# ERGODIC PROPERTIES OF INFINITE SYSTEMS

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## Abstract

Macroscopic systems are successfully modeled in statistical mechanics, at least in equilibrium, by infinite systems. We discuss the ergodic theoretic structure of such systems and present results on the ergodic properties of some simple model systems. We argue that these properties, suitably refined by the inclusion of space translations and other structure, are important for an understanding of the non-equilibrium properties of macroscopic systems.

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## I. Introduction

Statistical Mechanics attempts to account for the observed behavior of macroscopic physical systems on the basis of the microscopic laws which govern the behavior of their "elementary" constituents (particles or molecules). The central fact of the entire endeavor is that the number of particles is very large ($\sim 10^{26}$), so that the concepts of probability theory play an important role. (For large systems a statistical description is consistent with deterministic 'laws' and indeed is the only one feasible). It is an assumption, which can be proven in some cases, that insofar as many qualitative aspects of macroscopic behavior are concerned, the exact nature of the microscopic laws, even whether they are classical or quantum, is of little importance. Since the problems of classical statistical mechanics are mathematically interesting, and since the physical concepts are, in the classical framework, more transparent, classical statistical mechanics continues to be a subject of much interest. [See Lanford*, and Lanford and Lebowitz articles in this book].

The success of the program, at least from the mathematical or rigorous point of view, has so far been largely limited to the realm of equilibrium statistical mechanics, which deals with the description of matter in thermal equilibrium. It explains in particular how very complex macroscopic systems are susceptible to a complete macroscopic description involving only a small number of parameters (temperature, pressure, energy, entropy, volume,...), which satisfy the simple relationships of thermodynamics. The explanation is obtained by means of the identification, due to Gibbs, of macroscopic states with measures on the phase space of microscopic descriptions and the appropriate utilization of the fact that macroscopic systems are "big".

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* Lanford's lectures contain much of the statistical mechanical background for this article.

The situation with respect to nonequilibrium statistical mechanics is far more tentative. Here one is concerned with why and how systems come to equilibrium, i.e., we would like to account for the experimental fact that starting from a nonequilibrium state the system will evolve, under the action of its time evolution, to the appropriate equilibrium state, and furthermore that this approach to equilibrium satisfies the relevant kinetic (e.g. Boltzmann), hydrodynamic, and transport equations. We shall say that such systems exhibit "good thermodynamic behavior", (gtb). Here, too, it is argued that macroscopic states should be identified with measures on the phase space, but the problems of nonequilibrium statistical mechanics have proven, so far, less tractable than their equilibrium counterparts.

Because of the key role played by probability theory, it is to be expected that only in some appropriate thermodynamic limit (number of particles approaches infinity) is gtb precisely achieved. Hence, if exact mathematical results are desired, infinite systems of particles should be directly investigated; their very large yet finite counterparts, which may for all practical purposes exhibit behavior of the type described by macroscopic laws, will nonetheless exhibit it only approximately, precluding the formulation and proof of the appropriate theorems. (In the same way phase transitions, which are associated with non-analyticities in thermodynamic functions, do not occur for finite systems, though numerical computations on systems containing just a few hundred particles and experimental observations on macroscopic systems mimic this behavior very closely).

The key ingredients in equilibrium statistical mechanics, are the appropriate abstract framework - to isolate the elements of structure relevant to the phenomenon - and the right limit: the framework is that of measure theory and the limit is that of infinite volume (and particle number) usually referred to as the thermodynamic limit. A microstate of a finite system (in a finite volume) is completely described by a point x of the phase space $\Gamma$; $x = (q_1,\ldots,q_n, p_1,\ldots,p_n)$ gives the coordinates and momenta of all particles of the system. There exists a natural measure $d\underline{q}\, d\underline{p}$ (Lebesgue measure) on $\Gamma$ whose "projection" $\Gamma_E$ onto a surface of constant energy (Hamiltonian function) E, gives a measure $\mu_E$, the microcanonical measure. According to statistical mechanics expectations of observables (functions on $\Gamma_E$) are equated to the equilibrium values of thermodynamic quantities of the system at energy E. The exact macroscopic relations are obtained by taking the limit $N \to \infty$, $V \to \infty$, $E \to \infty$, $N/V \to \rho$, $E/V \to \epsilon$, where N is the particle number, V the volume, E the energy, $\rho$ the particle and $\epsilon$ the energy density of the macroscopic system.

A major difficulty in non-equilibrium statistical mechanics lies in finding the framework and limit appropriate to the task. We shall assume, as has often been assumed, that the same thermodynamic limit as used in equilibrium statistical mechanics is appropriate, at least for part of the problem. (Different or additional limits may have to be used for obtaining more specific kinetic behavior, see Lanford's lecture for the appropriate limit needed for the derivation of the Boltzmann equation. The underlying structure is therefore taken to be the phase space of infinite (but locally finite) configurations of particles in $\mathbb{R}^\nu \times \mathbb{R}^\nu$, (where $\nu = 3$ for realistic systems).

Before considering the nonequilibrium behavior of infinite systems, we will review briefly the situation for finite systems. Here, the nonequilibrium behavior is governed by the time evolution $T_t$ determined by the Hamiltonian in terms of which the equilibrium situation is described. Since Liouville's theorem, asserting the invariance of Lebesgue measure under $T_t$, implies that $\mu_E$ is also preserved by $T_t$, one finds oneself directly within the context of ergodic theory: one has a triple $(\Gamma_E, \mu_E, T_t)$ of just the right kind. Indeed, ergodic theory arose out of the attempt to justify the foundations of statistical mechanics. It has typically provided the framework for the investigation of the nonequilibrium behavior of finite systems [1].

It seems physically plausible to assume that events corresponding to subsets of the phase space which have vanishing Lebesgue measure have vanishing probability of occurring, and should therefore be irrelevant for experimentally observed behavior (which depends on reproducibility). We may then regard as physically reasonable only those measures which are absolutely continuous with respect to the Lebesgue measure (or $\mu_E$)$^+$. With this assumption the traditional ergodic properties have the following implications: (see Lanford's lectures and [1]).

a) If $(\Gamma_E, \mu_E, T_t)$ is ergodic then, for systems with energy E, $\mu_E$ is the only physically reasonable measure which is stationary (invariant) under the time evolution; furthermore, any physically reasonable state will tend, in the sense of time average, to this uniquely determined equilibrium state.

b) If the system is in addition mixing, this approach to equilibrium occurs in the strict sense, i.e.,

$^+$ While $\mu_E$ is itself singular with respect to the Lebesgue measure $d\underline{q}\, d\underline{p}$ it may be regarded as the limit of measures concentrated on the 'energy shell' $(E, E + \Delta E)$. (The energy being both smooth and universally conserved plays a special role; see Lanford's lectures).

$$\rho_t(A) = \rho(T_{-t}A) \xrightarrow[t \to \infty]{} \mu_E(A),$$

where $\rho$ is an absolutely continuo nonequilibrium measure and $A$ is a bounded observable.

