UNIQUENESS, ANALYTICITY AND DECAY PROPERTIES OF CORRELATIONS IN EQUILIBRIUM SYSTEMS*+

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Abstract: We give a survey of results relating to the topics mentioned in the title. We are particularly interested in the relationship between the analyticity properties of the free energy as a function of the thermodynamic parameters of the system and the clustering and uniqueness properties of the corresponding equilibrium states i.e. correlation functions. We concentrate on Ising spin systems with ferromagnetic pair interactions.

I. Introduction and Formulation

In order to keep the presentation simple I shall restrict myself, for most of this lecture, to the simplest non-trivial many-body system known to man; the Ising spin system with finite range ferromagnetic pair interactions on a ν-dimensional cubical lattice $(\mathrm{IFFI})_\nu$, $\nu = 1, 2, 3, \ldots$. (Interactions which fall off exponentially behave, as far as the problems discussed here are concerned, just like finite range interactions.) The Hamiltonian of this system in a finite domain $\Lambda \subset \mathbb{Z}^\nu$ with 'boundary conditions' corresponding to specifying the values of the spin variables outside Λ has the form [1,2],

$$H(\underline{\sigma}; \Lambda, b_\Lambda) = -\tfrac{1}{2} \sum_{i,j\in\Lambda} J(i-j)\sigma_i\sigma_j - \sum_{i\in\Lambda} [h + \sum_{j\notin\Lambda} J(i-j)\bar{\sigma}_j]\sigma_i, \quad (1.1)$$

where $\sigma_i = \pm 1$, $J(\underline{x}) \geq 0$, $J(\underline{x}) = 0$ for $|\underline{x}| > R$, $|\underline{x}|$ is the length of the lattice vector $\underline{x}$, h is the external magnetic field and $\{\bar{\sigma}_i\}$ is a specified set of values of σ_i for $i \notin \Lambda$, which constitute the set of boundary conditions $\{b_\Lambda\} = b$. (More generally only the 'distribution' of the σ_i for $i \notin \Lambda$ need be specified; we can also take into account 'free' boundary conditions by setting $\bar{\sigma}_i = 0$, and 'periodic' boundary conditions by modifying the definition of the $J(\underline{x})$.)

The partition function, free energy density and correlation functions of this system at temperature $T = \beta^{-1}$ are given by

$$Z(\beta,h'; \Lambda,b_\Lambda) = \sum_{\{\sigma_i=\pm 1\}} \exp[-\beta\, H(\underline{\sigma}; \Lambda,b_\Lambda)], \quad i\in\Lambda \quad (1.2)$$

$$\Psi(\beta,h'; \Lambda,b_\Lambda) = \{\ln Z(\beta,h'; \Lambda,b_\Lambda)\}/|\Lambda| \quad (1.3)$$

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$$\langle\sigma_A\rangle\ (\beta,h';\ \Lambda,b_\Lambda) = \sum_{\{\sigma_i=\pm 1\}} \{\sigma_A \exp[-\beta\ H(\underline{\sigma};\Lambda,b_\Lambda)]\}/Z(\beta,h';\Lambda,b_\Lambda), \qquad (1.4)$$

where $|\Lambda|$ equals the number of sites in Λ, $h' = \beta h$, and $\sigma_A = \prod_{i\in A} \sigma_i$, $A \subset \Lambda$. (We have absorbed a factor $-\beta$ in the free energy and shall drop the prime on h from now on.)

We are interested in the behavior of Ψ and $\langle\sigma_A\rangle$ in the thermodynamic limit, $\Lambda \to \infty$. The limit is taken in such a way that each site i eventually is (and remains) inside Λ. Let me present some questions of interest:

1. Does the limit $\Lambda \to \infty$ of $\Psi(\beta,h;\ \Lambda,b_\Lambda)$ exist and is the resulting function independent of the boundary conditions b and of the way in which $\Lambda \to \infty$, i.e. of $\{b_\Lambda\}$?

