t)$ be met. To produce $S(t)$, given its production function, the firm can choose to employ various combinations of capital and labor. This choice, which expresses itself through the firm's investment policy, will affect $R(t)$, and we wish to make it in such a way as to minimize the present worth of $R(t)$.

The capital entering the firm's production function is the economic or productive capital, which we shall denote by Z. This may differ substantially, for various reasons, from the rate base X. We shall be particularly interested in the difference between Z and X which occurs

as a consequence of inflation. This comes about in the
following way: in most U.S. jurisdictions the rate base X
is closely related to the total original cost of all vin-
tages of plant, less depreciation (Kahn, 1970, p. 40 and
footnote 44; Bauer, 1966, Chapter 4). Original cost is
also taken as the basis for depreciation charges. On the
other hand the price, $p(t)$, of each additional unit of
physical plant increases as inflation proceeds. (Techno-
logical change may be occurring simultaneously; this will
be expressed by the explicit appearance of time in the
production function.)

Economic capital $Z(t)$ and rate base $X(t)$ are deter-
mined by the construction budget, and the price level of
capital goods $p(t')$, for $t' \leq t$. In particular, let $C(t)$
denote the total amount spent on construction up to and
including time t; we will assume that investment is
irreversible, so that $C(t)$ is nondecreasing. Let $t=0$ be
the time when the firm was established, and suppose price
level to be normalized so that $p(0)=1$. Then, since
$dC(t) \geq 0$ represents the amount of new construction at t
we obtain[1]

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[1]If $C(t)$ changes smoothly at t, we have $dC(t)=C'(t)dt$,
and we can describe the quantities of interest directly
in terms of $C'(t)>0$, the rate of new construction. We
employ the formula given in order to allow $C(t)$ the
necessary freedom to be discontinuous.

$$X(t) = X(0)q(t) + \int_0^t q(t-y)dC(y), \tag{2.3}$$

$$Z(t) = Z(0)f(t) + \int_0^t f(t-y) \frac{1}{p(y)} dC(y), \tag{2.4}$$

and

$$D(t) = X(0)s(t) + \int_0^t s(t-y)dC(y). \tag{2.5}$$

Here $q(t)$ is the fraction of an initial dollar of net plant still undepreciated at age t, $f(t)$ is the fraction of an initial unit of physical plant still unretired at age t, both assumed to be independent of vintage, and $s(t)$, the unit depreciation schedule, is given by

$$s(t) = -dq/dt. \tag{2.6}$$

Since $p(0)=1$, $Z(0)=X(0)$ is the amount of plant with which the firm started.

The output $S(t)$ for given inputs of labor $L(t)$ and productive capital $Z(t)$ is determined by the firm's production function Q:

$$Q(L(t),Z(t);t) = S(t). \tag{2.7}$$

We assume that each isoquant (2.7) is convex, so that on it L is a decreasing function of Z, and dL/dZ is an increasing one, i.e.

$$\frac{dL}{dZ} < 0, \quad \frac{d^2L}{dZ^2} > 0. \tag{2.8}$$

Let us note also that if Z and L vary over the isoquant curve (2.7), then necessarily

$$\frac{dL}{dZ} = - \frac{\partial Q/\partial Z}{\partial Q/\partial L} \equiv - \frac{Q_2[L,Z;t]}{Q_1[L,Z;t]}. \tag{2.9}$$

In view of (2.2), the present worth of R(t) now becomes

$$\bar{R} \equiv \int_0^\infty \alpha(t)[a(t)X(t)+D(t)+w(t)L(t)]dt,$$

where $\alpha(t)$ is the present value, referred to t=0, of a unit expenditure at time t. To make the dependence of $\bar{R}$ on C(t) somewhat more explicit, we can substitute (2.3) and (2.5) and exchange the order of integration for the first two components of $\bar{R}$, obtaining

$$\bar{R} = X(0)M(0) + \int_0^\infty M(y)dC(y) + \int_0^\infty \alpha(t)w(t)L(t)dt. \tag{2.10}$$

Here

$$M(y) \equiv \int_y^\infty \alpha(t)[a(t)q(t-y)+s(t-y)]dt$$

represents the "cost," i.e., the contribution to the first two components of $\bar{R}$, of a unit of investment at t=y. C(y) also determines L by means of (2.4) and (2.7). Our goal

is to minimize $\bar{R}$ by choice of a suitable $C(t)$, subject
to the constraint of irreversible investment

$$dC(t) \geq 0. \qquad (2.11$$

We shall have occasion to discuss the consequences of
(2.11) later.