The stronger ergodic properties - K-system and Bernoulli - are not so directly susceptible to physical interpretation in terms of good thermodynamic behavior. We mention only that if $(\Gamma_E, \mu_E, T_t)$ is a K-system, its behavior will be completely nondeterministic in the following sense: A realistic measurement corresponds to a finite partition of the phase space of the observed system into subsets corresponding to the possible outcomes of that measurement. If we wish to obtain more information about the system we can repeat that measurement at discrete time intervals. If the system is K (and only if) then no such measurements even when carried out infinitely often in the past, determine with certainity the outcome of future measurements.

Thus, in a very strong sense, K-systems cannot be "finitely approximated", and since it is not unreasonable to expect of a finite system that it can be approximated by a sufficiently fine coarse graining (into position and momentum cells), the K-property has significant implications for finite systems. (For some interpretations of the Bernoulli property see [1,2]).

Now the KAM theorem and related results suggest that realistic models of finite physical systems may fail to be ergodic
Though this need not prevent large systems
from exhibiting, for all practical purposes, good thermodynamic behavior, it renders an exact result rather unlikely. It is therefore natural to investigate directly the nonequilibrium behavior of infinite systems.

The basic structural necessities for such an investigation - the equilibrium states $\mu$ and measure preserving Hamiltonian time evolutions $T_t$ - can be realized, as Lanford has indicated in his lectures, on the phase space X of infinite (locally finite) configurations. An obvious candidate for the abstract framework for the explication of the good thermodynamic behavior of infinite systems is therefore, as before, the (abstract) triple $(X, \mu, T_t)$. We are thus led to the investigation of the ergodic properties of infinite systems. However, some general observations indicate that such ergodic theory by itself is of rather limited potential:+

a) The previous remarks concerning the implications of ergodicity and mixing can be applied without modification to infinite systems if, by physically reasonable we understand absolutely continuous with respect to $\mu$ (the equilibrium measure). Because of the quasilocal nature of the $\sigma$-algebra of measurable sets employed in the description of these systems, measures absolutely continuous with respect to $\mu$ may be roughly interpreted as local perturbations of $\mu$. Therefore, insofar as return to equilibrium is concerned the situation is satisfactory. If we are interested, however, in understanding how a system ever comes to be in equilibrium - the problem of approach to equilibrium - then mixing says nothing, since the relevant initial states will typically be singular with respect to $\mu$. Similarly, ergodicity will be of little value in choosing one stationary state over another without some prior basis for selecting one ergodic state from the family of stationary (ergodic) states.

Consider, for example, the ideal gas: where particles undergo free motion in $\mathbb{R}^\nu \times \mathbb{R}^\nu$. $\mu$ is such that the particles are, independently of each other, uniformly scattered in space $(R^\nu)$ and their individual momenta have a Maxwellian (Gaussian)

+Some additional, hopefully relevant, structure will be discussed later.

distribution. This state as well as any other state differing from it only by having a different momentum distribution is stationary (since the momenta are constant). Furthermore, as we shall see later, the free time evolution on these states will typically be ergodic and, indeed, Bernoulli, the point, of course, being that states given by different momentum distributions are mutually singular.

b) For an infinite system to be K need not be a significant constraint upon its behavior, since it does not appear plausible that an infinite system should permit a finite approximation in the sense indicated above.

In view of these remarks it should not be very surprising to find that very simple infinite systems possess very good ergodic properties. We will now turn to the description of the ergodic properties of some such systems, particularly, non interacting systems.* It will then be shown how the introduction of additional structure (space translations) improves the situation.

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* See article by Lanford and Lebowitz for the ergodic properties of harmonic crystals.

## II. Ergodic Properties of the Time Evolution of Some Simple Infinite Systems

We shall be discussing two kinds of infinite systems, both of whose time evolutions have very good ergodic properties, but of which one exhibits distinctly better thermodynamic behavior than the other. Correspondingly, there are two different "mechanisms", which are responsible for these ergodic properties. The first of these, which is the only one possessed by the ideal gas corresponds to the escape of local information to infinity. The second, better mechanism, is a local dissipation of information.

Since we shall deal primarily with systems of non interacting particles, we first describe a convenient representation of such systems.

Let $(X_1, \Sigma_1, \mu_1)$ be a totally $\sigma$-finite nonatomic measure space. We regard $X_1$ as the phase space for a single particle and wish to construct from $(X_1, \Sigma_1, \mu_1)$ a probability space $(X, \Sigma, \mu)$, $\left(X \cong \left\{x = \{q_i, p_i\}, i \in Z^+\right\}\right)$ representing the independent distribution of particles in $X_1$ according to the measure $\mu_1$. Accordingly we let $X$ be the set of countable subsets of $X_1$ and define for any set $A \in \Sigma_1$, $N_A$ by

$$N_A(x) = \text{the cardinality of } (A \cap x)$$

for $x \in X$.

We then let $\Sigma = \sigma\{N_A\}$, be the smallest $\sigma$-algebra for which all of the $N_A$ are measurable, and define $\mu$ on $\Sigma$ as the probability measure representing a Poisson distribution of points in $X_1$, with density given by $\mu_1$. This is the measure for which the functions $N_A$ have distribution given by

$$\mu(\{x \in X | N_A(x) = m\}) = \exp[-\mu_1(A)][\mu_1(A)]^m/m! \quad ,$$

which, if true for all A, implies that disjoint regions of $X_1$ are independent [ 3].