The answer to this question for IFFI systems is an unqualified yes; all ways of going to the thermodynamic limit lead to the same $\Psi(\beta,h)$ for all real h and $\beta \geq 0$. (This is true generally for systems which are charge neutral [3,4]. The interesting exceptions are systems with net charge [4] and systems with dipolar interactions in an external field [4,5,6] where the thermodynamic free energy density is expected on the basis of simple magneto (electro) -static arguments, and known experimentally, to be shape dependent. The establishment of this shape dependence for dipolar systems on a rigorous mathematical basis is (to my knowledge) the last remaining interesting physical problem in "proving the existence of the thermodynamic limit of the free energy", a task begun over twenty years ago by van Hove, c.f. [1,3].)

2. Having now a unique thermodynamic free energy $\Psi(\beta,h)$ the next question is: what are its analyticity (real) properties? Non-analyticites in β,h are connected with phase transitions and are, from both a physical and mathematical point of view, the 'interesting' features of Ψ for real macroscopic systems. True singularities clearly occur only in the thermodynamic limit $\Lambda \to \infty$, their 'precursors' however are indistinguishable from them quantitatively for macroscopic systems studied experimentally, $|\Lambda| \sim 10^{23}$. The 'finite size' corrections, (surface effects) are an interesting problem in themselves which needs much further investigation, c.f. [7]. They will not be discussed here.

3. Is the thermodynamic limit of $\langle\sigma_A\rangle\ (\beta,h;\ \Lambda,b_\Lambda)$ unique? Since $|\langle\sigma_A\rangle| \leq 1$ we can always find subsequences b_Λ on which the limit exists for all finite sets. We denote these by $\langle\sigma_A\rangle(\beta,h;b)$. The measure 'corresponding' to any such set of correlations defines an 'equilibrium state' which satisfies the DLR (Dobrushin-Lanford-Ruelle) eqs. and vice versa, c.f. [8,9]. If the limit is not unique then there are the further questions: i) Are all the limit functions translation invariant? i.e., does $\langle\sigma_{A+\underline{x}}\rangle(\beta,h;b) = \langle\sigma_A\rangle(\beta,h;b)$? ii) How many 'extremal' translational invariant equilibrium states are there for each β,h, and a given interaction? Translation invariant equilibrium states can always be decomposed uniquely into extremal states [1,8], e.g. if there were only two such extremal states, designated by subscripts 1 and 2, then each set of translation invariant correlation functions would have the

form $\langle\sigma_A\rangle(\beta,h;b) = \alpha_b\langle\sigma_A\rangle_1(\beta,h) + (1-\alpha_b)\langle\sigma_A\rangle_2(\beta,h)$ where $0 \leq \alpha_b \leq 1$ is independent of A. ('Similar' statements hold for non translation invariant equilibrium states.)

4. What are the analyticity properties of the, possible boundary dependent, $\langle\sigma_A\rangle(\beta,h;b)$? Like the free energy the finite volume correlation functions are clearly real analytic in β and h.

5. What are the cluster properties of the infinite volume correlation function, e.g. does $[\langle\sigma_A\,\sigma_{B+\underline{x}}\rangle - \langle\sigma_A\rangle\langle\sigma_{B+\underline{x}}\rangle] \to 0$ as $|\underline{x}| \to \infty$, if yes does it do so exponentially fast, etc.? This question is connected to the questions about analyticity [10,11].

