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Cost-Minimization by a Regulated Firm in the Presence of Inflation

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ABSTRACT

We investigate the optimal capital-labor ratio for a regulated firm in the presence of inflation. The firm is required to meet a given, price-inelastic demand for its services. These services are produced by a combination of labor and of durable physical plant (i.e., physical plant can be reduced only through its innate retirement schedule). The firm collects revenues sufficient to cover wages, depreciation, taxes and a return on investor-supplied capital at a rate set by the regulatory body. The unit costs of labor and plant are inflating, at possibly unequal rates. Our problem is then to find the construction budget, as a function of time, which minimizes the present worth of revenue requirements.

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Introduction

The familiar computation of the cost-minimizing amounts of capital and labor required for a given level of output (whose result is that the marginal products of capital and labor should be proportional to the prices of these two factors) treats the cost of capital as a rent and hence is symmetric in the two factors of production. In the present paper, by contrast, we imagine the stock of economic (physical) capital to increase according to a construction budget, C(t), and to decrease only through retirements (which are determined by a life table). The labor costs consist, as always, of wages, but the capital costs include a return on rate base (in the case of a regulated firm, this is the return allowed by the regulator), income taxes, and depreciation. The total of all capital and labor costs is called revenue requirements. The persistence of economic capital leads us naturally to pose a dynamic rather than a static problem; thus we seek to minimize the present worth of revenue requirements by choosing an optimum function C(t).

The problem is particularly interesting in the presence of inflation. In this case we obtain results which cannot be obtained from a static analysis, even one parametrized by time. In particular, our firm minimizes the present worth of revenue requirements by becoming more capital-intensive than a static analysis would suggest.

The present calcualtion is not of the Averch-Johnson[(1)] type, in that we are minimizing the present worth of total costs (including taxes and depreciation), rather than maximizing instantaneous profit. These two objective functions are not equivalent, since the Averch-Johnson firm is optimizing its output as well as its inputs, whereas our firm is constrained to meet a given demand. This may be called the franchise constraint. We show that this cost-minimizing firm, subject to the franchise constraint, is substantially more capital intensive than would be revealed by a static analysis.

Review of the Static Model

We shall now give a brief review of the usual static treatment, so as to contrast it with the dynamic one. In the static model, we consider a firm constrained to meet a given (price-inelastic) demand Q, in a certain period. It meets this demand by using K units of capital and L units of labor. The firm's output is determined by a production function:

$$Q = F(L, K). \tag{1}$$

The unit costs of labor and capital are w and r respectively, where r is thought of as a rent. The total cost is thus,

$$C = wL + rK. \tag{2}$$

We seek to minimize (2), subject to (1).

One technique is as follows. Form the function

$$I = wL + rK - \lambda[F(L, K) - Q]. \tag{3}$$

(λ is a Lagrange multiplier.) At the minimum we should have

$$I_L = 0, \tag{4a}$$

$$I_K = 0, \tag{4b}$$

$$I_\lambda = 0, \tag{4c}$$

Where the subscripts denote partial derivatives.

Applying (4a) and (4b) to (3), we obtain respectively

$$r = \lambda\, F_K, \tag{5a}$$

$$w = \lambda\, F_L. \tag{5b}$$

Application of (4c) to (3) recovers the constraint (1). Assuming that the second-order conditions indicate that Eqns. (5) represent a minimum, then for cost minimization we require

$$\frac{r}{w} = \frac{F_K}{F_L}. \tag{6}$$

Eqn. (6) can then be combined with (1) to give the optimal amounts of capital and labor for a given demand, Q. Consider, for example, a Cobb-Douglas production function (3):

$$F(L,\ K) = \lambda_1\ L^{\lambda_2}\ K^{\lambda_3}\ . \tag{7}$$

(If $\lambda_2 + \lambda_3 > 1$, we have increasing returns to scale.)
For this particular case*, (6) and (7) yield for the optimal labor-capital ratio:

$$\frac{L}{K} = \frac{\lambda_2}{\lambda_3}\,\frac{r}{w}. \tag{8}$$

So far, since the cost of capital is thought of as a rent, the treatment is completely symmetric in capital and labor.

