Phase transitions of a multicomponent Widom–Rowlinson model*

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We study a multicomponent version of the "$A-B$" model of Widom and Rowlinson, generalized in a symmetric way: There is an infinite repulsive interaction between any two unlike particles. We consider both lattice and continuum versions of the model and show that the "demixing" transition occurs for any finite number M of components, all having the same activity. No conclusion can be drawn about this transition in the limit $M \to \infty$. It is shown, however, that another transition, in which the density is greater on one of the sublattices, appears at a finite value of M which persists for all larger M at any fixed value of the activity. In the limit $M \to \infty$, $z \to 0$, $Mz = \zeta$, const, this system apparently becomes "equivalent" to a one-component system with activity ζ in which there is an exclusion for occupancy of nearest neighbor sites. The latter transition then becomes the "hard square" transition.

I. INTRODUCTION

Widom and Rowlinson introduced a model[1] for fluid systems which has been quite fruitful. The model postulated two types of particles (A and B), each of which had no interaction with other molecules of the same kind. Between unlike molecules, however, there was an infinite repulsive interaction. Widom and Rowlinson discussed the thermodynamics and symmetry of the demixing transition predicted to occur at high activities, and also showed the equivalence of the two-component model with a one-component model having many-body forces.

Lebowitz and Gallavotti[2] constructed a lattice version of the $A-B$ model in which the $A-B$ interaction was $+\infty$ for separations of one (or zero) lattice units, and vanished otherwise. (This was their Model 1; other variations were also discussed.) The Peierls' contour argument[3] was employed to prove rigorously that the lattice version of the $A-B$ model does in fact have a demixing transition.

Ruelle[4] then extended the proof to the original continuum model, where the only nonzero interaction was an infinite repulsion if the separation between an A particle and a B particle became less than some fixed distance R. Lebowitz and Lieb[5] then showed that Ruelle's proof could be modified to cover the continuum case with a soft $A-B$ repulsion, at sufficiently low temperature. Physically these $A-B$ systems are analogous to the ferromagnetic transition in Ising spin systems and many results (e.g., inequalities) proven for the latter can be carried over to the former.

In theories of liquid crystals it is often convenient to identify the various possible orientations of the asymmetric molecules with the components of a mixture.[6] As far as the elongated cores of the molecules are concerned, the interactions between the "components" are repulsive and greater (in the sense of the excluded covolume of two molecules) for molecules with more dissimilar orientations. We have thus been led to consider a caricature of this situation in the form of a multicomponent Widom—Rowlinson model. The number M of components is arbitrary; particles interact only with dissimilar particles, and then repulsively but symmetrically: the identity of the unlike species is unimportant. To produce a more realistic model for liquid crystals it would be necessary for the repulsive interaction to vary with some appropriate measure of the difference in orientations of the molecules.

For our purpose the various species will simply be numbered $1, 2, \ldots, M$. We will primarily be concerned with the lattice version of the model, with infinite repulsion between unlike molecules occupying neighboring sites. For simplicity we explicitly discuss the two-dimensional case. Some observations about the continuum version will also be offered.

We will first show that the "demixing" transition of Widom and Rowlinson persists for any finite number of components, for either the lattice or the continuum version. Not surprisingly, our upper bound on the common critical activity of the components tends to infinity as $M \to \infty$.

For the lattice model, however, there is another transition that appears for large but finite M and remains at finite activities as $M \to \infty$. In this transition, the symmetry between the two sublattices is broken, one of them having a higher density of particles. We call this the crystal (or "hard square") transition due to its apparent relationship with the phase transition of the hard square lattice gas.[7] This transition has no analog in the continuum system—at least none that is demonstrable at the present time.

A related but not equivalent model is the M-state Potts model.[8] In its simplest form the model postulates M states for each lattice site, nearest neighbor interactions being zero for like states and $W \neq 0$ for unlike neighboring states. The "ferromagnetic" case $W > 0$ has been most studied; the expected "Curie point" in zero field has been located as the self-dual temperature of the dual transformation. The ordered phases of the Potts model at low temperature are probably analogous to the almost-one-component phases of the present model at high activity, but the absence of a vacuum state in the former, i.e., empty sites which do not interact with any component, prevents an exact isomorphism.