2. Survey of Selected Results

We would of course like to be able to answer the above questions for any particular interaction. Even more important we want to understand, in as much generality as possible, the connection between the properties of the free energy density which is a macroscopic thermodynamic quantity, and the correlation functions which describe the microscopic structure of the system. Thus we would like to know the general class of systems for which the analyticity of $\Psi(\beta,h)$ in some region of the $R\ell\beta - R\ell h$ plane implies unique equilibrium states with strong cluster properties and vice versa. While little is known about the answer to this question in general quite a lot is known for IFFI systems (there is still quite a lot unknown too). It will generally be clear which results are specific to these systems and which have analogies for other systems. (If Ψ is required to be analytic in 'all interactions' then the above question can be answered fairly generally, c.f. Ruelle [12], but our point of view here is to think of the interactions as given while β,h are 'externally controlled' variables. Finding the set of relevant variables is of course a basic part of the problem.)

Let me begin then with a theorem which relates derivatives of Ψ to spatial 'averages' of some correlation functions.

<u>Theorem 1:</u> If $\Psi(\beta,h)$ is differentiable in h at some value of β and h then the thermodynamic limit of the magnetization is equal to the derivative of Ψ,

$$m(\beta,h;b) \equiv \lim_{\Lambda\to\infty} |\Lambda|^{-1} \sum_{i\in\Lambda} \langle\sigma_i\rangle(\beta,h;\Lambda,b_\Lambda) = \partial\Psi(\beta,h)/\partial h \tag{2.1}$$

independent of b, c.f. [1,2].

<u>Proof:</u> $\Psi(\beta,h;\Lambda,b_\Lambda)$ is a convex function of h hence so is $\Psi(\beta,h)$ and the limit of its derivative

$$m(\beta,h;\Lambda,b_\Lambda) \equiv \frac{\partial}{\partial h}\Psi(\beta,h;\Lambda,b_\Lambda) \tag{2.2}$$

equals the derivative of its limit whenever the latter exists.

Since the limit function $\Psi(\beta,h)$ is independent of $\{b_\Lambda\}$ and because of convexity, exists for almost all h we also have

Corollary 2. If $m(\beta,h;b)$ depends on $\{b_\Lambda\}$ then the right and left derivatives of $\Psi(\beta,h)$ are unequal, $\frac{\partial\Psi(\beta,h)}{\partial h+} > \frac{\partial\Psi(\beta,h)}{\partial h-}$, and there exist sequences $\{h_i\}$ decreasing (increasing) to h such that

$$\lim_{i\to\infty} \frac{\partial\Psi(\beta,h)}{\partial h}\Big|_{h_i=h} = \partial\Psi(\beta,h)/\partial h_{(\pm)} .$$

Clearly similar results hold for differentiation with respect to β (or other parameters in the Hamiltonian) and also for general systems, c.f. [13]. The next theorem, however, which is a very strong 'converse' of Theorem 1 is (at the present time) essentially restricted to IFFI 'type' of systems and I don't see any way of extending it to general systems. (Indeed there are some counter examples for antiferromagnetic systems [14] unless h in Theorem 3 is replaced by the 'staggered' field).

Theorem 3. If $\partial\Psi(\beta,h)/\partial h$ exists (i.e. is continuous) then all the correlation functions are independent of boundary conditions [15], i.e.

$$\langle\sigma_A\rangle(\beta,h;b) = \langle\sigma_A\rangle(\beta,h).$$

Theorem 3 may be usefully combined with the following results about IFFI systems.

Theorem 4. a) $\Psi(\beta,h)$ is jointly analytic in h and (real) β for $h \neq 0$, $\beta \geq 0$, [16]. b) $\exists$ a $\beta_a > 0$ such that for $\beta \leq \beta_a$ (high temperature region) $\Psi(\beta,h)$ is real analytic in β and h [17].

Thus for $h \neq 0$ or $\beta \leq \beta_a$ there is only one equilibrium state. This state being unique is translation invariant. It is furthermore true that the correlations in this state have all the desired properties:

Theorem 5. For $h \neq 0$ or $\beta \leq \beta_a$ the $\langle\sigma_A\rangle(\beta,h)$ are (real) analytic in β and h and have exponential clustering properties [16,17,18]. (For even stronger cluster properties see [11]).