The static model can be parametrized by time. We can write

$$w = w(t)$$
$$r = r(t)$$

So that (8) reads

$$\frac{L}{K} = \frac{\lambda_2}{\lambda_3}\,\frac{r(t)}{w(t)}. \tag{9}$$

For example, we might have steady inflation in the unit costs of labor and capital:

$$w(t) = w_o e^{I_L t}, \tag{10a}$$

$$r(t) = r_o e^{I_K t}. \tag{10b}$$

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*(to which we shall not confine ourselves in the dynamic model)

In the real world, one often finds

$$I_L > I_K. \tag{11}$$

Substituting Eqns. (10) in (9), we obtain

$$\frac{L}{K} = \frac{\lambda_2}{\lambda_3} \frac{r_0}{w_0} e^{-(I_L - I_K)t}. \tag{12}$$

Thus when labor inflation is faster than capital inflation, the cost-minimizing firm, even in the static model, is motivated to become increasingly capital intensive.

Note that, according to (12), the capital and labor shares of the total cost are in constant proportion:

$$\frac{wL}{rK} = \frac{\lambda_2}{\lambda_3}. \tag{13}$$

This is true for the Cobb-Douglas function, but not for an arbitrary production function.

The cost of a unit of plant, $r(t)$, may be regarded as already including the effect of technological change. That is, the capital inflation rate, I_K, may be thought of as a dollar inflation rate net of the technological rate of increase of the productivity of a unit of physical plant.

Taxes, Debt, and Regulation

Still within the context of the static model, we now start to move closer to a realistic picture of the regulated firm. In particular, we become a little more explicit about the cost of capital.

Suppose the firm collects just enough revenues, R, to cover all its costs:

$$R = \mathcal{C}$$

The firm's earnings are taxable; let us denote the income tax rate by τ. The firm's taxes are

$$T = \tau(R - wL - i\delta K). \tag{14}$$

Here δ is the debt ratio, so that δK is the amount of debt capital. The interest rate is i. Eqn. (14) expresses the fact that wages and interest are tax deductible. We have not yet included depreciation in the model, so the deductibility of tax depreciation from taxable income is not expressed in (14).

Since revenues must cover all costs, including taxes,

$$R = \rho K + wL + T. \tag{15}$$

Here the new symbol, ρ, denotes the "fair rate of return" on total capital allowed by the regulator, and assumed to be earned by the firm.

Solving (14) and (15) for revenue requirements, we obtain

$$R = wL + aK, \tag{16}$$

where we have abbreviated

$$a = \frac{\rho - \tau i \delta}{1-\tau} \tag{17}$$

The quantity a is the pre-tax rate of return, taking account of the tax-deductibility of interest.

Eqn. (16) resembles (2), and we could go ahead and minimize (16) subject to the constraint (1). However, the rental cost of capital, r, has now been replaced by a rather different quantity, the pre-tax allowed rate of return, a. We are in fact in transition from a cost minimization objective to a revenue requirement minimization objective. To clarify the process, we must now distinguish between economic capital and rate base. This will also show us how to include depreciation in the objective function.

Economic Capital, Rate Base, and Depreciation

The firm's capital enters into its revenue requirements (i.e., its total costs) in two ways: first as the rate base on which it is allowed a rate of return ρ, and second, as the basis for depreciation (see Eqn.(16)). It is important to specify the definition of capital, or rate base, used in this context. In most U.S. jurisdictions,(2) what is meant is the total original cost of all vintages of plant, less depreciation. (*Modulo* certain accounting and other technicalities, this is the same as the accumulation of stocks, bonds and retained earnings, again measured in historical dollars.)

We denote the rate base by X. It should be emphasized* that this is not the reproduction cost of the plant, nor is it the value of the firm's shares in the real stock market. (In fact, this latter quantity nowhere enters our analysis.) In fact, if the rate base were replacement cost, the heavy capitalization (illusary over-capitalization) which we derive, would not exist.

On the other hand, it produces service by the use of economic capital, i.e., physical plant. This is the kind of capital which should appear in a production function constraint, such as (1). A surrogate for economic capital, in the absence of technological change, is the gross plant measured in constant dollars. We call this quantity Z. Having made this distinction, we shall now drop the symbol K.

*We are grateful to the referee for this point.