The problem of raising the money needed for new con-
struction through external sources (sale of bonds and shares)
and internal sources (retained earnings and depreciation) will
not be discussed here explicitly. It can enter indirectly by
affecting the debt ratio $\nu$ and possibly also the interest rate
i, and thus the pre-tax rate of return a. We shall, however,
not consider any dependence of a on the construction budget.

3. Solution of Variational Problem

The natural way to determine the infimum of $\bar{R}$ is to
show that the minimizing value is attained by some con-
struction budget, and to characterize the optimal budget
by means of a variational argument. We do not quote the
proof of the first step here; to carry out the second, let
us consider the effect, for a particular construction bud-
get, of a marginal change in $dC(t)$ of $\varepsilon\Delta$ over the interval
$\tau-\Delta/2 \leq t \leq \tau+\Delta/2$. This variation alters the rate base
and economic capital; the former affects the cost, the

latter the wage bill.  In view of (2.4), (2.7) and (2.9) we find that, to first order in $\delta Z$, for $t \geq \tau$.

$$\delta Z = \epsilon \Delta^{-1} \int_{\tau - \Delta/2}^{\tau + \Delta/2} f(t-y) \frac{1}{p(y)} dy$$

$$\delta L = \frac{\delta L}{\delta Z} \delta Z = - \frac{Q_2}{Q_1} \delta Z,$$

so that by (2.10)

$$\delta \bar{R} = \epsilon V(\Delta) \tag{3.1}$$

where

$$V(\Delta) = \Delta^{-1} \int_{\tau - \Delta/2}^{\tau + \Delta/2} M(y) dy + \int_0^\infty \alpha(t) w(t) \frac{dL}{dZ}(t) \frac{\delta Z}{\epsilon} dt$$

$$= \Delta^{-1} \int_{\tau - \Delta/2}^{\tau + \Delta/2} dy \left\{ M(y) + \int_y^\infty \alpha(t) w(t) \frac{dL}{dZ}(t) f(t-y) \frac{1}{p(y)} dt \right\} .$$

Thus if $V(\Delta) < 0$, (3.1) shows that, by a choice of $\epsilon > 0$, $\bar{R}$ can be decreased without violating the requirement that $dC(t) \geq 0$; if $V(\Delta) > 0$ and $dC(t) > 0$ in some neighborhood of $\tau$, a choice of $\epsilon < 0$ with $|\epsilon|$ sufficiently small will have the same effect.  Since $\Delta$ can be made arbitrarily small, the sign of $V(\Delta)$ is given by that of

$$J(y) \equiv M(y) + \int_y^\infty \alpha(t) w(t) \frac{dL}{dZ}(t) f(t-y) \frac{1}{p(y)} dt. \tag{3.2}$$

The integral in this expression represents the present value of the change in wages produced by an additional unit of construction expenditure at time y, for this converts into $1/p(y)$ of equipment, of which $f(t-y)/p(y)$ survives at time t, and modifies the amount of labor required by $\frac{dL}{dZ}(t)f(t-y)/p(y)$. Thus $J(y)$ measures the difference between cost and benefit, and so the total net effect on $\bar{R}$, of a marginal change in construction expenditure at time y. Since, by (2.8), $J(y)$ varies mono-tonically with marginal changes in construction at $t=y$, we conclude that for the optimal construction budget $C(t)$, either

$$J(y) = 0, \tag{3.3}$$

or

$$dC(y) = 0 \text{ and } J(y) > 0, \tag{3.4}$$

for each instant y. Of these, (3.3) represents the desired state (henceforth denoted by a subscript D); (3.4) corresponds to the next best case when, due to our constraint (2.11), condition (3.3) cannot be attained.