We thus see that for our systems everything is fine (and thus uninteresting) for $h \neq 0$ or $\beta \leq \beta_a$. We must therefore, look at $h = 0$ and low temperatures to have a chance of finding something interesting. Now it can be shown, [1], that in one dimension, $(\text{IFFI})_1$, $\Psi(\beta,h)$ is (real) analytic for all β and h. We must therefore look to higher dimensional system, which is what Peierls did and found that there was indeed something interesting happening for $\nu \geq 2$ [1,2]. (Peierls' method of proof has been extremely fruitful for our understanding of phase transitions. It was invented one evening after an unnamed mathematician gave, in a lecture, an elaborate proof of the nonexistence of a phase transition in one dimension and concluded by saying that 'clearly the same is true in higher dimensions', [19]).

Theorem 6. If $\nu \geq 2$ and the nearest neighbor interactions do not vanish then there exists a $\beta_p < \infty$ such that

$$m(\beta,0;\Lambda,b_+) \equiv -m(\beta,0;\Lambda,b_-) \geq \epsilon > 0, \quad \text{for } \beta \geq \beta_p, \tag{2.3}$$

independent of Λ, where $b_+(b_-)$ corresponds to all spins outside Λ being set equal to $1(-1)$.

The exact value of β_p is not too important here. It is generally a poor lower bound on the values of β for which the inequality $m(\beta,0;\Lambda,b_+) > 0$ holds. What is important is that for $\nu \geq 2$ we have at sufficiently low temperatures, $T \leq T_p = \beta_p^{-1}$, that $m(\beta,0;b_+) \neq m(\beta,0;b_-)$, i.e. $m(\beta,0;b)$ depends on the boundary conditions. Thus according to Corollary 2, $\partial\Psi(\beta,h)/\partial h$ will be discontinuous at $h = 0$ for $\beta \geq \beta_p$. We can actually say more.

Theorem 7. a) $\langle\sigma_A\rangle(\beta,h;\Lambda,b_\Lambda)$ is monotone nondecreasing in β and h whenever the b_Λ are such that all the external fields appearing in the Hamiltonian (1.1) are non-negative, i.e. $h_i \equiv h + \sum_{j\notin\Lambda} J(i-j)\,\bar{\sigma}_j \geq 0$. b) For $h \geq 0$ the correlation functions $\langle\sigma_A\rangle(\beta,h;\Lambda,b_+)$ are monotone nonincreasing in Λ and their limits, $\langle\sigma_A\rangle(\beta,h;b_+)$ are translation invariant even when there is more than one equilibrium state, i.e. at $h = 0$ and low temperatures. Furthermore

$$\lim_{h\downarrow 0} \langle\sigma_A\rangle(\beta,h) = \langle\sigma_A\rangle(\beta,0;b_+) \tag{2.4}$$

c) $$m(\beta,0;b_+) = \langle\sigma_i\rangle(\beta,0;b_+) = \lim_{h\downarrow 0} \frac{\partial\Psi(\beta,h)}{\partial h} = m^*(\beta) \tag{2.5}$$

and $m^*(\beta)$, the 'spontaneous magnetization', is monotone nondecreasing in β.

Part a) of the theorem is a special case of the basic Griffiths inequalities [1,2]. b) and c) follow from a) and some other arguments, c.f. [15]. (Clearly $\langle\sigma_A\rangle(\beta,h;\Lambda,b_+) = (-1)^{|A|}\langle\sigma_A\rangle(\beta,-h;\Lambda,b_-)$ and $\Psi(\beta,h) = \Psi(\beta,-h)$ so similar results hold for negative fields.) It follows directly from c) that we can define a unique critical temperature $T_c = \beta_c^{-1}$ for spontaneous magnetization by the relation $m^*(\beta) = 0$ for $\beta < \beta_c$, $m^*(\beta) > 0$ for $\beta > \beta_c$.