II. THE MODELS

In the following two sections we will describe the de-

mixing transition and the crystal transition. It is first useful to establish some general terms which will be used in all cases, lattice or continuum and for either transition in the lattice case.

The interaction potential between a particle of type i and one of type j, at a separation $\mathbf{r}$, is given by

$$\varphi_{ij}(\mathbf{r}) = \begin{cases} 0 & \text{for } |r| > R, \\ +\infty & \text{for } |r| \leq R, \end{cases} \tag{1a}$$

for $i \neq j$, $1 \leq i, j \leq M$. For $i = j$ we have

$$\varphi_{ij}(\mathbf{r}) = \begin{cases} 0 & \text{for } \mathbf{r} \neq 0, \\ +\infty & \text{for } \mathbf{r} = 0. \end{cases} \tag{1b}$$

In the continuum case R is the "hard core diameter" between unlike particles; in the nearest neighbor lattice case R is simply the lattice spacing.

For any configuration of particles there is a unique decomposition of the particles into groups which we call *clusters*: Two particles belong to the same cluster if the particle configuration *requires* the two to be of the same type. Equivalently, given the set of particle locations, each cluster contains particles all of the same type.

In each case to be discussed there is a particular way of defining an "outer contour piece," γ. Once that is done we will denote by *boundary cluster of* γ a cluster containing a particle interior to γ whose center is no farther than R from γ.

III. DEMIXING TRANSITION

A. Lattice model

We consider a rectangular region Λ of the two-dimensional square lattice. Each site can be occupied by any of the M components—all of which have the same activity z. According to Eqs. (1), neighboring occupied sites must carry the same type of particle. We represent the particles as squares whose centers reside at the centers of the sites of the square lattice. If the lattice is completely filled, the corners of these squares define the dual lattice.

We employ the Peierls argument to show that there is an activity $z'(M)$ such that a phase transition occurs for some $z < z'(M)$. The technique is to impose a homogeneous boundary condition—say a band of particles of type 1 all around the perimeter of Λ—and show that this boundary condition prejudices the equilibrium state throughout Λ. Specifically, we can show that for some $z'(M)$, the total density of all components $j \neq 1$ is a *decreasing* function of z, whereas we know the total density is an increasing function of z.

The proof is virtually already done in Ref. 2. On rereading that proof (for Model 1), wherever "A" is mentioned, we read "component-1"; wherever "B" occurs, we read "other-than-component-1."

The only change is in the definition of a "cluster" and the multiplicity of the configuration transformation. "Cluster" is defined in the preceding section; it is the same as in Ref. 2 except that the translation of "B" to "other-than-component-1" is not quite accurate. The bound on the multiplicity, $m^{|G|}$, in Ref. 2 for the present model becomes $M^{3|G|}$. Each boundary cluster after the transformation is composed wholly of component-1 particles, whereas prior to the transformation it was of some other component. Clearly there are no more than $M^{3|G|}$ boundary clusters of outer boundary G.

This dependence of the multiplicity on M (see Eq. (3.5), Ref. 2) means that z' depends on M and in fact tends to infinity as $M \to \infty$.[9] We cannot therefore, make any statement about a phase transition for the limiting case $M \to \infty$.

B. Continuum model

In the continuous case again there is very little that needs to be changed from the two component proof of Ruelle.[4] Again, we reread Ruelle's proof, inserting "other-than-component-1" wherever "B" occurs. There is a change needed in the configuration transformation, however. In the two component case, to "remove" an outer contour piece it suffices to simply interchange interior A and B particles. In the present case we modify only particles in boundary clusters. We need the observation that if outer piece γ has length l (in units of "little" square edge length d), then the number of boundary clusters of γ cannot exceed $l/3$.

The change in the contour transformation of Ref. 4 is in its step (a) which is changed to read: "*All* particles in any boundary cluster of γ are changed to component 1." As in the lattice case this introduces a multiplicity to the transformation and changes the estimate of the probability $p(\gamma)$ of outer piece γ to

$$p(\gamma) \leq M^{l/3} \exp(-ld^2 z/2).$$

This probability replaces Eq. (2) of Ref. 4 in the rest of the development. We can then show that if the activity z is sufficiently high (depending on M) the expectation value of the density of other-than-component-1 particles is strictly less than that of component 1. Again, however, no conclusion can be drawn for the limiting case $M \to \infty$.