We note certain consequences which flow from this variational characterization. Firstly, if the discount and inflation measures $\alpha(t)$, $w(t)$ and $p(t)$ are assumed known for all t, the integral equation (3.3) can be solved for $\frac{dL}{dZ}(t)$, and thus fixes the slope of the iso-quant at the optimum operating point. Moreover, if, as

is often reasonable, we choose the life table $f(t)$ to be an exponential, (3.3) can be differentiated to obtain an ordinary equation for $\frac{dL}{dZ}$. (See (3.12).) Since the demand $S(t)$ prescribes the choice of isoquant, the desired amount of economic capital $Z_D(t)$ is now likewise determined, using no future values of $w(t)$ and $p(t)$.

The most interesting feature here is that future values of demand play no role in this calculation. In turn, past values of $Z_D(s)$, $s \leq t$, determine the desired construction budget $C_D(t)$. If the $C_D(t)$ so found is nondecreasing, it evidently defines the optimal construction budget. Thus we conclude that as soon as demand is sufficient to keep $C_D(t)$ from falling, the optimum budget can be determined independently of future demand, hence is myopic in Arrow's terminology (1964). However, should demand slacken sufficiently that the $C_D(t)$ it defines in conjunction with (3.3) decreases in certain periods, we are prevented by (2.11) from following $C_D(t)$, and the alternative (3.4) will come into play in determining optimum investment. In such a situation, since we are not permitted to divest ourselves of capital as the best policy dictates, but can only stop construction, the best stopping time will generally come prior to that when $C_D(t)$ first decreases. Put differently, since we cannot recover costs by selling superfluous equipment, we are

better advised to acquire less of it to begin with.  In
such a case, choosing the correct times at which to cease
construction will require watching the behavior of $C_D(t)$
over a certain future horizon.

## Capital Intensivity

Let us next, keeping the various discount and life
schedules the same, consider the effect of inflation in
factor prices on the optimal construction budget.  If
the rates of inflation for labor and capital are unequal,
then the firm relies ever more heavily on the factor whose
price rises more slowly.  We do not consider this case
further but assume henceforth that the inflation rates of
labor and capital are equal.   We therefore now consider
two inflation regimes, designated by subscripts 1 and
2, such that

$$\frac{w_1(t)}{p_1(t)} = \frac{w_2(t)}{p_2(t)} = r(t). \qquad (3.5)$$

Then Eq. (3.3) dictates, for the two situations,

$$\int_y^\infty \alpha(t)r(t)\,\frac{p_1(t)}{p_2(y)}\left[\frac{dL_1}{dZ_1}(t)\right]f(t-y)dt = -M(y), \qquad (3.6)$$

$$\int_y^\infty \alpha(t)r(t)\,\frac{p_2(t)}{p_2(y)}\left[\frac{dL_2}{dZ_2}(t)\right]f(t-y)dt = -M(y). \qquad (3.7)$$

If we take $p_2(t)=p_1(t)e^{\beta t}$, with $\beta>0$, the known part of the integrand of (3.7) differs from that of (3.6) by the factor $e^{\beta(t-y)}\geq 1$; since the right-hand sides are the same, we can expect the solutions to satisfy $\left|\frac{dL_2}{dZ_2}(t)\right|\leq\left|\frac{dL_1}{dZ_1}(t)\right|$.

Thus at the optimal operating point the isoquant is flatter when the inflation rate increases, so that by (2.8) the firm becomes more capital-intensive. An explanation is that, with (3.5), there is no inducement for the firm to make instantaneous changes in the capital labor ratio, but there remains an incentive to shift some of the investment to an earlier time so as to take advantage of the lower price.

Example

To illustrate these general considerations, we now specialize the problem, first by choosing an exponential retirement function

$$f(t) = \begin{cases} e^{-\lambda t}, & t>0 \\ 0, & t\leq 0, \end{cases} \qquad (3.8)$$

then by letting the discount rate and rate of return be constant

$$\alpha(t) = e^{-\sigma t}, \tag{3.9}$$

$$a(t) = a,$$

and finally by choosing for Q the Cobb-Douglas production function

$$Q(Z,L;t) = \mu_1 L^{\mu_2} Z^{\mu_3}, \tag{3.10}$$

with the $\mu_i$ possibly depending on t.