To summarize then: for IFFI systems we already know, i) For $h \neq 0$ or $T > T_c$ we have unique states. For $h \neq 0$ or $T > T_a$ we have in addition analyticity and exponential clustering. ii) For $h = 0$ and $T < T_c$ we have nonunique states, discontinuity of $\partial\Psi/\partial h$ and of $\langle\sigma_i\rangle(\beta,h)$ (and all $\langle\sigma_A\rangle$ for $|A|$ odd).

A question which naturally arises now is what happens to analyticity and cluster properties at $h = 0$ and $T_c < T \leq T_a$, e.g. is there a temperature T_b, $T_c < T_b \leq T_a$ where Ψ and/or the $\langle\sigma_A\rangle$ stop being analytic in h and or β. (The nature of the singularities occuring at T_c will not be discussed here). This is a most interesting question and the only additional information known to me about general IFFI systems is given in the following theorem.

Theorem 8. $\Psi(\beta,h)$ and $\langle\sigma_A\rangle(\beta,h)$ are infinitely differentiable in β and h for $\beta < \beta_F$ where $\beta_F^{-1} = T_F$ is (essentially) the 'mean field' critical temperature, defined by the relation $\sum_{\underline{x}} \tanh \beta_F J(\underline{x}) = 1$.

Proof: The proof of Theorem 8 is based on the following result, [10]:

Theorem 9. If the pair correlation $\langle\sigma_i\sigma_{i+\underline{x}}\rangle\ (\beta,0)$ (independent of b for $\beta < \beta_c$) has a bound of the form

$$0 \leq \langle\sigma_i\sigma_{i+\underline{x}}\rangle\ (\beta,0) \leq K\ |\underline{x}|^{-(k\nu+\epsilon)},\ K < \infty,\ \epsilon > 0,\ k \in \mathbb{Z}_+ \tag{2.6}$$

then $\Psi(\beta,h) \in C^{k+1}$ and $\langle\sigma_A\rangle\ (\beta,h) \in C^k$ in both variables. (The inequality on the left of (2.6) is part of Griffiths first inequality).

The existence of an exponential bound (corresponding to $k = \infty$ in (2.5)) follows from Griffiths third inequality [20]

$$\langle\sigma_i\sigma_j\rangle\ (\beta,0) \leq \sum_{k\neq i} \langle\sigma_j\sigma_k\rangle\ \tanh[\beta J(j-k)],\ \text{for } j \neq i \tag{2.7}$$

and can be improved slightly using generalizations of this inequality c.f. [21]. An alternative method which bounds the pair correlation function using self-avoiding walks is due to Fisher [22]. It can sometimes yield an exponential bound for temperatures considerably lower than the mean field critical temperature. It is not known however at the present whether even exponential decay and thus/or C^∞ persist for all $(\text{IFFI})_\nu$ systems down to T_c.

(To obtain an explicit bound from (2.7) we note that if (2.7) is solved as an equality then, for $\beta < \beta_F$, the solution is an upper bound to $\langle\sigma_i\sigma_j\rangle$, c.f. [21]. This yields, (for simplicity use periodic boundary conditions),

$$\langle\sigma_i\sigma_j\rangle\ (\beta,0) \leq \langle y_i y_j\rangle_g(t)/\langle y_i^2\rangle_g(t),\quad \text{for } \beta < \beta_F \tag{2.8}$$

where $\langle y_i y_j\rangle_g(t)$ are the correlations in a "Gaussian" model [23] with the interactions $\beta J(i-j)$ replaced by $\tanh[\beta J(i-j)]$,

$$\langle y_i y_j\rangle = (\underline{t}^{-1})_{ij}\ ,\quad i,j \in \mathbb{Z}^\nu \tag{2.9}$$

where $\underline{t}$ is the matrix with elements

$$(\underline{t})_{ii} = 1,\quad (\underline{t})_{ij} = \tanh \beta J(i-j),\ i \neq j. \tag{2.10}$$

(2.9) can be readily shown to decay exponentially (use Fourier transforms).