Economic capital and rate base may be unequal both because of inflation in the cost of physical plant and because of depreciation. Depreciation should be included as a component of revenue requirements, and is also tax deductible. For our present purposes we shall assume that book and tax depreciation are identical, and denote both by D. Then Eqns. (14) and (15) are replaced by:

$$T = \tau(R - wL - i\delta X - D), \tag{18}$$

$$R = \rho X + wL + T + D. \tag{19}$$

Again solving for R,

$$R = wL + aX + D. \tag{20}$$

We wish to minimize (20) subject to

$$Q = F(L, Z). \tag{21}$$

The Dynamic Model

We cannot solve the problem of minimizing (20) subject to (21), without specifying the relationships among X, Z, and D. These three quantities are linked through the construction budget, C(t), the rate of expenditure (in current dollars) on new plant at time t.

Suppose the firm is founded at t = 0 with Z(0) units of new physical plant. Let f(t) be the life-table, i.e., the fraction of a vintage still unretired at age t. (The life-table is assumed invariant over vintages.) Then the physical plant at time t comprises the surviving plant of all previous vintages. That is,

$$Z(t) = Z(0)f(t) + \int_0^t [C(y)/r(y)]f(t-y)dy, \tag{22}$$

since at time y one dollar of construction budget is enough to buy [1/r(y)] units of physical plant. (Here r(y) still denotes the cost of a unit of plant purchased at time y, even though this plant is no longer thought of as rented.)

The rate base or net plant at time t comprises the undepreciated parts of all previous investments; if the initial dollar investment is X(0), then

$$X(t) = X(0)q(t) + \int_0^t C(y)q(t-y)dy. \tag{23}$$

Here q(y) is the fraction of the initial dollar cost of a vintage still undepreciated at age y; this function is assumed invariant over vintages.

The rate of depreciation of a unit vintage at age y is $[-\dot{q}(y)]$, where the dot denotes the time derivative. It follows by superposition that the total depreciation at time t is

$$D(t) = -\dot{q}(t) + \int_0^t C(y)[-\dot{q}(t-y)]dy. \tag{24}$$

Our problem is now to minimize (20) subject to (21), where Z, X, and D are defined by (22), (23), and (24) respectively. The point is that all three of these quantities are determined by the construction budget, C(t). Moreover, C(t) is not quite arbitrary; it must satisfy

$$C(t) \geq 0, \quad \text{for all } t \geq 0. \tag{25}$$

We might also in the interests of realism, impose an upper bound on C(t), but we do not do so here.

Since the evolution in time of economic capital, rate base, and depreciation are now all dependent on the construction budget, it is no longer appropriate to ask for the amounts of capital and labor which will enable us to meet a given demand at minimum cost at each instant. Arbitrary amounts of plant, Z, at successive instants may not be obtainable using a permissable construction budget. Therefore, we reformulate our problem, and seek that construction budget, C(t), which will minimize

$$PW_\sigma(R) \equiv \int_0^\infty e^{-\sigma t} R(t)dt. \tag{26}$$

Here the letters PW stand for present worth, and σ denotes the discount rate, taken to be exogenously given.

It should be observed that in our previous static problem, in which capital, $Z(t)$, was not constrained to functions which satisfy (22) for a feasible $C(t)$, the new objective of minimizing present worth was satisfied _a fortiori_, since one cannot do better in minimizing the present worth of a function than to minimize the function at all points.

At this point we make several simplifying assumptions, just for analytical convenience. First, we assume steady wage and capital inflation, as in Eqns. (10). Second, we assume that the life-table is exponential, i.e., the fraction of a vintage of plant unretired at age y is

$$f(y) = e^{-\beta y}, \tag{27}$$

where $(1/\beta)$ is the mean life of plant. Third, we suppose that the firm is using straight-line depreciation, that is that the rate of depreciation of a unit vintage at age y is

$$-\dot{q}(y) = \beta f(y). \tag{28}$$

It follows from (27) and (28) that depreciation is also exponential, i.e.,

$$q(y) = e^{-\beta y}, \tag{29}$$

where we have used the fact that $q(0) = 1$. From (29), it is easy to show that

$$D(t) = \beta X(t). \tag{30}$$

Substituting (30) into (20), we find that our revenue requirements are now

$$R(t) = (a+\beta)X(t) + w(t)L(t) \tag{31}$$

Finally, we assume that the exogenously demand is growing exponentially with time,