As a consequence of (3.8), (3.3) becomes

$$\int_y^\infty \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] e^{-\lambda t} dt = -e^{\lambda y} p(y)M(y), \tag{3.11}$$

and if this is true in the neighborhood of some point $y$, differentiation shows that necessarily

$$e^{-\lambda y}\alpha(y)w(y) \left[\frac{dL}{dZ}(y)\right] = \frac{d}{dy}\left[e^{-\lambda y} p(y)M(y)\right],$$

whence

$$\frac{dL}{dZ}(y) = \frac{1}{\alpha(y)} \frac{p(y)}{w(y)} \left[M'(y) + \frac{p'(y)}{p(y)} M(y) - \lambda M(y)\right]; \tag{3.12}$$

here, to be consistent with (2.8), it is clearly necessary that the right hand side of (3.12) be negative; therefore we restrict the rate of inflation by requiring that $p'(y)/p(y)$ never exceed $\lambda - M'(y)/M(y)$. Thus we observe explicitly the

two phenomena described earlier: that the desired slope $\frac{dL}{dZ}$ (y) can be determined myopically, independently of the future course of inflation or demand, and that the effect of an increase in the capital inflation rate $p'(y)/p(y)$ in the presence of (3.5) increases the (negative) slope $\frac{dL}{dZ}$, hence results in greater capital intensivity.

In passing, let us add the following fuller explanation of the myopic character of the desired $\frac{dL}{dZ}$ (y). Suppose that for an increasing demand $S_1(t)$ equation (3.11) has been solved as in (3.12). Suppose now that $S_2(t)$ coincides with $S_1(t)$ for $t \leq T$ but becomes identically zero thereafter. Since it serves no purpose here to produce after $t=T$, the left-hand side of (3.11) now becomes $\int_y^T \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] e^{-\lambda t}dt$, a quantity distinctly smaller in absolute value than before, while the right-hand side remains unchanged, and it may seem surprising that the same solution serves for both equations. We can resolve this puzzle by noting that if $\frac{dL}{dZ}$ (t) continues to be defined by (3.12) for $t \leq \tau < T$, but is chosen in the remaining interval $\tau < t < T$ so that

$$\int_\tau^T \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] e^{-\lambda \tau}dt = -e^{-\lambda \tau}p(\tau)M(\tau), \tag{3.12a}$$

then

$$\int_y^T \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] e^{-\lambda t}dt = \int_y^\tau + \int_\tau^T ,$$

and evaluating these two integrals by (3.12) and (3.12a)
respectively, we find

$$\int_y^T \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] e^{-\lambda y}dt = -e^{-\lambda y}p(y)M(y),$$

as (3.11) dictates. However close τ may come to T, the
requirement (3.12a) can be met by making $\frac{dL}{dZ}(t)$ sufficiently
negative in τ<t<T, that is by lowering the capital stock,
possibly very rapidly. Put in other words, the firm in the
second case can act as though the future were perfectly
secure, with the idea of rectifying any resulting miscal-
culation by a sharp change in policy at the very end. If,
however, capital divestiture is prohibited, this strategy
is excluded; the correct solution, which is no longer myopic,
is discussed in Section 4.

Under the additional hypothesis (3.9),

$$M(y) = g(\sigma)e^{-\sigma y}$$

where, by (2.6) and an integration by parts,

$$g(\sigma) = \int_0^\infty e^{-\sigma t} [aq(t)+s(t)]dt$$

$$= \int_0^\infty e^{-\sigma t}s(t)dt + \frac{a}{\sigma}\left[1 - \int_0^\infty e^{-\sigma t}s(t)dt\right] .$$

We may interpret $g(\sigma)$ as the present value, at discount rate $\sigma$, of a unit of investment in equipment, which returns the stream of depreciation payments $s(t)$, while the unde-preciated investment, having present value $\left[1-\int_0^\infty e^{-\sigma t}g(t)dt\right]$, earns the rate of return a. Therefore, (3.12) becomes

$$\frac{dL}{dZ}(t) = - g(\sigma) \frac{p(t)}{w(t)} \left[(\lambda+\sigma) - \frac{p'(t)}{p(t)}\right].$$ (3.1 )