IV. CRYSTAL TRANSITION

We turn now to the "new" transition for the multicomponent model. In the previous section we discussed the demixing transition induced by high activity and the packing requirement that particles be of the same type in order to achieve high densities. The "driving force" behind the present transition is somewhat different. The idea is that for modest activities and large M the chance is small that nearest neighbor sites will be occupied. Instead the particles will preferentially occupy one of the sublattices, since when only one sublattice is occupied there is no restriction on the species occupying any site, with a subsequent gain in entropy. In this way the "ordered" state of this model is similar to that of the nearest neighbor exclusion problem on the square lattice. The latter system has a well-known transition associated with sublattice ordering.[7] It must be shown, however, that this transition does actually occur at a bounded activity for finite M and persists in some well-defined sense as $M \to \infty$. We also suspect (for fixed large M) an upper activity limit on the stability of this sublattice ordering.

We use the technique employed by Dobrushin[7b] to prove the nonuniqueness of the equilibrium state for the nearest neighbor exclusion problem. Specifically, we shall show that for any positive activity z there is an $M_0 = M_0(z)$ such that the multicomponent lattice model with at least M_0 components has the crystal transition. The criterion is that M_0/z^4 be sufficiently large if $z \geq 1$ or that $M_0 z$ be sufficiently large if $z < 1$. The latter case has a limit which we believe represents the hard square system: $M \to \infty$, $z \to 0$, $Mz = \zeta =$ activity of the hard square gas. The existence and identification of this limit can be proven explicitly in one dimension. (There is, of course, no phase transition in one dimension.) Combining this result with that of the previous section we conclude that the multicomponent model with large but finite number of components has two quite different ordered phases at finite activities. We do not have a very useful estimate of the minimum value of M for which both may be observed.

A. Definitions

To facilitate the proof it is convenient to introduce the following definitions, which are illustrated in Fig. 1.

1. Contour segments: "bonds" of the square lattice, dual to the lattice of sites, separating two sites which are both empty or both occupied. (If both are occupied, then both particles must be of the same type.) This definition differs from that of Peierls (for the ferromagnetic Ising model) but is the same as used by Dobrushin[7b] for the antiferromagnetic Ising model and the hard square lattice gas.

2. Contour Γ: union of all contour segments, consisting of various connected components.

3. Pieces γ_i: smallest set of connected components of Γ, such that if two connected components are separated by a distance of no more than R (= lattice spacing) they belong to the same piece. Γ is then the union of the disjoint pieces $\gamma_1, \ldots, \gamma_n$.

4. Outer piece γ: one of the pieces such that there is a path from the boundary reaching a segment of γ without crossing Γ.

5. Interior site x: a lattice site such that a path from the boundary crosses γ an odd number of times before reaching x. Otherwise a site is exterior to γ.

6. Boundary conditions, with checkerboard coloring of square lattice with black and white squares: white boundary condition means black squares on outer *two* rows and columns are *vacant*. White squares on very outer-most rows and columns are populated arbitrarily, i.e. each site contains any one of the M species. Black boundary condition: white squares are vacant on two outer rows and columns and black squares on very outer-most rows and columns are populated arbitrarily. See the comment below in subsection B about these boundary conditions.

7. Bottom segment of piece γ: a horizontal segment adjacent to and beneath an interior site of γ. Any other horizontal segment is a top segment.

8. Distinguished sites of a configuration X producing contour Γ with outer piece γ:

a. A-site (annihilation): interior occupied site beneath a top segment.

b. L-site (liberated): an interior vacant site above a bottom segment (L_0-site); or an exterior occupied site adjacent to a bottom segment or a vertical segment of γ (L_1-site).

c. G-site (generator): interior occupied site adjacent to a bottom or vertical segment, but not adjacent to a top segment (G_0-site); or exterior occupied site adjacent to a top segment but not adjacent to a bottom or vertical segment (G_1-site). *Note*: Accordingly, every *occupied* site adjacent to a contour segment has exactly one kind of distinguished site designation. An interior occupied site is an A-site if adjacent to a top segment and is otherwise a G_0-site. An exterior occupied site is an L_1-site if adjacent to a bottom or vertical segment and is otherwise a G_1-site.

9. Cluster and boundary cluster: as defined in Sec. II.

In connection with these definitions we will need the following two observations, which we state as lemmas.