$$Q(t) = e^{\gamma t}, \tag{32}$$

Where units have been chosen in such a way that $Q(0) = 1$. Any reasonable production function, (21), can be uniquely inverted to yield

$$L(t) = Y[Q(t), Z(t)] = Y[e^{\gamma t}, Z(t)]. \tag{33}$$

The revenue requirements are now

$$R(t) = (a+\beta)X(t) + w_0 e^{I_L t} Y(e^{\gamma t}, Z(t)). \tag{34}$$

Our problem can now be stated in its entirety:

find the function $C(t) \geq 0$ which minimizes $PW_\sigma(R(t))$, where $R(t)$ is given by (34), and where $X(t)$ and $Z(t)$ are related to $C(t)$ by

$$Z(t) = Z(0)e^{-\beta t} + \int_0^t C(y)e^{-I_K y} e^{-\beta(t-y)} dy \tag{35}$$

and

$$X(t) = Z(0)e^{-\beta t} + \int_0^t C(y)e^{-\beta(t-y)} dy. \tag{36}$$

In writing (35) and (36) we have implicitly made another choice of units; i.e., we have without loss of generality, set $r_0 = 1$ (in Eqn. (10b)) and hence $X(0) = Z(0)$.

<u>The Solution</u>

The minimization itself is a straightforward exercise in the calculus of variations[4].

Let us introduce the following functions:

$$\psi(t) = e^{\beta t} Z(t) \tag{37}$$

$$\phi(t) = e^{\beta t} X(t) \tag{38}$$

$$\Omega(t) = e^{\beta t} C(t). \tag{39}$$

In these terms, (35) and (36) become respectively,

$$\psi(t) = Z(0) + \int_0^t \Omega(y) e^{-I_K y} dy, \tag{40}$$

and

$$\phi(t) = Z(0) + \int_0^t \Omega(y) dy. \tag{41}$$

Using (34), our problem can now be stated as follows:

$$\underset{\Omega(t)\geq 0}{\text{Min}} \int_0^\infty e^{-\sigma' t}(a+\beta) \left\{ \phi(t) + b(t)\ Y\left(e^{\gamma t}, e^{-\beta t}\psi(t)\right) \right\} dt. \tag{42}$$

In (42), $\phi(t)$ and $\psi(t)$ are understood to be related to $\Omega(t)$ through (41) and (40) respectively. We have introduced the abbreviations

$$\sigma' = \sigma + \beta, \tag{43}$$

and

$$\underline{b}(t) = \frac{w_0 e^{\left[I_L+\beta\right] t}}{a+\beta}. \tag{44}$$

The positive constant factor $(a+\beta)$ in (25) has no effect on the minimization. Thus we can find the extremum by setting

$$\delta M = 0,$$

Where

$$M\left\{\Omega(t)\right\} = \int_0^{\infty} e^{-\sigma' t}\left\{\phi(t) + b(t)\quad Y\left(e^{\gamma t},\ e^{-\beta t}\psi(t)\right)\right\} dt. \tag{45}$$

Here the δ is the conventional symbol for the variation of a function (there is no danger of confusion with the debt ratio). The variation in M is induced by one in $\Omega(t)$:

$$\Omega(t) \to \Omega(t) + \delta\Omega(t).$$

The variation may be carried inside the integral. Note that

$$\delta Y = \frac{\partial Y}{\partial[e^{-\beta t}\psi(t)]}\quad \frac{\partial[e^{-\beta t}\psi(t)]}{\partial\psi(t)}\ \delta\psi(t).$$

From Eqn. (40),

$$\delta\psi(t) = \int_0^t e^{-I_K y}\,\delta\Omega(y)dy.$$

Therefore,

$$\delta Y = \frac{\partial Y}{\partial Z}\ e^{-\beta t}\int_0^t e^{-I_K y}\,\delta\Omega(y)dy. \tag{46}$$

From Eqn. (41),

$$\delta\phi(t) = \int_0^t \delta\Omega(y)dy. \tag{47}$$

From (45), (46) and (47),

$$\delta M = 0 = \int_{t=0}^{\infty} e^{\sigma' t}\left\{\int_{y=0}^{t}\delta\Omega(y)dy + b(t)e^{-\beta t}\,\frac{\partial Y}{\partial Z}\int_{y=0}^{t} e^{-I_K y}\,\delta\Omega(y)dy\right\}dt. \tag{48}$$