For any automorphism $T_1$ of (flow $T_1^t$ on) $(X_1, \Sigma_1, \mu_1)$ we define an automorphism T of (flow $T^t$ on) $(X, \Sigma, \mu)$ by

$$Tx = T_1 x \qquad (T^t x = T_1^t x)$$

where the right hand side of the equation is to be understood as the action of the (one-particle) automorphism (flow) on the subset x of $X_1$. Thus we obtain a time evolution on the infinite system by letting all the particles move independently according to the one-particle time evolution. We call $(X, \mu, T(T^t))$[+] the Poisson system built over $(X_1, \mu_1, T_1(T_1^t))$.

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[+]We will for the most part drop the reference to the $\sigma$-algebra.

The ideal gas and the Bernoulli construction

In In its simplest manifestations, such as in the ideal gas, the 'escape to infinity' mechanism mentioned earlier has a formal representation directly exhibiting the Bernoulli character of the flow. We call this representation the Bernoulli construction:

For $C \subset X_1$, let $\Sigma_C = \sigma\{N_A | A \subset C\}$ be the local $\sigma$-algebra ($\subset \Sigma$) of C. If $\{A^n\}$ is a family of disjoint sets ($\in \Sigma_1$) such that $\bigcup_{n=-\infty}^{\infty} A^n = X_1$ and $T_1 A^n = A^{n+1}$ then $\Sigma_{A^o}$ is an independent generator for T. This means that the sequence $\Sigma_{A^o}, \Sigma_{A^1}, \ldots$ is an independent sequence of $\sigma$-algebras and that $\Sigma = \sigma\{\Sigma_{A^n} | n \in Z\}$. Since a generalized Bernoulli shift may be characterized by the existence of an independent generator, these systems give rise to Bernoulli shifts; we shall say that the Poisson system built over a such a system admits of a Bernoulli construction.

(We remind the reader (see [2]) that if $T^t$ is a flow such that $T^1 (\equiv T)$ is a Bernoulli shift then $T^\tau$ is Bernoulli for all $\tau$; thus, insofar as the Bernoulli property is concerned it suffices, when investigating a flow $T^t$, to investigate $T^1$.)

The simplest example of this type of system is the infinite ideal gas (IG). We can realize IG $(X^I, \mu^I, T^{It})$ as the Poisson system built over $(X_1^I, \mu_1^I, T_1^{It})$ where

$$X_1^I = \mathbb{R}^\nu \times \mathbb{R}^\nu$$

$$d\mu_1^I = \rho\, (\beta/2\pi)^{\frac{1}{2}\nu} \exp\left[-\tfrac{1}{2}\beta p^2\right] dq\, dp$$

and $$T_1^{It}\,(q,p) = (q + pt,\ p).$$

Here $(q,p) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$, $\rho$ is the particle density, $\beta$ the inverse temperature (of the ideal gas equilibrium state), and we have taken all the masses to be unity.

That IG admits of a Bernoulli construction follows from the fact that the set

$$A^o = \{(q,p) = ((q_1,\ldots,q_\nu),\ (p_1,\ldots p_\nu)) \in X_1^I \,|\, q_1 + p_1 t = 0 \text{ for some } 0 \leq t < 1\}$$

has, essentially by construction, the necessary properties.

Thus, we construct a partition of $X_1^I$ on the basis of the time at which the particle has vanishing first coordinate. We could also have partitioned according to the time of nearest approach to the origin. More generally, if our system $(X, \mu, T)$ were such that the latter did not determine a unique time, we could try to partition according to the last time of closest approach to the origin. It turns out, by virtue of 2) below, that if this too is essentially inadequate then no Bernoulli construction is possible.

This is related to the observation (suggested by the fact that though the one particle ideal gas has rather poor mixing properties, the Poisson system built over it has such properties of the strongest kind) that insofar as the Bernoulli construction is concerned, the ergodic properties of a Poisson system are roughly inversely proportional to those of the one particle system over which it is built. This is stated more precisely in the following proposition:

<u>Proposition</u>: 1) If $(X_1, \Sigma_1, \mu_1, T_1)$ is ergodic in the sense that if a set $A \in \Sigma_1$ is invariant, either it or its complement has zero measure, no Bernoulli construction is possible.

2) If there exists a set $A \subset X_1$ of finite measure which is recurrent in the sense that points in a non null $\Sigma_1$ subset of $A$ return to $A$ infinitely often under the action of $T_1$, no Bernoulli construction is possible. If all sets of finite measure are non-recurrent, one can perform a Bernoulli construction [4].

These possibilities are illustrated by the random walk models (RW$\nu$). RW$\nu$ denotes the Poisson system built over the automorphism representing a particle undergoing a symmetric random walk on $\mathbb{Z}^\nu$. Thus RW1 is built over $(B \times \mathbb{Z}, T_1)$, where $B$ is the space of doubly infinite sequences $\xi_i = \pm 1$ equipped with the measure obtained by assigning equal probability to + 1 and - 1 and forming the infinite product measure, and $T_1$ is given by

$$T_1(\xi, n) = (S\xi, \xi_o + n),$$

where $S$ is the shift on $B$ and $\xi = \{\xi_i\} \in B$. $B \times \mathbb{Z}$ is to be understood as equipped with the natural product measure. The models RW$\nu$, $\nu > 1$, are similarly formed.

Since, with probability one, a particle undergoing a random walk on $\mathbb{Z}^\nu$ will return to the origin (infinitely often) only if $\nu \leq 2$, we have that $RW^\nu$ is recurrent if and only if $\nu \leq 2$, so that a Bernoulli construction (based, say, upon the last time of nearest approach to the origin of $Z^\nu$, using only the $Z^\nu$ coordinate to measure distance) is possible only for $\nu \geq 3$.

We observe that $RW^\nu$ has much better "mixing" properties than IG and that this is reflected by the presence in $RW^\nu$ of a local mechanism of mixing, to which we earlier referred. Before pursuing this further, we indicate, in the next section, how the properties of IG are affected if we allow the particles to interact in the simplest possible way. We shall see that while the flow of local information to infinity is hindered by the interactions, such a loss of local information still takes place by a mechanism similar to the diffussion in the $RW^\nu$ models (without however having a local mechanism for mixing) and gives rise to good ergodic properties (at least K). To make this clearer we digress briefly with a comment about K-systems.

A general characterization of a K-system is by the existence of a measureable partition (or $\sigma$-algebra) $\xi$, called a K-partition, with the following properties.

1) $T^t \xi \geq \xi$ for $t \geq 0$.

2) $\bigvee_t T^t \xi = \epsilon$.