Lemma 1: Any A-site or L_1-site of an outer piece γ belongs to some boundary cluster with two or more particles.

Proof: By definition a particle on either type of site is adjacent to a contour segment which must have an occupied site on the other side.

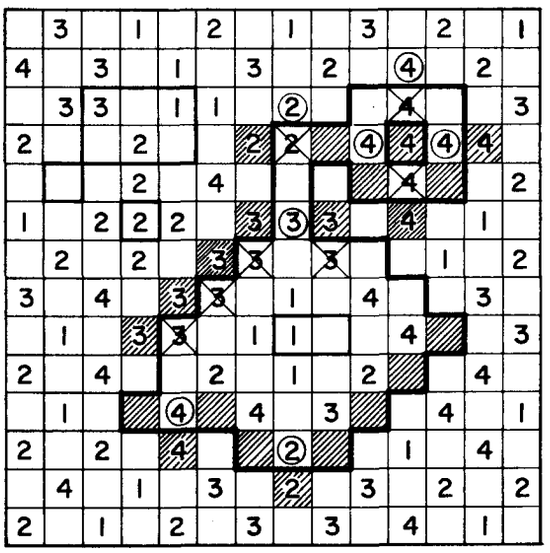

FIG. 1. A configuration whose contour consists of three pieces, two of which are outer pieces. The heaviest lines are the contour segments of one piece γ_1, consisting of two connected components. The numerals represent particle illustration. Distinguished sites associated with outer piece γ_1 are identified as follows: A-site particles are x'd, G-site particles are circled, and L-sites are shaded. The particle of type 4 at the center of the "square doughnut" portion enclosed by γ_1 is exterior to γ_1. The particle of type 1 contained in the small piece enclosed by γ_1 is interior to γ_1. The piece γ_1 has five boundary clusters; each must contain at least one G-site, according to Lemma 2. It should be noticed that the sites of one sublattice are vacant on the two outer rows and columns.

Lemma 2: Given an outer piece γ, one of whose interior sites belongs to a boundary cluster C, the cluster C must contain at least one G-site.

Proof: Regarding only the cluster C, we locate its highest site or highest set of contiguous sites. (In case of ties any highest site or highest set of contiguous sites will do.) If there is a highest single site, the site below it must be occupied and the other three neighboring sites empty. Hence the site in question must be either: (a) interior, adjacent to a bottom segment and not to a top segment, or (b) exterior, adjacent to a top segment and not to a bottom or vertical segment. In the first case the site is a G_0-site, and in the second case it is a G_1-site. If there is not a highest single site we consider the highest set of contiguous sites. The site above each must be vacant and alternate members of the set of contiguous sites must be interior sites, and adjacent to vertical contour segments. Each such interior site is a G_0-site.

B. Configuration transformation

We now define a one-to-many transformation among the allowed configurations on Λ. With configuration X producing contour Γ having outer piece γ we associate a class of X^* of configurations, in three steps:

a. particles at A-sites are annihilated;

b. all remaining particles at sites interior to γ are displaced upward by one unit;

c. L-sites are arbitrarily populated.

We notice that the inverse transformation is well defined by virtue of the original configurations at the G-sites and Lemmas 1 and 2. (The G_0-site particles have been displaced upward one unit by step b.) This means that for any configurations Y and X producing the same outer piece γ, $Y^* \cap X^* = \emptyset$ if $Y \neq X$. In the third step (arbitrary population of L-sites) alterations are made in the occupancy of some sites exterior to γ (the L_1-sites). According to our definitions these sites do not belong to boundary clusters of any different piece γ', since that would require γ' to be within one lattice spacing of γ and hence united into one piece. The boundary condition consists of vacant sites and thus an L_1-site is never part of the boundary condition.

Figure 2 shows the class of configurations produced by this transformation from the configuration shown in Fig. 1.

C. Probability of outer contour piece

We can now calculate a bound on the grand canonical probability of an outer piece γ, in the following steps.

1. Length. If γ contains l segments and c connected components, it can be traversed by a k-step lattice walk, where $k \leq l + 2(c-1)$. Since $l \geq 4c$ we have $k \leq 3l/2$.