Changing the order of integration in (48), we obtain

$$\delta M = 0 = \int_{y=0}^{\infty} \delta\Omega(y) \left\{ \int_{t=y}^{\infty} e^{-\sigma' t} \left[1 + b(t) e^{-I_K y} e^{-\beta t} \frac{\partial Y}{\partial Z} \right] dt \right\} dy. \tag{49}$$

Now the construction budget must be non-negative. If the optimum construction budget is such that the function $\Omega(t)$ is everywhere "interior" to its permitted region, i.e., if $\Omega(t) > 0$, for all $t>0$, then we may set the term in braces in (49) identically equal to zero; that is

$$\int_{t=y}^{\infty} e^{-\sigma' t} dt = - e^{-I_K y} \int_{t=y}^{\infty} e^{-\sigma' t} b(t) e^{-\beta t} \frac{\partial Y}{\partial Z} \, dt. \tag{50}$$

The integral on the left can of course be performed, giving

$$\frac{-e^{-(\sigma' - I_K)y}}{\sigma'} = \int_{t=y}^{\infty} e^{-\sigma' t} b(t) e^{-\beta t} \frac{\partial Y}{\partial Z} \, dt. \tag{51}$$

Differentiating both sides of (51) with respect to y gives

$$\frac{(\sigma' - I_K) e^{-(\sigma' - I_K)y}}{\sigma'} = - e^{-\sigma' y} b(y) e^{-\beta y} \frac{\partial Y}{\partial Z}$$

or

$$\frac{\partial Y}{\partial Z} = \frac{-(\sigma' - I_K)e^{(I_K+\beta)y}}{\sigma' b(y)} . \tag{52}$$

Substituting into (52) for σ' and b we obtain

$$\frac{\partial Y}{\partial Z} = \frac{-(a+\beta)(\sigma+\beta-I_K)e^{-(I_L-I_K)y}}{w_0(\sigma+\beta)} ,$$

or

$$\frac{\partial Y}{\partial Z} = -\alpha e^{-(I_L-I_K)y} , \tag{53}$$

where we have defined the quantity α, to which we shall refer later, by:

$$\alpha = \frac{(a+\beta)(\sigma+\beta-I_K)}{w_0(\sigma+\beta)} .$$

The function Y was introduced when we inverted the production function to solve for labor (Eqn. 33). Using the identity

$$\left.\frac{\partial L}{\partial Z}\right|_{\phi} = - \frac{F_Z}{F_L} ,$$

where F still denotes the production function, Eqn. (53) may be rewritten as follows:

$$\frac{F_Z(y)}{F_L(y)} = \alpha e^{-(I_L-I_K)y} . \tag{54}$$

Discussion of the Solution

Eqn. (54) is our essential result. Let us now compare it to the result of the static model, as expressed by Eqn. (6). When the static model was elaborated to include taxes, depreciation, and the time-dependence of the unit costs of capital and labor, the problem was expressed by Eqns. (20) and (21). As we have seen, the static model cannot properly take account of the time-dependent distinction between economic and financial capital. If, for example, economic capital and rate base were equated in the static model, then Eqn. (20) would correspond to Eqn. (2) with

$$w \quad \text{replaced by} \quad w_0 e^{I_L t},$$

and (assuming an exponential life table and straight-line depreciation, and thus Eqn. (30))

$$r \quad \text{replaced by} \quad (a+\beta)e^{I_K t}.$$

Hence, (6) would become

$$\frac{F_Z(y)}{F_L(y)} = \frac{a+\beta}{w_0} e^{-(I_L - I_K)y}. \tag{55}$$

Comparison of (54) and (55) reveals that the effect of going to the dynamic model (i.e., of replacing instantaneous cost minimization by a present-worth criterion, and constraining economic and rate base to those functions permitted by a non-negative construction budget) has been the introduction of the factor

$$(\sigma+\beta-I_K)/(\sigma+\beta) \tag{56}$$

in the α of Eqn. (54). Clearly the parameter σ, the discount rate, cannot appear in the static treatment.