When (2.8) is combined with Newman's inequalities [24]

$$\langle\sigma_1\ldots\sigma_{2n}\rangle(\beta,0) \leq \sum_{\text{pairings}} \langle\sigma_{i_1}\sigma_{j_1}\rangle\ldots\langle\sigma_{i_n}\sigma_{j_n}\rangle \tag{2.11}$$

we obtain

$$\langle\sigma_A\rangle(\beta,0) \leq \langle y_A\rangle_g(t)/[\langle y_i^2\rangle_g(t)]^{\frac{1}{2}|A|}\ ,\quad \beta < \beta_F \tag{2.12}$$

'Similar' inequalities hold for $h \neq 0$.)

It should be noted here that while it is very natural to conjecture that analyticity persists for $(\text{IFFI})_\nu$ systems up to $T = T_c$ any proof of it will require making

some use of the lattice structure since the statement is not true for the Bethe lattice [25] and for random spin systems [26].

Let me come now to a discussion of what additional information is known for $h = 0$ and $T < T_c$. I am of course considering the case where $T_c > 0$, for $\nu = 1$ we have $T_c = T_a = 0$.

Theorem 10. $\exists$ a temperature T_e, $0 < T_e < T_c$ ($T_e \sim T_p$) such that for $h = 0$, $0 < T \leq T_e$ the following statements are true:

a) There are only two extremal translation invariant states and these are obtained from b_+ and b_- boundary conditions.

b) The correlations in the extremal states decay exponentially.

c) $\Psi(\beta,h)$ and $\langle\sigma_A\rangle(\beta,h)$ are infinitely differentiable in β and h as $h \to 0$ from either side and so are $\Psi(\beta,0)$ and $\langle\sigma_A\rangle(\beta,0;b_+)$.

The proof of part a) can be obtained in different ways, [27,28] and the same is true of b), [28,29]. It follows from a) and b) that every set of translation invariant correlation functions can, for $\beta \geq \beta_e = T_e^{-1}$, be written as a linear combination of correlation functions which have very strong cluster properties,

$$\overline{\langle\sigma_A\rangle}(\beta,0);b) = \alpha\,\langle\sigma_A\rangle(\beta,0;b_+) + (1-\alpha)\,\langle\sigma_A\rangle(\beta,0;b_-),\quad 0 \leq \alpha \leq 1 \tag{2.13}$$

where the bar indicates translational averaging (if necessary).

Theorem 10c) follows from b) and a variation of Theorem 9 which applies for $h \geq 0$ when (2.6) is replaced by the condition, c.f. [30],

$$0 \leq \langle\sigma_i\sigma_{i+\underline{x}}\rangle(\beta,0;b_+) - [m^*(\beta)]^2 \leq K\,|\underline{x}|^{-(k\nu+\epsilon)} \tag{2.14}$$

where, by (2.5), $m^*(\beta) = \langle\sigma_j\rangle(\beta,0;b_+)$ for all j.

(A very interesting question, related to the problems we are discussing is whether $\Psi(\beta,h)$ and the $\langle\sigma_A\rangle(\beta,h)$ can be analytically continued (below T_c) across the line $h = 0$. If such an analytic continuation were to exist one would be tempted to identify the Ψ and $\langle\sigma_A\rangle$ so obtained with metastable states as may be done for systems with 'very long range' potentials [31]. For this reason this question has generated more interest in the general statistical mechanics community than most of the other questions I have raised. The situation at present is unclear (even forgetting about proofs). There appear to be some strong arguments for expecting an essential singularity in Ψ and $\langle\sigma_A\rangle$ at $h = 0$, [32]. Very recently however these arguments have been questioned [33]. What we know rigorously [8b] is that for systems like IFFI the 'equilibrium state' cannot be continued analytically across $h = 0$ for $T < T_c$. This however does not rule out at all the possibility that Ψ and some $\langle\sigma_A\rangle$, say for all $|A| \leq 137$, may be continued analytically or even that Ψ and all $\langle\sigma_A\rangle$ can be continued analytically but that some $\langle\sigma_A\rangle$ will fail to satisfy some positivity conditions. In summary then the question is wide open.)