2. Number of L-sites. Let n_V and n_H denote the number of vertical and horizontal segments, respectively, of γ. Let n_{L_0}, n_{L_1}, n_L denote the number of L_0-sites, L_1-sites, L-sites, respectively. Half of the horizontal segments produce an L-site (each of those at the bottom), so $n_L \geq n_H/2 \geq l/4$ *if* $n_H \geq n_V$. Notice that an L-site cannot thereby by counted twice. If, however $n_V > n_H$, we first notice that each vertical segment is followed (in a circuit around a connected component of γ) either (a) by another vertical segment or (b) by a horizontal segment. In case (a) one of the two vertical segments must produce an L_1-site, while in case (b) the horizontal segment produces "half" of an L-site (it might be a top segment, but there must be as many bottom segments as top segments). By this method of counting it is possible for some L_1-sites to be counted twice, so we can only conclude that $n_L \geq n_V/4 > l/8$. Regardless of the ratio n_H/n_V we can always assert that $n_L > l/8$.

3. Number of A-sites. Since A-sites occur only beneath top segments, which must be equaled in number by bottom segments, we clearly know that the number of A-sites, n_A, cannot exceed $l/2$.

4. Probability of outer piece γ. Let Z denote the partition function (with all M components having the same activity z), let $Z(\gamma)$ denote the partition function restricted to configurations X producing a contour with outer piece γ, and let Z_{X^*} denote the sum over configurations in class X^* derived from configuration X by the transformation defined above. We have then that the probability $p(\gamma)$ of outer piece γ is

$$p(\gamma) = Z(\gamma)/Z \leq \frac{\sum_{X \supset \gamma} z^{N(X)}}{\sum_{X \supset \gamma} Z_{X^*}} . \tag{2}$$

Here $N(X)$ is the total number of particles present in configuration X. As pointed out previously the inverse transformation $X^* \to X$ is unique so there is no overcounting. The L-sites are independent and in Z_{X^*} each contributes a factor $1 + Mz$; in the configuration X each L_1-site contributed a factor z while the L_0-sites contributed the factor 1. Finally each A-site had a factor z in X and has a factor 1 in Z_{X^*}.

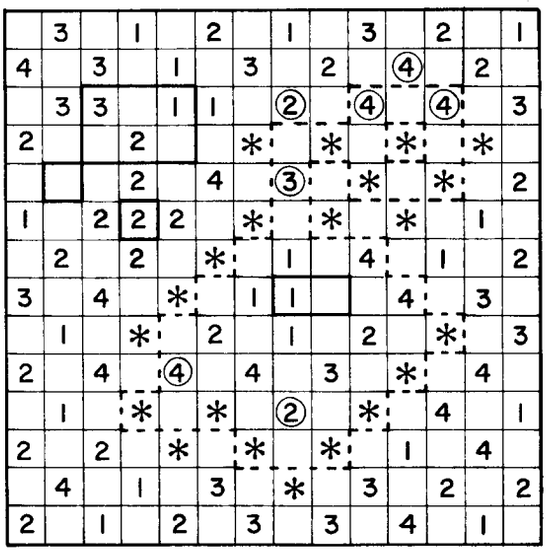

FIG. 2. The effect of the configuration transformation on the configuration shown in Fig. 1. Sites labeled * are arbitrarily occupied by any of the species. Given the outer contour γ_1, here shown dotted, the configuration before the transformation had to be the one shown in Fig. 1; it can be reconstructed from the G-sites.

Thus we have

$$z^{N(X)}/z_{X^*} = z^{n_{L_1}} z^{n_A}/(1+Mz)^{n_L}$$
$$\leq (1+Mz)^{-n_L}$$
$$\leq (Mz)^{-l/8} \quad \text{for } z<1. \tag{3a}$$

On the other hand, if $z \geq 1$ we have

$$z^{N(X)}/Z_{X^*} \leq z^{n_L} z^{n_A}/(1+Mz)^{n_L}$$
$$\leq M^{-n_L} z^{n_A}$$
$$\leq M^{-l/8} z^{l/2}$$
$$= (M/z^4)^{-l/8} \quad \text{for } z \geq 1. \tag{3b}$$

According to Eq. (2) the right-hand sides of Eq. (3) are also upper bounds on $p(\gamma)$.

D. Nonuniqueness

The standard arguments will be used to show that for white boundary conditions the probability that a black square is occupied may be made arbitrarily small, by choosing sufficiently large values of M. However, to complete the demonstration of nonuniqueness we must show that the *total* density is bounded below for fixed z as $M \to \infty$.