For the Cobb-Douglas production function, on the isoquant

$$\mu_1 L^{\mu_2} Z^{\mu_3} = S(t),$$

$$\frac{dL}{dZ} = - \frac{\mu_3}{\mu_2} \frac{L}{Z} = - \frac{\mu_3}{\mu_2} \frac{[S(t)/\mu_1]^{\frac{1}{\mu_2}}}{Z(t)^{\frac{\mu_2+\mu_3}{\mu_2}}} ,$$

so that $Z_D$, the desired solution of (3.3), is described explicitly by

$$Z_D(t) = \left\{\frac{\mu_3}{\mu_2} \frac{[S(t)/\mu_1]^{\frac{1}{\mu_2}}}{g(\sigma) \frac{p(t)}{w(t)} \left[(\lambda+\sigma) - \frac{p'(t)}{p(t)}\right]}\right\}^{\frac{\mu_2}{\mu_2+\mu_3}} ,$$ (3.1 )

with $\mu_1$ possibly depending on t. Finally, recalling (2.4), we see that the corresponding construction budget $C_D(t)$ is given by

$$dC_D(t) = e^{-\lambda t}p(t)d[Z_D(t)e^{\lambda t}].$$ (3.1 )

This rule determines the variation, possibly discon-
tinuous, of $C_D(t)$, but it does not define the starting
value $C_D(0+)$. If

$$\frac{d}{dt}\left[Z_D(t)e^{\lambda t}\right] \geq 0,$$

(3.1

as will be true even in a steady-state condition of con-
stant $Z_D(t)$, then if we start with the capital stock $Z_D(0+)$
defined by (3.14), the construction budget (3.15) maintains
equipment at the level $Z_D$ and hence is optimal. However,
the firm's initial plant $Z(0)$ is specified. If $Z(0)<Z_D(0+)$,
we can attain the required optimum by letting $C(t)$ jump
instantaneously from its value of 0 at 0 to $Z_D(0+)-Z(0)$
at t=0+.[2] This immediately puts us on $Z_D(t)$, where we
can remain, without violating the constraint (2.11). If
however, the initial plant $Z(0)$ exceeds the desired value
$Z_D(0+)$, the constraint prevents us from moving to $Z_D$ imme-
diately. All we can do is cease new investments until the
time $\tau$ when deterioration of initial stock brings us to $Z_D$,
and stay on $Z_D$ thereafter. This time can evidently also be
determined myopically, depending as it does only on knowing
the value of $Z_D(\tau)$.

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[2] It is for the sake of this possibility that we described
construction in terms of the cumulative budget $C(t)$,
rather than by means of the construction rate $C'(t)$, for
in the latter formulation this optimal policy could only
be approximated, but not attained.

Next we discuss the case in which (3.16) is violated.

## 4.  Irreversibility of Investment

To observe the effect of a violation of (3.16), let us again envision the extreme case in which (3.16) is satisfied for all t less than some time $T_0$, but, perhaps because of new technology or fashion, the demand $S(t)$ suddenly drops to zero for $t \geq T_0$. To avoid extraneous complication, let us suppose that for $Z_D(t)$ defined by (3.14), $Z_D(0+)=Z(0)$; this ensures that $Z_D(t)$ describes the desired capital stock and $C_D(t)$, as defined by (3.15) with $C_D(0+)=0$, the desired construction budget. However, $Z_D(t)$ becomes 0 for $t \geq T_0$, while $C_D(t)$ calls for instantaneous liquidation of all plant at $t=T_0$. Thus, because of the constraint (2.11), optimal construction cannot everywhere coincide with $C_D(t)$.

The dichotomy we have established in (3.3) and (3.4) shows that, for the optimal policy, the only alternative to following $C_D(t)$ is to cease new construction, whereupon $C(t)$ remains constant. Let us assume that this happens first during an interval $\tau_1 < t < \tau_2$: $C(t)=C_D(\tau_1)$ for $\tau_1 < t < \tau_2$, while for some time after $\tau_2$, we again follow the unconstrained path, i.e. $C(t)$ may jump at $\tau_2$ so that $C(\tau_2)=C_D(\tau_2)$. It is easy to see that an interval where this happens

cannot be one over which $C_D(t)$ steadily increases, for then we can replace the constant $C(t)$ by $C_D(t)$ in the interval without violating our constraint (2.11), and thereby lower $\bar{R}$. Since in the present case $C_D(t)$ increases until $t=T_0$, the interval in which $C(t)$ is constant must have the form $t \geq T$, i.e., we stop new construction at $t=T$ and bear the cost of existing plant (depreciation and return on capital) until it has completely depreciated. The remaining problem is to determine the value of T.