Again it is not known for general $(\text{IFFI})_\nu$ systems whether Theorem 10 remains

valid for all $T < T_c$ or does there exist some T_d, $T_e \leq T_d < T_c$, where matters change. More information is known about a special class of $(IFFI)_\nu$ systems, those with nearest neighbor interactions only; $J(\underline{x}) = 0$ for $|\underline{x}| > 1$, $J(\underline{1}) = J$ independent of directions. (This is actually more restrictive than necessary but we want to keep matters simple). We shall denote these systems by I_ν and their critical temperatures by $T_{c,\nu}$. As is well known I_2 was solved by Onsager [34] for h = 0 (the first non-trivial many body problem ever solved). Onsager obtained $\Psi(\beta,0)$ for I_2 and found that it was singular (second derivative diverges logarithmically) at β_o given by $[\sinh 2\beta_o J] = 1$. Onsager and Kaufman [34] also showed that the pair correlation (for 'periodic boundary conditions') decays exponentially above T_o and approaches (also exponentially) $m_o^2(\beta)$ below T_o with $m_o(\beta)$ the Onsager-Yang magnetization $m_o(\beta) = [1-\sinh^{-4}(2\beta J)]^{1/8}$ for $\beta \geq \beta_o$.

It follows from these (and other) facts that T_o coincides with the critical temperature for spontaneous magnetization $T_{c,2}$, [10] that $m_o(\beta) = m^*(\beta)$, [35], and that Ψ and $\langle\sigma_A\rangle$ are C^∞ in β and h for $\beta < \beta_c$ and are C^∞ in β and infinitely differentiable in h as $h \to 0\pm$ for $\beta > \beta_c$ [10,30]. It is also known [36] that I_2 has <u>only</u> translation invariant states for $T \leq T'$ where $T' \sim .7T_c$. Furthermore it has been shown recently [37] that I_2 has exactly two extremal states (+, - boundaries) for all $T < T_c$.

Thus of the questions raised in the beginning of my talk the only ones left unanswered for I_2 are a) whether analyticity holds for all $T > T_c$ and b) whether the nonexistence of non-translation invariant states is true for all T (The question is really just for $T' < T < T_c$). It would be most surprising indeed if the answer to both questions were not yes, [36]. (I make no conjecture about analytic continuation below T_c).

This brings me to a most interesting point; the property of I_2 of having only translation invariant states (at least at low temperatures) definitely does not hold for $\nu > 2$. This was first shown by Dobrushin [38]. The result has been improved and the proof greatly simplified by van Beijeren [39], who proved:

<u>Theorem 11.</u> For h = 0 and $T < T_{c,\nu-1}$, I_ν <u>has</u> non translational invariant states.

Since $T_{c,1} = 0$ this is consistent with the results for I_2. Indeed the difference between $\nu > 2$ and $\nu = 2$ with regard to the existence of non translation invariant states is very similar to the difference between $\nu > 1$ and $\nu = 1$ with regard to the existence of a $T_{c,\nu} > 0$. This becomes clearer when we consider the nature of the Dobrushin and van Beijeren proofs. They consider a sequence of ν-dimensional cubes Λ_N, $\nu \geq 2$ with 2N horizontal layers labeled by k = -N, -N+1,...,N, and boundary conditions b_N corresponding to + spins, $\bar{\sigma}_i = 1$, on the 'top half' N layers and -spins, $\bar{\sigma}_i = -1$, on the bottom half layers; $\{b_N\} = b^{\pm}$.