3) $\bigwedge_t T^t \xi = \nu$.

Here $\xi \geq \eta$ means that the elements of the partition $\eta$ are unions of elements of $\xi$ ($\xi$ is finer than $\eta$), $\bigwedge_\alpha \xi_\alpha$ is the finest measurable partition coarser than all the $\xi_\alpha$, $\epsilon$ is the partition into points (corresponding to the full $\sigma$-algebra $\Sigma$), $\nu$ is the partition whose sole element is the entire space X (corresponding to the $\sigma$-algebra of trivial sets, which have measure zero or measure one), and all statements here and to follow are to be understood as up to a set of measure zero.

The connectinn between this characterization and the definition (for finite entropy) in terms of the existence of a finite generating partition P with trivial tail ($\bigwedge_j \bigvee_{k=j}^{\infty} T^{-k}P = \nu$) is obtained by setting $\xi = \bigvee_{k=1}^{\infty} T^{-k}P$. It should thus not be surprising that $\xi(\bigcup_{k=1}^{\infty} T^{-k} \Sigma_{A^o})$ (for the one dimensional IG), the partition according to "origin events" (see below) occurring after t = 1, should be a K-partition. (This is just the zero-one law for tail events.)

Ergodic properties of the time evolution of some one dimensional systems of hard rods

The independent generator $\Sigma_{A^o}$ which we obtained for IG may be described for $\nu = 1$ as the $\sigma$-algebra determined by events at the origin between times zero and one. The two crucial facts concerning these "origin events" are that

a) they generate: two phase points having the same origin events must coincide (except perhaps for a set of zero measure), and

b) origin events in disjoint time intervals are independent.

We wish to know how altering our system by introducing a pair interaction, and making the corresponding change of measure, will affect the ergodic properties of the system, and in particular how it will affect a) and b).

We may roughly describe the systems we wish to consider - one dimensional infinite systems of hard rods of diameter $d > 0$ - as having measure spaces which are obtained from that of IG (with general momentum distribution given by f(p)) by merely deleting phase points for which some pair of particles is closer than d (we ignore the difficulty that what remains has zero measure), and as having a dynamics differing from free motion by the stipulation that when a pair of particles collide (i.e., approach within a distance d of each other) they exchange momenta (all masses are the same). The dynamics may equivalently be described by requiring particles to exchange positions during collisions, their momenta being unaffected. Since, if we adopt the latter description, the hard rod system (HR) goes, in the limit $d = 0$, into IG even insofar as single particle motion is concerned, this description is the more convenient for the investigation of its ergodic properties.

Sinai [5] has shown that the partition according to (appropriately defined) future origin events are K-partitions for HR. Thus a) is preserved by the introduction of the hard core interaction, and though b) will no longer hold, the K-property implies an approximate independence between sufficiently separated (in time) past and future origin events. If f (which, for simplicity we take to be even) vanishes in a small neighborhood of the origin, it can be shown [6] that enought of b) remains for HR to be Bernoulli.

More precisely, what is shown is that if one defines "origin event" appropriately, there exist arbitrarily "fine" finite partitions determined by origin events between times zero and one, which satisfy a strong property of approximate independence of past and future called the weak Bernoulli property [7]. Now, if a partition P of X satisfies this property, then $(X, \bigvee_{j=-\infty}^{\infty} T^j P, \mu, T)$ is a Bernoulli shift. Since an increasing union of Bernoulli shifts is Bernoulli [2] (i.e., if $(X, \Sigma_n, \mu, T)$ is Bernoulli for each n and $\Sigma_n \nearrow \Sigma$, then $(X, \Sigma, \mu, T)$ is Bernoulli) these Hard Rod systems will form Bernoulli flows.

The point is that what is going in this case in HR is not very different from what is going on in IG. A particle of velocity v moves at this velocity except for moments of collision, when it jumps the distance d in the direction of the other colliding particle. However, the "free distance" between two pulses (obtained by subtracting the total length of rods between them) behaves linearly in time. Thus, if one uses a "reduced description" obtained by removing the volume occupied by the hard cores from the distances between the particles ...

the exact IG time evolution. If there were a reflecting wall at the origin, this reduction would be straightforward and one would immediately obtain an isomorphism between HR and IG [8]. In our case, however, the reduction is problematical, having the following defects:

i) The image of the hard rod measure in the reduced description is Poisson only outside a neighborhood of the origin.

ii) The location of the unreduced origin jumps in the reduced description whenever a particle crosses it.

Because of ii) the same particle (momentum pulse) may appear at the origin more than once, precluding the independence of future origin events from past ones. However, if the momentum distribution is as described, all particles will eventually leave the origin, never to reappear, since the origin undergoes essentially a random walk in the reduced description. The measure in the reduced description is then sufficiently close to Poisson to allow the properly defined "origin event" partitions to satisfy the weak Bernoulli property.

For more general velocity distributions, in particular when there are a finite fraction of particles with zero velocity, the loss of local information is no longer fast enough to prove the Bernoulli property. Nevertheless a 'diffusive' type of loss to infinity is present giving rise to the K-property without a good local mechanism for loss of information. This is made more precise in the next chapter, where we indicate that a quantity, which may be interpreted as the local rate of dissipation (of, say, information) is zero for both IG and HR. We thus wish to consider systems for which a different mechanism, which is local, is responsi for the "mixing" which occurs, and it is to these that we next devote out attention.

The Lorentz gas and systems of periodic-K-type

Instead of turning on a pair interaction, which, except in the most trivial cases, would make the investigation of ergodic properties exceedingly difficult, we turn on an external field. The particles thus remain independent, though the one particle systems are modified. We thus consider the Lorentz gas (LG), which is like IG except that the particles are excluded from a disconnected region whose components have disjoint convex closures and smooth boundaries from which the particles are elastically reflected. These convex barriers are distributed throughout $\mathbb{R}^{\nu}$ in a periodic array (with periods $L_1,\ldots,L_{\nu}$). We will primarily consider the case when all particles have unit speed, so that the momentum distribution, instead of Maxwellian, is merely spherically symmetrical. These two systems, Maxwellian and constant speed, which may of course be represented as Poisson systems in an obvious manner, are identical insofar as any of the ergodic properties which we investigate are concerned. Therefore we will denote both systems by LG, though the arguments which we give will directly apply only to the latter system.