The factor (56) is less than unity; it seems clear that it is also positive, since the discount rate should contain an additive term in some measure of inflation, say I, not very different from I_K:

$$\sigma = \sigma_0 + I.$$

The effect of the factor (56) is to increase the captial intensivity of the firm <u>beyond</u> that predicted by the static model. This may be seen as follows: any reasonable production function has isoquants convex towards the origin in the L-Z plane; therefore, L/Z is a decreasing function of $\partial Y/\partial Z$, and consequently an increasing function of F_Z/F_L. Since the factor (56) diminishes α, it diminishes F_Z/F_L, and hence L/Z; i.e., it makes the firm more capital intensive. This effect reflects the tendency of the firm in an inflationary situation to build plant earlier rather than later, when its continuing capital costs will be greater.

We note that the factor (56) is unity in the absence of capital inflation. It goes to unity as σ becomes very large, i.e., as the firm becomes extremely myopic. It also goes to unity when β becomes extremely large, i.e., when the life of plant is very short. Consider the following plausible numerical values:

$$\sigma = .08$$

$$\beta = .05 \text{ (20 year mean life of plant)}$$

$$I_K = .04$$

In this case the factor (56) is 9/13; the labor-capital ratio is at each instant only about 69% as high as the static analysis would indicate to be optimal.

It is interesting to note that an observer of the regulated firm which we are describing might (if he were aware only of the static analysis) attribute the firm's high capital intensivity to the Averch-Johnson effect. In fact, however, our firm is really cost-minimizing, in a present-worth or dynamic sense, while the profit-maximizing Averch-Johnson firm, subject to an inequality constraint, does not in general operate at the minimum-cost point. A straightforward extension of the Averch-Johnson model to a (discrete-time) dynamic case, with a price adjustment mechanism, has been worked out by E. Davis.(5)

The above optimization has concerned itself with present worth of revenue requirements at $t = 0$. It may be asked whether, if the firm reconsiders the decision as to its future construction budget at any subsequent time $t_1 > 0$, a similar optimization will lead to the decision to continue on the course upon which it has already embarked. The answer to this question is affirmative; this may be seen as follows. The minimum present worth of revenue requirements at $t = 0$ can be broken into two parts:

$$PW_\sigma \left.(R_0)\right|_{t=0} = \int_{t=0}^{t_1} e^{-\sigma t} R_0(t)dt + \int_{t_1}^{\infty} e^{-\sigma t} R_0(t)dt, \qquad (57)$$

where $C_0(t)$ is the construction budget that minimizes this quantity, and $R_0(t)$ is the corresponding revenue requirement. Suppose that an optimizer at t_1 can do better with a different construction budget $C_1(t)$ (for $t > t_1$) and corresponding revenue requirements $R_1(t)$. This would mean that

$$\int_{t_1}^{\infty} e^{-\sigma(t-t_1)} R_1(t)dt < \int_{t_1}^{\infty} e^{-\sigma(t-t_1)} R_0(t)dt. \tag{58}$$

But since the factor $\exp(\sigma t_1)$ cancels out of (58) transforming the right-hand member of (58) to the second term on the right of (57) this would contradict the assumption that $C_0(t)$ was a valid cost-minimizing construction budget to begin with. Therefore, re-optimization at a later time will not lead to any change in policy.

Special Production Functions

Let us now find the labor-capital ratio explicitly for two commonly used production functions.

1) Cobb-Douglas: $f(Z, L) = \lambda_1 L^{\lambda_2} Z^{\lambda_3}$

Solving the equation $Q = f(Z, L)$ for L, we obtain

$$L = Y(Q, Z) = \left(\frac{Q}{\lambda_1}\right)^{\frac{1}{\lambda_2}} Z^{-\lambda_3/\lambda_2}$$

Therefore

$$\frac{\partial Y}{\partial Z} = \frac{-\lambda_3}{\lambda_2} \left(\frac{Q}{\lambda_1}\right)^{\frac{1}{\lambda_2}} Z^{\left(\frac{-\lambda_3}{\lambda_2} - 1\right)}$$

$$= \frac{-\lambda_3}{\lambda_2} \left(\frac{Q}{\lambda_1}\right)^{\frac{1}{\lambda_2}} Z^{\frac{\lambda_3}{\lambda_2}} \Big/ Z$$

$$= - \frac{\lambda_3}{\lambda_2} \frac{L}{Z} \qquad (59)$$

Hence, from (53) we have

$$\frac{L}{Z} = \frac{\lambda_2}{\lambda_3}\,\alpha e^{-(I_L - I_K)t} \tag{60}$$

2) Constant elasticity of substitution[6]:

$$f(Z,L) = \left[\lambda_1\left(\lambda_2 Z^{-\eta} + (1-\lambda_2)L^{-\beta}\right)\right]^{-\frac{1}{\eta}}.$$