If we let $\sigma_{k,\alpha}$ designate the spin variable at site α in the k<u>th</u> layer then by symmetry $\langle\sigma_{k,\alpha}\rangle(\beta,0;\Lambda_N, b_N) = -\langle\sigma_{-k,\alpha}\rangle(\beta,0;\Lambda_N,b_N)$. At $\beta^{-1} = T = 0$ we clearly have

$\langle\sigma_{k,\alpha}\rangle = 1$ for $k > 0$ and all N. The question now is whether as $N \to \infty$, and the boundaries recede, does $\langle\sigma_{k,\alpha}\rangle$ for $k > 0$ remain positive for some temperature $T \neq 0$ and thus $\langle\sigma_i\rangle(\beta,0;b^{\pm})$ would not be translation invariant. We already know that for $T > T_c$, $\langle\sigma_i\rangle(\beta,0;b) = 0$ for all boundary conditions. For I_2 we also know that for $0 < T \leq T'$ all states are translation invariant, and thus $\langle\sigma_{k,\alpha}\rangle(\beta,0;b^{\pm}) = \langle\sigma_{-k,\alpha}\rangle(\beta,0;b^{\pm}) = 0$ by symmetry. For $\nu \geq 3$ however it is shown by van Beijeren that

$$\langle\sigma_{1,\alpha}\rangle(\beta,0;b^{\pm}) \geq m^*_{\nu-1}(\beta) > 0 \text{ for } T < T_{c,\nu-1} \tag{2.15}$$

(van Beijeren also shows that, as expected, $\langle\sigma_{k,\alpha}\rangle(\beta,0;b^{\pm})$ is monotone non-decreasing in k.)

Physically (2.15) means that the $\nu-1$ dimensional surface which separates up spins from down spins and intersects the boundary of Λ_N between the $k = 1$ and $k = -1$ level (this is a connected piece of the union of 'faces' separating cells with $\sigma_i=1$ from cells with $\sigma_i = -1$) does not fluctuate too widely as $N \to \infty$ when $\nu-1 \geq 2$ and $T < T_{c,\nu-1}$ (for $\nu - 1 = 1$ it does so fluctuate).

The interesting question for I_ν, $\nu \geq 3$, is now whether these non-translation invariant states persist up to $T_{c,\nu}$ or is there a 'roughening temperature' $T_{r,\nu} < T_{c,\nu}$ where the dividing surface roughens and disappears, as in I_2. There appears to be some evidence, [40] based on extrapolations from low temperature expansions and numerical computations, that $T_{r,3} \sim .57T_{c,3} \sim 1.1\, T_{c,2}$, i.e. quite close to van Beijeren's lower bound. Again however definite results are sorely lacking.

3. Concluding Remarks

We have seen that while much remains to be done all the information available for IFEI systems are consistent with our hopes for simplicity. In lattice gas language there is a line in the chemical potential μ-temperature T plane along which two phases, liquid and gas, coexist ending in a critical point. For values of μ - T not on this line we have unique equilibrium states with (very likely) exponential decay of correlations and analyticity in μ and T. On the line itself we have (very likely) only two translation invariant states. The question now is how much of this simplicity remains for even simple continuum systems, e.g. for atoms interacting with Lenard-Jones type potentials. Since the interaction potential is now long range, falling off as r^{-n}, we cannot expect and do not get exponential decay of correlations [11,18]. On the other hand we expect to have also a solid-fluid transition and a triple point. Taking these into account however does the rest of the picture remain simple or are there real surprises, e.g. two dimensional regions of the μ-T plane where the equilibrium state is not unique and/or the free energy not analytic. We don't seem to be anywhere near getting an answer to these questions. All we do know is that if we permit a large amount of arbitrariness in the potentials then we can get very strange things indeed [41], e.g. one dimensional Heisenberg spin systems with spontaneous magnetization.

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