An important feature of LG is its periodic structure. A Poisson system will be called <u>periodic</u> if a) its one-particle measure space can be represented in the form $(X_0, \mu_0) \times \mathbb{Z}^{\nu}$, when $\mu_0$ is a finite measure and $\mathbb{Z}^{\nu}$ is regarded as equipped with counting measure, and b) the representation of the one-particle dynamics in $X_0 \times \mathbb{Z}^{\nu}$ commutes with the translations $S_{\ell}$, $\ell = 1,\ldots,\nu$, of $X_0 \times \mathbb{Z}^{\nu}$ induced by the natural unit translations of $\mathbb{Z}^{\nu}$. For LG we may take $(X_0, \mu_0)$ to be the restriction of the one particle measure to $V_0 \times \mathbb{R}^{\nu}$ (momentum space), where $V_0$ is the $\nu$-dimensional "rectangle" $\displaystyle\mathop{\times}_{i=1}^{\nu} [-\tfrac{1}{2} L_i, \tfrac{1}{2} L_i)$ of sides $L_i$ centered at the origin. Furthermore, since the dynamics of LG is invariant under translation by $L_i$ in the <u>ith</u> direction, we also have b).

For periodic systems a set of the form $X_o \times \Lambda$ will be called a rectangle R if $\Lambda$ is a (bounded $\nu$-dimensional) rectangle of $Z^\nu$. ($\Lambda$ must be a product of intervals). We equip any rectangle R with the natural probability measure ${}_R\mu_1$ induced by the one-particle measure $\mu_1$ and with the automorphism ${}_RT_1$ obtained from the one-particle time evolution $T_1$ by replacing $Z^\nu$ by $Z^\nu/Z^\nu_R$, where $Z^\nu_R$ is the subgroup of $Z^\nu$ corresponding to the rectangle R. Finally, we let $R_o = X_o \times \{0\}$.

For LG any rectangle R (whose projection in $Z^\nu$ has sides of length $n_1, .., n_\nu$) is represented in $\mathbb{R}^\nu$ by a "rectangle" (with sides of length $n_1L_1, ..., n_\nu L_\nu$) The induced dynamics ${}_RT_1^t$ is obtained from $T_1^t$ by identifying the opposite sides of R - periodic boundary conditions. Thus the systems (R, ${}_R\mu_1$, ${}_RT_1$)are just the finite volume one-particle systems of which the Poisson system built over ($X_1$, $\mu_1$, $T_1$) is the infinite volume (and particle number) limit.

Though all the noninteracting systems considered here have been periodic, they can be distinguished by the degree of mixing exhibited by their finite volume time evolutions ${}_RT_1$. For IG ${}_RT_1$ will be "far from mixing", while for LG, as Sinai has shown, (explicitly for $\nu$=2),the ${}_RT_1$ K-automorphisms (and are in fact, Bernoulli) [see Gallavotti's lecture and references there ]. RW$\nu$ is similar to LG in this respect: for RW1, for example, ${}_RT_1$ is Bernoulli if R has odd length [4]. The good mixing behavior of these one particle systems should give rise to a (local) mechanism for the production of good mixing properties of the Poisson systems built over them.

LG and RW$\nu$ thus satisfy:

a) There exists a sequence $R_i$ of rectangles, whose volumes (in $\mathbb{Z}^\nu$) approach infinity, such that the ${}_{R_i}T_1$ are K-automorphisms.

They also satisfy:

b) $T_1(R_o)$ is bounded.

c) ${}_{R_o}T_1$ has finite entropy (see [2] or section III below for definition).

Periodic systems satisfying a) - c) will be said to be of <u>periodic-K-type</u>. b) and c) are of purely technical significance.) The main result concerning them is contained in the

<u>Theorem:</u> Systems of periodic-K-type form K-systems.

We give a sketch of the proof. The basic idea is to "lift" the structures responsible for the mixing behavior of the one-particle system to the Poisson system built over it. The essence of the "lifting" is embodied in the following construction: If $\xi$ is a measurable partition of $(X_1, \mu_1)$, we obtain a measurable partition $\bar{\bar{\xi}}$ of $(X,\mu)$ by partitioning X according to the number of particles in the fibers (elements) of $\xi$.

The following lemma is crucial:

<u>Lemma</u>: If $\xi$ satisfies,

1') $T_1 \xi \geq \xi$

2') $\bigvee_n T_1^n \xi = \epsilon$,

3') the fibers of $T_1^{-n} \xi$ expand toward infinity as $n \to \infty$,

then $\bar{\xi}$ is a K-partition for $(X, \mu, T)$.

3') means that for any point $x \in X_1$ and any bounded set $A \subset X_1$, the fraction of the fiber of $T_1^{-n} \xi$ containing x which overlaps A approaches zero.

<u>Proof</u>. 1) and 2) of the definition of K-partition follow easily from 1') and 2'), while 3) holds because 3') guarantees that the only information contained in $\bigwedge_n T^{-n}\bar{\xi}$ must be measurable with respect to the local

$\sigma$-algebra of the complement of any bounded region of $X_1$, which, since our system is Poisson, implies 3) [4].

To obtain a partition satisfying 1') - 3') we utilize the K-structure possessed by $(X_1, \mu_1, T_1)$. Let P be a finite generator for ${}_{R_o}T_1$. We form the "product" $Q_oP$, where $Q_o$ is the partition of $X_1$ induced by the partition of $Z^\nu$ into points. $Q_oP$ is the partition obtained by refining $Q_o$ by partitioning each of its elements "according to P". Let $\xi = \bigvee_{j=0}^{\infty} T_1^{-j}\, Q_oP$ (the partition according to future $Q_oP$-names). $\xi$ satisfies 1') - 3'). 1') and 2') are immediate. 3') is satisfied because $\xi/R$ (the restriction of $\xi$ to the rectangle R) is a K-partition for ${}_RT_1$ if ${}_RT_1$ is a K-automorphism. The Doob Martingale Theorem then implies 3') [4].

Finally, we observe that the partition $\xi$ which we have constructed also satisfies

4') $\xi$ is invariant under translations $(S_\ell)$,

5') $\xi \geq Q_o$.

Thus systems of periodic-K-type possess translation invariant K-partitions ($\xi$) reflecting a local mechanism for the dissipation of disturbances.