Suppose that, as in (3.1), we contemplate the effect of increasing construction marginally at $t=T$, and thereby marginally increasing T. The unit cost of additional investment is $M(T)$, as before, but the corresponding change in the wage bill is now

$$\int_T^{T_0} \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] f(t-T) \frac{1}{p(T)} \, dt,$$

with

$$Z(t) = Z(0)f(t) + \int_0^T f(t-y) \frac{1}{p(y)} \, dC_D(y), \quad T \leq t \leq T_0. \tag{4.1}$$

Here we have modified the expression appearing in (3.2) by using explicitly the assumption that construction stops at T (so that subsequent plant is determined), and production stops at $T_0$. The value of T is now determined by the

requirement that cost and benefit balance, i.e., by the
equation

$$M(T)p(T) + \int_T^{T_0} \alpha(t)w(t) \left[\frac{dL}{dZ}(t)\right] f(t-T)dt = 0. \qquad (4.?$$

The integrand here is fixed, since plant size and isoquant
are prescribed by (4.1) and (2.7) respectively, so that, in
contrast to the case of (3.11) where the equation determines
an unknown function, (4.2) here determines T as the (single)
zero of a given function of T. However, this function can
no longer be computed myopically, for, although (4.1) uses
data only from the past, the value of $T_0$, as well as $S(t)$
for $T \leq t \leq T_0$, enters (4.2) in an essential way.

To illustrate this let $f(t)$, $\alpha(t)$, and $a(t)$ be as in
(3.8) and (3.9), let Q be a Cobb-Douglas production function
with constant $\mu_i$, suppose wage and capital inflation rates
to be constant,

$$w(t) = w_0 e^{I_L t},$$

$$p(t) = e^{I_k t}, \quad I_k \leq \lambda + \sigma, \qquad (4.3$$

and finally, take demand to be exponential for $t < T_0$,

$$S(t) = \begin{cases} d_0 e^{\gamma t}, & t \leq T_0 \\ 0, & t > T_0, \end{cases} \qquad (4.?$$

with $\gamma/\mu_2 \geq I_k - I_L$. Thereupon, by (3.14),

$$Z_D(t) = \begin{cases} c_1 e^{\delta t}, & t \leq T_0 \\ 0, & t > T_0 , \end{cases} \tag{4.5}$$

with

$$c_1 = \left[ \frac{\mu_3}{\mu_2} \frac{w_0}{g(\sigma)(\lambda+\sigma-I_k)} \left(\frac{d_0}{\mu_1}\right)^{1/\mu_2} \right]^{\frac{\mu_2}{\mu_2+\mu_3}} , \tag{4.6}$$

$$\delta = \left( \frac{\gamma}{\mu_2} + I_L - I_k \right) \frac{\mu_2}{\mu_2+\mu_3} .$$

(We assume here for simplicity that $c_1 = Z(0+)$.) If construction stops at T, we are left with capital which, in view of the choice of f, is described by

$$Z(t) = c_1 e^{\delta T} e^{-\lambda(t-T)} \qquad T \leq t. \tag{4.8}$$

For the Cobb-Douglas production function we then have, by (3.14)

$$\frac{dL}{dZ} = - \frac{\mu_3}{\mu_2} \left(\frac{d_0}{\mu_1}\right)^{\frac{1}{\mu_2}} c_1^{-\frac{\mu_2+\mu_3}{\mu_2}} e^{\left(\frac{\gamma}{\mu_2} + \lambda \frac{\mu_2+\mu_3}{\mu_2}\right)t} e^{-(\delta+\lambda)\frac{\mu_2+\mu_3}{\mu_2} T} , \tag{4.9}$$

which converts (4.2) into an equation of the form

$$c_2 e^{\zeta T} = \int_T^{T_0} e^{\zeta t} dt, \tag{4.10}$$

with $\zeta = -\sigma + I_L + (\gamma+\lambda\mu_3)/\mu_2$, and $c_2$ a positive constant. This in turn becomes

$$e^{\zeta(T_0-T)} = 1 + \zeta/c_2, \tag{4.11}$$

which yields

$$T_0 - T = \frac{1}{\zeta} \log(1+\zeta/c_2). \tag{4.12}$$

We have thus determined how long in advance of the instant when demand drops the firm should stop construction. This duration itself evidently depends on $S(t)$ in the period $T \leq t \leq T_0$.

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