Imagine the lattice Λ paved with "Red Cross symbols" of five sites, and focus on one (K) consisting of the site (x, y) and the four neighboring sites $(x \pm 1, y)$ and $(x,\ y \pm 1)$.

Lemma 3: *For any configuration on* $\Lambda \setminus K$, the expected number of particles n_K in K is no less than $Mz/(1+Mz)$.

Proof: For any configuration on $\Lambda \setminus K$, the partition function on K has the form

$$\xi_K = 1 + \sum_{i=1}^{5} a_i z^i$$

where $a_i \geq 0$, and the expected number of particles on K is $n_K = z(\partial \ln \xi_K/\partial z)$. Now by algebra we show

$$n_K \geq a_1 z/(1+a_1 z) \geq \alpha^* z/(1+\alpha z)$$

for any $\alpha \leq a_1$. We can always take $\alpha = M$ [from the configuration with (x, y) occupied and the other four sites empty]. Since this holds for any configuration $\Lambda \setminus K$ we know that in K the average density must not be less than $(1/5)Mz/(1+Mz)$. The same reasoning applies to each of the other "Red Cross symbols" paving Λ and so we obtain the lower bound on the total density p_t,

$$p_t \geq (1/5)Mz/(1+Mz). \tag{4}$$

Now with white boundary conditions if a black square is occupied it must be enclosed in some outer contour piece γ. Equations (3) give upper bounds on the probability $p(\gamma)$. There are no more than $(k/4)^2\, 3^{k-2}$ pieces of length l around any given site, where $k = 3l/2$ is the upper bound on the length of a lattice walk circumnavigating γ. This means that the probability p_b that a black site is occupied is bounded above:

$$p_b \leq \frac{1}{36} \sum_{j=2}^{\infty} j^2 y^j = \frac{y^2(4-3y+y^2)}{36(1-y)^3} \tag{5}$$

for $y < 1$. Here $j = k/2$ and $y = 9/(Mz)^{1/6}$ for $z < 1$, while $y = 9(z^2/M)^{1/6}$ for $z \geq 1$.

To demonstrate the influence of the white boundary conditions we must show that $p_b < p_t/2$ for sufficiently large M. For any $z > 0$, Eqs. (4) and (5), together with $y < 1$, yield a minimum value M_0 for which $p_b < p_t/2$ is satisfied. For $z < 1$ the requirement is that the product $M_0 z$ be sufficiently large, while for $z \geq 1$ the requirement is that M_0/z^4 be sufficiently large. Numerically Eqs. (4) and (5) are not very helpful for determining the minimum M_0 for which this transition would be observed. (They show that $M_0 \approx 27^6$ is sufficiently large!)

It seems likely, but is not proven, that for fixed large M and increasing activity z, the "hard square" sublattice ordering will break down *before* the demixing phase separation occurs. That is, we expect the "phase diagram" to appear as shown schematically in Fig. 3. We have actually proven only that phase transition lines lie below the "Crystal" region and to the left of the "Demixing" region. If these phase transition lines have the same general shape as shown in Fig. 3, then the above assertion would be correct. With increasing activity, then, such a system would undergo *three* phase transitions.

E. The "hard square" limit

The case $z < 1$ is particularly interesting. In this case the variables M and z enter Eqs. (4) and (5) only as the product Mz. This is consistent with the statement that the present model becomes isomorphic to the hard square lattice gas in the limit $M \to \infty$, $z \to 0$, $Mz = \zeta$ = activity of the one-component hard square lattice gas.

To be more precise we believe that in the above limit the thermodynamic properties of the system as well as its "equilibrium measure" defined on the set of "equivalence classes of configurations" $\tilde{\mathcal{A}}$ becomes the same as for the hard square system. Two configurations X and Y belong to the same equivalence class $\tilde{X} \in \tilde{\mathcal{A}}$ if they have the same set of occupied sites, i.e., if they differ only by the labeling of the species at each occupied site. To see how such an isomorphism would come about we note