## III. Space-Time Ergodic Theory of Infinite Systems

We denote by IG1 the system which differs from the one-dimensional IG only in that all particles are constrained to move with unit velocity to the "right", say. In exactly the same manner as for IG, IG1 forms a Bernoulli flow.

We are now in a position to further evaluate the adequacy of the $(X,\mu,T^t)$ framework for the account of the emergence of good thermodynamic behavior. Two comments are relevant:

a) Since IG1, whose thermodynamic behavior must be regarded as rather poor, forms a Bernoulli flow - the strongest of ergodic properties -

the ergodic properties associated with the $(X, \mu, T^t)$ framework cannot possibly bear an adequate relation to gtb.

b) The situation is even worse. It appears likely that LG forms a Bernoulli flow. (If it does not it affords a rather natural example of a K-system which is not Bernoulli (see [2]). It is also rather plausible that realistic models of physical systems which exhibit gtb should have ergodic properties as strong as those of systems which don't, and hence should be Bernoulli. If this is so, then Ornstein's isomorphism theorem for Bernoulli flows would imply that these systems are completely identical from our abstract point of view, rendering the $(X, \mu, T^t)$ framework entirely inadequate for the formalization of the relevant thermodynamic distinctions.

(The relevant theorem asserts that two Bernoulli flows of infinite (K-S) entropy are isomorphic. All infinite systems to which we have thus far referred have infinite entropy. That systems which permit a Bernoulli construction have infinite entropy may be seen by noting that $(X, \Sigma_{A^o}, \mu)$ is a continuous measure space, and hence given any n we may find a partition P of X having n elements of equal measure which are measurable with respect to $\Sigma_{A^o}$. Since $\Sigma_{A^o}$ is an independent generator, we have that $h(T) \geq h(P,T) = \log n$.)

Perhaps the simplest way to expand our abstract framework is to replace T by $\mathcal{J}$ in the triple $(X, \mu, T)$, where $\mathcal{J}$ is the abelian group generated by space translations and time evolution. This is sensible because translation invariant equilibrium states have the greatest physical significance, since they represent a homogeneous macroscopic situation. Furthermore, if the temperature and fugacity are such that no phase transition occurs (e.g., a low density gas), the unique equilibrium state must be translation invariant. In addition, since physical interactions are translation invariant, space translations and time evolution will commute. Finally, since the localizing (physical) space ($\mathbb{R}^\nu$) is a crucial element in the underlying structure of infinite systems [see Lanford's Lectures], it is natural that some token of that structure-space translations- should be embodied in the abstract framework. We also note that the systems which we are considering - the periodic systems - fit naturally into the new framework, if by $\mathcal{G}$ we understand the group generated by T and the $S_i$, i.e., the discrete space-time (translation) group.

For simplicity we consider only the discrete space-time group. We shall further limit the detailed description to the case of one spatial dimension, though everything easily extends, at the expense of somewhat more cumbersome notation, to higher dimensions.

Introduction of space translations produces the immediate benefit of alleviating the embarrassment posed by Ornstein's theorem: though the theorem should be generalizable to multi-dimensional abelian groups [ 9], it should be much more difficult, and, as we shall argue, of much greater significance, for an infinite system to be Bernoulli under the space-time group.

## Space-time ergodic properties of periodic systems

We will show that the two mechanism (the local and the non local) for the production of the "mixing" behavior of our systems, which lead to similar properties within the $(X,\mu,T)$ framework, lead to very different sorts of ergodic properties in the space-time framework.

We will denote by $(X, \Sigma, \mu, \mathcal{G})$ both the general dynamical system and specific infinite particle systems, with $\mathcal{G}$ the abelian group generated by two automorphisms, S and T, of the probability space $(X, \Sigma, \mu)$. If X is the phase space of a system of particles, S and T will denote respectively unit space translation and unit time evolution. If $(X, \Sigma, \mu, T)$ is a periodic (Poisson) system, $(X, \Sigma, \mu, \mathcal{J})$ will also be called a Poisson system, and we will continue to designate the corresponding one-particle system by means of the subscript "1".

Though the properties which we shall discuss will be formulated in terms of thr pair (S,T), they will depend only upon $\mathcal{J}$, unless we indicate otherwise. We shall have to distinguish between the system $(X, \mu, \mathcal{J})$ and the system $(X, \mu, (S,T))$ only if S and T are not independent (e.g., IG1) or when considering (S,T) K-systems.

A system $(X, \mu, \mathscr{J})$ is ergodic if measurable sets invariant under $\mathscr{J}$ have measure zero or measure one. Since this property is weaker than the ergodicity of T alone, we shall have nothing more to say about it.

$(X, \mu, (S,T))$ is said to be mixing if

$$\lim_{(n,m) \to \infty} \mu (S^n T^m A \cap B) = \mu(A) \mu(B)$$

for all $A, B \in \Sigma$. By $(n,m) \to \infty$ we mean that $n^2 + m^2 \to \infty$. Thus if a system is (S,T) mixing it is not only mixing under both S and T; after a sufficiently long "time" an event A - and all its translates-will become approximately independent of a fixed event B. In particular, an observation will tend to become uncorrelated not only with future performances of that observation but, uniformly, with similar observations performed at different locations. Thus systems in which the dissipation of disturbances is purely non local, i.e. in which the disturbances travel unhindered to infinity and are otherwise unaffected, are not (S,T) mixing.

This state of affairs is illustrated by IG1, for which S and T coincide. Here $S^n T^{-n} = I$ (the identity) so that IG1, with purely non-local dissipation, is not (S, T) - mixing.

On the other hand, IG is (S, T) - mixing. This is so because

$$\lim_{(n,m) \to \infty} \mu_1 (S_1{}^n T_1{}^m A_1 \cap B_1) = 0$$

if $A_1$ and $B_1$ are bounded measurable subsets of $X_{\mathbb{I}}$ $(= \mathbb{R}^2)$. Thus the space-time framework distinguishes two systems identical in the framework of the time evolution. That the mixing which occurs in IG is not purely non local is a consequence of the dispersion due to the continuous momentum distribution.

The mixing which occurs in IG is nontheless of a very poor sort and we would thus like to be able, within the expanded framework, to distinguish IG from systems like LG in which the mixing is of a much "better" sort. To do so we will use the notions of (S, T) - K-system and $\mathcal{G}$-entropy.