By a calculation similar to the above, we find that at the optimum

$$\frac{L}{Z} = \left(\frac{1-\lambda_2}{\lambda_2}\,\alpha\right)^{\frac{1}{\eta+1}} e^{\frac{-(I_L-I_K)t}{\eta+1}}: \tag{61}$$

A special property of the Cobb-Douglas production carries over from the static model (see Eqn. 13); namely, that the capital and labor shares of the total revenue requirements are asymptotically in constant proportion; that is, the ratio $(a+\beta)X/wL$ approaches, for large t, a value independent of time. This can best be shown by displaying the explicit form of Z(t) for this production function. Solving Eqn. (60) for L, and substituting into the production function constraint,

$$e^{\gamma t} = \lambda_1 L^{\lambda_2} Z^{\lambda_3},$$

we obtain

$$Z = Ae^{\frac{\lambda_2}{\lambda_2+\lambda_3}(I_L - I_K + \gamma/\lambda_2)t}, \tag{62}$$

Where

$$A = \left(\frac{\lambda_3}{\alpha \lambda_2 \lambda_1{}^{1/\lambda_2}} \right)^{\frac{\lambda_2}{\lambda_2+\lambda_3}} . \tag{63}$$

It is not difficult to show (we do not so here) that asymptotically as $t \to \infty$,

$$\frac{\dot{X}(t)}{X(t)} \to \frac{\dot{Z}(t)}{Z(t)} + I_K . \tag{64}$$

Hence, asymptotically,

$$X \propto \exp\left\{ \left[\frac{\lambda_2}{\lambda_2+\lambda_3} (I_L - I_K + \gamma/\lambda_2) + I_K \right] t \right\} . \tag{65}$$

We can also find from (62) and the production function constraint, the explicit form of $L(t)$. We obtain

$$L(t) = \left(\frac{A^{-\lambda_3}}{\lambda_1} \right)^{1/\lambda_2} e^{\frac{1}{\lambda_2+\lambda_3}[\gamma - \lambda_3(I_L - I_K)]t} . \tag{66}$$

From (65) and (66), plus the assumption that $w(t) \propto \exp(I_L t)$, it follows that $(a+\beta)X/wL$ is asympototically constant.

Non-Optimal Initial Condition

A comment is in order at this point. We started with initial physical plant of amount $Z(0)$, which was positive but otherwise arbitrary. Yet Eq. (62) now seems to determine $Z(0)$, namely $Z(0) = A$, where the constant A depends on the given parameters of the problem (and in particular on the discount rate σ). This apparent paradox is explained by the fact that only when $Z(0) = A$ is the optimal $\Omega(t)$ everywhere interior, and hence only if

$Z(0) = A$ is the variational procedure which led to (62) justified. We now explore this aspect of the problem a little further.

We have found the cost-minimizing growth path under the circumstance that the initial amount of capital is optimal, i.e., that $Z(0) = A$. Suppose that this is not the case. Then it can be demonstrated that the optimal construction budget is that which returns the firm to the growth path described above as quickly as possible. More specifically, let the solution expressed in Eqn. (62) be represented as in Figure 1.

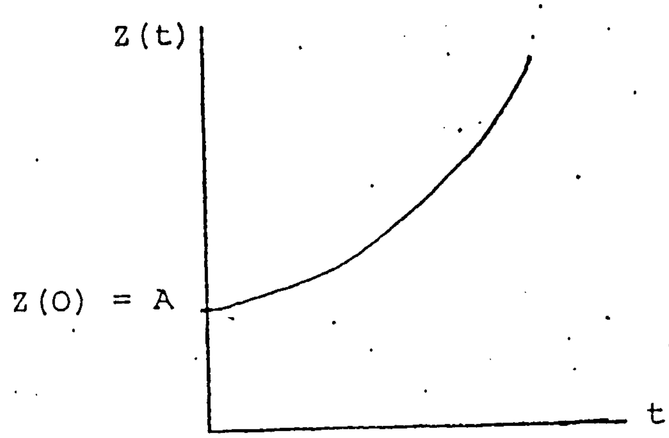

Fig. 1

If $Z(0) < A$, the best one can do is to increase plant to A immediately, by instantaneous construction in the amount $[A-Z(0)]$, and then to proceed as before. This is shown in Fig. 2.