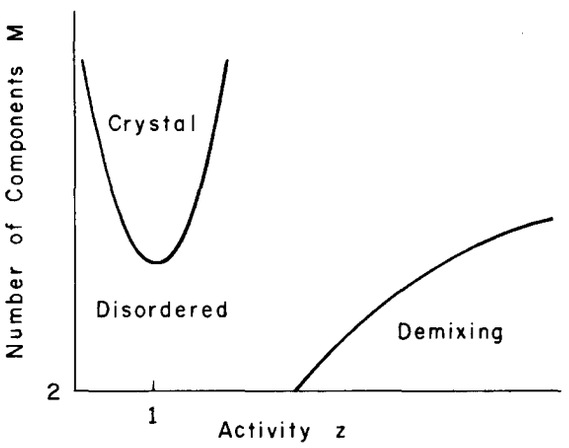

FIG. 3 ("Phase Diagram"). Shown schematically are the lines proven to lie within regions belonging to the two types of ordered phases: the crystal ("hard square") phase and the "demixed" phases of predominantly one component. The actual extent of the incursion of the disordered region into these two areas is not known.

first that a "fully restricted system" of M components in which no adjacent sites can be occupied, i.e., in which Eq. (1a) hold for all i and j, is obviously isomorphic, in the sense defined above to the one-component hard square system with fugacity $\zeta = Mz$. It seems reasonable to expect that in the limit $M \to \infty$, $z \to 0$, $Mz = \zeta$, the multicomponent Widom–Rowlinson model considered in this paper has the same property. We give an explicit computation of the thermodynamic properties for a one-dimensional system in the Appendix.

APPENDIX: ONE-DIMENSIONAL LATTICE SYSTEMS

Consider a one-dimensional lattice containing L sites, $L \geq 3$, with periodic boundary conditions. (Similar results hold for other boundary conditions.) Let $Z_\alpha(z, M; L)$ $\alpha = 0, 1, 2$, be the partition function for the "hard rod" system ($M \equiv 1$), the fully restricted system, and the Widom–Rowlinson model, considered in this paper, respectively. In all cases

$$Z_\alpha(z, M; L) = \operatorname{tr} T_\alpha^L = \sum_{k=1}^{M+1} \lambda_k^L(\alpha, z, M)$$

where T_α is the transfer matrix. T_α is a symmetric matrix of dimensionality $M+1$ (with $M \equiv 1$ for $\alpha = 0$) and $\lambda_k(\alpha, z, M)$ are its eigenvalues. The forms of these matrices are

$$T_0 = \begin{pmatrix} 1 & \sqrt{z} \\ \sqrt{z} & 0 \end{pmatrix}, \quad 2 \times 2,$$

$$T_1 = \begin{pmatrix} 1 & \sqrt{z} & \cdots & \sqrt{z} \\ \sqrt{z} & & & \\ \vdots & & \bigcirc & \\ \sqrt{z} & & & \end{pmatrix}, \quad (M+1) \times (M+1)$$

$$T_2 = \begin{pmatrix} 1 & \sqrt{z} & \cdots & \sqrt{z} \\ \sqrt{z} & & & \\ & z & \bigcirc & \\ \sqrt{z} & & & z \end{pmatrix}, \quad (M+1) \times (M+1)$$

with eigenvalues

$$\lambda_{1,2}(0, z) = [1 \pm (1 + 4z)^{1/2}]/2,$$

$$\lambda_{1,2}(1, z, M) = [1 \pm (1 + 4Mz)^{1/2}]/2, \quad \lambda_k(1, z, M) = 0, \quad k = 3, \ldots, M+1,$$

$$\lambda_{1,2}(2, z, M) = \{1 + z \pm [(1+z)^2 + 4(M-1)z]^{1/2}\}/2, \quad \lambda_k(2, z, M) = z, \quad k = 3, \ldots, M+1.$$

In the limit $z \to 0$, $M \to \infty$, $Mz = \zeta$, we clearly have $Z_\alpha(z, M; L) \to Z_0(\zeta, L)$ for $\alpha = 1, 2$ (and we have omitted the M from Z_0). The same thing happens if we first take the thermodynamic limit $L \to \infty$ of the pressure $L^{-1} \ln Z_\alpha$ and then take the limit on z and M. The isomorphism of the equilibrium measures (as defined at the end of the paper) can presumably also be shown readily for the one-dimensional system and probably remains valid also in higher dimensions.

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[9] From our bound on the multiplicity and Eq. (3.5) of Ref. 2 it follows that a bound of the form $z' >$ (const) $\times M^{12}$ is sufficiently large. It is unlikely that an optimal bound would increase so rapidly with M.