The concept of (S, T) - K-system which we will define (see [ ]) is a property of an ordered pair (S, T) of automorphisms rather than of the group $\mathcal{G}$ which they generate. Insofar as space translations and time evolution play rather different roles in statistical mechanics, this distinction is quite appropriate. The definition may be regarded as obtained by means of a generalization of the natural ordering of $\mathbb{Z}$, on the structure of which the notion of K-system for a single automorphism is implicitly based, to an ordering of $\mathbb{Z}^2$. We will write $(n,m) \leq (p,q)$ if $m < q$ or if $m = q$ and $n \leq p$. $(X, \mu, (S,T))$ is said to be a K-system if there exists a measurable partition $\zeta$ which

1) is increasing: $S^n T^m \zeta \geq \zeta$ i $(n, m) \geq (0,0)$

2) generates: $\bigvee_{(n,m)} S^n T^m \zeta = \varepsilon$,

3) has trivial tail: $\bigwedge_{(n,m)} S^n T^m \zeta = \nu$, and

4) has "continuity": $\bigwedge_n S^n \zeta = T^{-1} \bigvee_n S^n \zeta$.

1)-3) imply that there exist an S- invariant K-partition for T. As it does not appear likely that IG should possess such a partition, it is suggested that IG is not a (S, T) - K-system. That this is so we shall later show using $\mathcal{G}$-entropy. Furthermore, if $(X, \mu, T)$ is of periodic K-type, it is not difficult to construct from the K-partition $\bar{\xi}$ obtained for this system, using the fact that $\xi$ satisfies 4') and 5') (see end of section II), a partition satisfying 1)-4) above. Thus systems of periodic - K-type, and in particular LG, form space-time K-systems, and IG is not isomorphic to LG in the space-time framework.

We mention that we are interested in the concept of (S, T)- K-system primarily because it is a property which is easy to check for systems of periodic - K-type and which implies that the system have completely positive $\mathcal{G}$-entropy, of which we shall soon say more.

The entropy of is defined in a manner completely analagous to the definition of the entropy of an automorphism T (see [2]). Recall that the entropy H(P) of a partition $P = \{P_i\}$ is given by

$$H(P) = -\sum \mu(P_i)\log \mu(P_i).$$

$h(P,\mathcal{G})$ is defined by

$$h(P,\mathcal{G}) = \lim_{\alpha \to \infty} 1/|\alpha| \, H\Big(\bigvee_{(n,m)\in\alpha} S^n T^m P\Big)$$

where $\alpha \subset Z^2$ is a parallelogram, $|\alpha|$ is the number of points in $\alpha$, and $\alpha \to \infty$ means that the distances between parallel sides approach infinity. That this limit exists is a consequence of the subadditivity of the entropy of a partition:

$$H(P_1 \vee P_2) \leq H(P_1) + H(P_2).$$

The entropy $h(\mathcal{G})$ of $\mathcal{G}$ is then given by

$$h(\mathcal{G}) = \sup_P \; h(P,\mathcal{G}).$$

If P generates (i.e.. if $\bigvee_{(n,m)} S^n T^m P = \varepsilon$) we have that $h(\mathcal{G}) = h(P,\mathcal{G})$, just as for a single automorphism. Recall, too, that for a single automorphism we have that

$$h(P,T) = H(P/\bigvee_{j=1}^{-\infty} T^j P) = H(P/\bigvee_{j<0} T^j P).$$

Using the ordering of $Z^2$ introduced above, one may formulate (see [10]) a similar expression for the $\mathcal{G}$-entropy of a partition P:

$$h(P,\mathcal{G}) = H(P/\bigvee_{(n,m)<(0,0)} S^n T^m P).$$

This expression implies that $h(\mathcal{L})$ is, rather roughly, a measure of the local (asymptotic) rate of loss of information.

Another, apparently better, measure of the local rate of information dissipation may be obtained as follows: If $(X, \mu, T)$ is the time evolution of an equilibrium state of an infinite system of particles (translation invariant) and if $\Lambda$ is a finite rectangular volume, we may consider the projection $(X_\Lambda, \mu_\Lambda, T_\Lambda)$ of $(X, \mu, T)$ into $\Lambda$ obtained by ignoring all particles not in $\Lambda$ and by defining $T_\Lambda$ unambiguously by means of periodic boundary conditions (say) and by allowing the motions of the individual particles to be influenced only by the mutual interaction between particles in $\Lambda$ and possibly by an external field. If $(X, \mu, T)$ is a periodic (Poisson) system, $\Lambda$ should be taken to be a rectangle (see Section II.)

We define the T-entropy per unit volume to be the

$$\lim_{\Lambda \to \infty} h(T_\Lambda)/|\Lambda|,$$

where $|\Lambda|$ denotes the volume of $\Lambda$ and $\Lambda \to \infty$ in the sense that the smallest side of $\Lambda$ approaches infinity. This quantity may be regarded as a measure of the loss of information (due to "collisions") occurring in the system per unit volume per unit time (local rate of information loss).+

It is not obvious from the definition that the local rate of information loss is an invariant of the (S, T) - framework, i.e. that two systems which are abstractly isomorphic have the same rate of information loss. If this quantity were given by $h(\mathcal{L})$, this of course would be so, and, at the same time, we would have a nice interpretation for $h(\mathcal{L})$. In this direction we have the following result:

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+ We are indebted to D. Penrose for very valuable suggestions and discussions on this point.

Proposition: If $(X, \mu, \mathcal{J})$ is a Poisson system (satisfying some rather mild additional conditions (see [11])), then $h(\mathcal{J}) = h(T_\Lambda)/|\Lambda|$ for any rectangle $\Lambda$ (no limit necessary).

We conjecture that a similar proposition holds for a general (interacting) infinite system; of course, the limit $\Lambda \to \infty$ would then be essential.

The above proposition implies that IG and LG (say) can also be distinguished on the basis of their space-time entropies: $h(\mathcal{J})$ must be finite and nonvanishing for LG since this is so for the finite volume projections (K-systems have positive entropy) while the finite volume IG has vanishing entropy. We now give a sketch of a direct proof of the vanishing of $h(\mathcal{J})$ for IG. We do this by finding a partition P such that

$$\bigvee_{j=-\infty}^{-1} T^j (\bigvee_n S^n P) = \varepsilon$$

This will suffice because we would have

$$h(\mathcal{J}) = h(P, \mathcal{J}) = H(P/\varepsilon) = 0,$$

since $P \leq \varepsilon$.