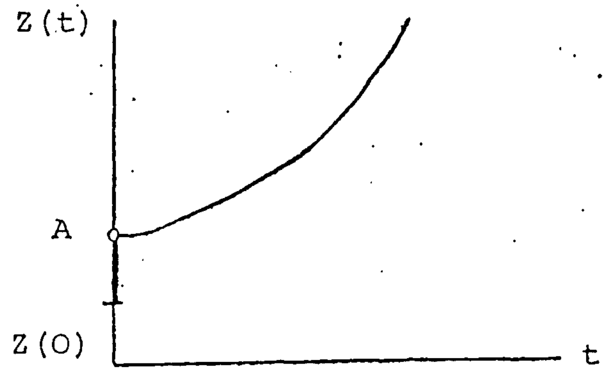

Fig. 2

If on the other hand the firm starts with too much capital, $Z(0) > A$, its best policy is to build nothing until that time, t_0, when Z returns to the path of Fig. 1, and then to proceed as before. This is shown in Fig. 3.

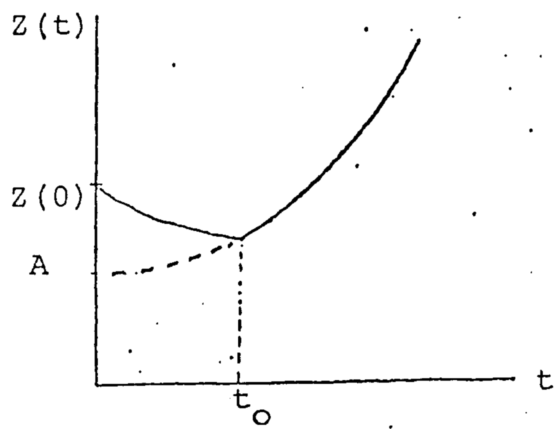

Fig. 3

The time t_0 is the solution of the equation

$$Z(0)f(t_0) = Ae^{\frac{\lambda_2}{\lambda_2+\lambda_3}\left(I_L - I_K + \frac{\gamma}{\lambda_2}\right)t_0} \tag{67}$$

where f(t) is the life table.

Remark

We have, throughout this paper, taken both the rate of return, ρ, and the demand $Q = \exp(\gamma t)$ as given. There is no guarantee, however, that this rate of return will provide the company with sufficient funds to build plant to meet this demand, even at minimum cost. Whether the rate of return is sufficient depends in part on the firm's external financing policy, since new money is also an ingredient of the construction budget. It is therefore interesting to ask what is the optimum construction budget when the optimization is restricted to those functions which correspond to a fixed rate of return and a given reasonable financing policy. We hope to treat this question in a future paper.

Acknowledgement

The authors are indebted to M. A. Chaudry and J. H. Rohlfs for helpful comments.

References

(1) H. Averch and L. L. Johnson: "Behavior of the Firm under Regulatory Constraint," American Economic Review, Vol. 52, December, 1962, pp. 1053-69.

For a summary of more recent development, see W. J. Baumol and A. K. Klevorick, "Input Choices and Rate-of-return Regulation: an Overview of the Discussion," The Bell Journal of Economics and Management Science, Vol. I, Autumn, 1970, pp. 162-190.

(2) J. Bauer; "Updating Public Utility Regulation," Public Administration Service, Chicago, 1966, Chapter 4.

(3) C. W. Cobb and P. H. Douglas: "A Theory of Production," American Economic Review, Supplement, March, 1928, pp. 139-265.

(4) See R. Courant and D. Hilbert: "Methods of Mathematical Physics," Vol. I, Interscience Publishers, NY, 1955, Chapter IV.

(5) E. Davis: "A Dynamic Model of the Regulated Firm with a Price Adjustment," The Bell Journal of Economics and Management Science, Vol. 4, Spring, 1973; pp. 270-282.

(6) H. Uzawa: "Production Functions with Constant Elasticity of Substitution," Review of Economic Studies, Vol. 29, October, 1962.