We choose for P the partition with elements of the form

$$P\ \{n;\, m_1, \ldots, m_n;\, k_1, \ldots, k_n\}.$$

Here n is the number of particles in $[0,1)$ (i.e., in $[0,1) \times R$, the unit cell at the "right" of the origin in the one-dimensional IG). The particles in any cell $[\ell, \ell+1)$ are indexed according to their distance from the "left" boundary of that cell. $m_i$ labels the cell which at $t = 1$ will contain the particle with index i in $[0,1)$ (the ith particle from the origin) and $k_i$ is the index at $t = 1$ in $[m_i, m_i + 1)$ of that particle.

Thus the partition $\bigvee_{j=-\infty}^{\infty} S^j P$ gives not only the number of particles in each "unit cell", but also provides some velocity information by giving the cell membership of each particle at $t = 1$ along with sufficient index information to keep track of the identity of each particle so that upon successive applications of T we obtain partitions giving information about the trajectories with respect to unit cells of the $t = 0$ indexed particles. The velocity and the exact position of each particle may then be determined (using, e.g., the Jacobi theorem for the irrational rotation of the circle), proving the desired result. (We remark that though the partition P just defined is not finite, its entropy $H(P)$ is finite [11], and partitions of finite entropy have essentially the same properties as finite partitions [12]). By a similar argument it may be shown that $h(\mathcal{G}) = 0$ for the hard rod systems as well, providing some justification for our assertion that the mixing which occurs in the hard rod systems is of essentially the same sort as occurs in IG.

Just as for a single automorphism $(X, \mu, \mathcal{G})$ is said to have completely positive entropy if $h(P, \mathcal{G}) > 0$ for all non-trivial partitions P. For a single automorphism it is this concept, which is equivalent to that of K-system, upon which the characterization of K-systems in terms of finite approximation and completely non-deterministic behavior given in the introduction is based. It is therefore of interest that (S, T) - K-systems have completely positive $\mathcal{G}$-entropy. (We don't know whether the converse holds). IG thus does not form a (S, T) - K-system. Systems with completely positive $\mathcal{G}$-entropy are completely non-deterministic in a much stronger sense than would be implied by completely positive T-entropy. These systems do not admit of an approximation, in the sense given in the introduction, by the outcomes of a measurement (coarse graining) which is global and only locally finite, in the sense that it is performed by making a doubly infinite sequence of measurements (corresponding to $\bigvee_n S^n P$) each of which is a translate of some finite measurement (corresponding to a finite partition P). This is evidently a reasonable extension to infinite systems of the concept of "finite approximation" for finite systems.

The final property we wish to mention is that of $\mathcal{G}$-Bernoulli system. This may be defined by the existence of a measurable partition P which is an independent generator for $\mathcal{G}$, i.e. $\{S^n T^m P\}$, $n \in Z$, $m \in Z$, forms an independent family of partitions and generates. If $(X, \mu, \mathcal{G})$ is Bernoulli, then a) $(X, \mu, S)$ is Bernoulli and b) $(X, \mu, T)$ has an S-invariant independent generator (for T, namely, $\bigvee_n S^n P$). $\mathcal{G}$-Bernoulli systems can, in fact, be characterized by a) and b), since factors of Bernoulli shifts are Bernoulli [2 ]. As the space-time Bernoulli property implies the existence of a (generating) <u>global</u> measurement whose successive (time) iterates are independent, it appears to be an appropriate extension to infinite systems of the time Bernoulli property for finite systems. We have not yet found any particle systems to be $\mathcal{G}$-Bernoulli.

<u>Concluding remarks</u>

i) The $\mathcal{G}$-ergodic properties are invariant under (integral) Galilean transformations. This need not be the case for T-ergodic properties - consider IG1. This is so because from the abstract standpoint we may view the sole effect of a Galilean transformation as the replacement of the pair (S, T) by another pair of generators (S', T') of $\mathcal{G}$ (see [11]).

ii) Space-time ergodic theory, like time ergodic theory, cannot deal directly with the approach to equilibrium, as opposed to the return to equilibrium, problem (see section 1). Thus, if the LG starts out with an initial velocity distribution f(p) which is not spherically symmetric (but is otherwise Poisson) we cannot, on the basis of its good $\mathcal{G}$-ergodic properties say anything about approach to a stationary state; since this initial state is not absolutely continuous with respect to any stationary state. Nevertheless, it is easy to see that the same local mixing mechanism which is responsible for LG being a $\mathscr{S}$ K-system will also bring the system to a stationary state, with a velocity distribution $\bar{f}(p)$ obtained from $f(p)$ by angular averaging. Similar results can be proven for more general initial states of this kind. Indeed we expect an approach to a stationary state to occur whenever the initial state is 'singular' with respect to the good stationary measure only in a global sense, i.e. its projection on any finite region $\Lambda$, would be absolutely continuous with respect to a correspondingly projected stationary measure (with good mixing properties). Of course in LG all Poisson measures with spherical symmetric velocity distributions are stationary and there is therefore no approach to a 'unique' equilibrium state. We may expect however that in a system with mutual particle interactions which also possesses a good local dissipation mechanism, e.g. a hard sphere system, that a true approach to equilibrium will take place from initial states 'locally absolutely continuous' with respect to the equilibrium state. We conjecture that good $\mathscr{S}$-ergodic properties (e.g. completely positive $\mathscr{S}$-entropy) will be somehow related, as in LG, to such good (non ergodic theoretic) gtb behavior.

iii) In attempting to formulate, in the infinite system context, additional properties of physical import similar to that of mixing for finite systems several vague possibilities suggest themselves. We mention a couple:

a) One might consider some sort of topological mixing: Infinite systems come equipped with a natural topology within whose framework measure zero difficulties or consequences thereof may not appear.

b) the framework of something like Axiom A Attractors might be useful. Pehaps the phase space is largely decomposable into disjoint regions, each with their own "attractor" and a unique limit measure on each "attractor".

iv) Finally, we note that to completely account for gtb one would almost surely have to use the limit employed for the derivation of the Boltzmann equation [see Lanford's Lectures], in which, in particular, the particle density becomes infinite while the size of the particle vanishes. We expect, however, that such a limit should not be necessary for the problem of (non detailed) approach to equilibrium - it should be possible to formulate the infinite system "physical" analog of mixing for finite systems.


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