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WHAT IS NEW IN THE ISING MODEL*

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ABSTRACT

We give a brief review of some recent developments and discuss some open questions for Ising spin systems with ferromagnetic pair interactions.

It is a measure of the diversity of Onsager's contributions to the whole field of science that Runnels and I are the only speakers at this symposium who are discussing so called exact results in statistical mechanics. As everyone here knows, Lars Onsager whom we are honoring today, was the first to solve exactly a non-trivial many-body problem[1]. What he found was nothing less than a new world of cooperative phenomena, a world

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which is still mysterious and largely unexplored. I think it is no exaggeration to rank the solution of the Ising model as one of the great intellectual feats of our time. It required daring to be the first to tackle such a problem and it required deep insight and great technical wizardry to solve it. I can still remember some of the words used by Marc Kac, in a lecture he gave when I was a first year graduate student in Syracuse, to describe this work: "finally Onsager brought in all the mathematical machinery and the problem simply collapsed under their weight."

I should like now to talk briefly about some recent results, and even more unsolved problems, for general Ising systems and also for the two-dimensional Ising system on a square lattice with nearest neighbor interactions which Onsager did not solve or at least has not published so far. I hope, and half expect, that when my talk is over Lars will get up, erase everything from the board, and write down the answers.

For simplicity of exposition I shall restrict myself here to Ising spin systems with ferromagnetic pair interactions whose Hamiltonians have the form,

$$H_\Lambda = -\frac{1}{2}\sum_{\substack{i,j\in\Lambda \\ i\neq j}} J_{ij}\,\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i - \sum_{\substack{i\in\Lambda \\ j\in\tilde{\Lambda}}} J_{ij}\,\sigma_i\,\bar{\sigma}_j \qquad (1)$$

Here Λ is a finite subset of a ν-dimensional lattice $\mathbb{Z}^\nu$, $\sigma_i = \pm 1$ is an Ising spin variable at each lattice site, $i\in\Lambda$, the J_{ij} are translation invariant ferromagnetic interactions, $J_{ij} \geqq 0$, $\sum_j J_{oj} = \alpha < \infty$, h is a uniform external field, and the variables $\bar{\sigma}_j$ in $j\in\tilde{\Lambda} = \mathbb{Z}^\nu\setminus\Lambda$, are specified as a boundary condition b_Λ, $\bar{\sigma}_j = (\pm 1, 0)$. The case $\bar{\sigma}_j = 0$ for all j corresponds to "free boundaries". We shall also consider "periodic" boundary conditions.

The free energy per spin (multiplied by $-\beta \equiv -(kT)^{-1}$) of this system is

$$\Psi(\beta,\tilde{h};\Lambda,b_\Lambda) = |\Lambda|^{-1} \ln Z_\Lambda \tag{2}$$

where

$$Z_\Lambda = \sum_{\{\sigma_i, i\varepsilon\Lambda\}} \exp(-\beta H_\Lambda) \tag{3}$$

$|\Lambda|$ is the number of sites in Λ and we have set $\tilde{h} = \beta h$ (we shall always deal with $\tilde{h}$ from now on and drop the tilde). The correlation functions are defined by

$$\langle\sigma_A\rangle\ (\beta,h;\Lambda,b_\Lambda) = \sum_{\{\sigma_i\}} \sigma_A \exp(-\beta H_\Lambda)/Z_\Lambda \tag{4}$$

where $\sigma_A = \prod_{i\varepsilon A} \sigma_i$, A a subset of Λ. These functions are known to be real analytic in β and h for all finite Λ. The question however arises of what happens when the size of the system increases, formally when $\Lambda \to \infty$. It is only in this limit when phase transitions show up qualitatively as non-analyticities in the free energy and/or the correlation functions[2].

The following statements are known to be true for the thermodynamic limit, $\Lambda \to \infty$, of the free energy density

$$\Psi(\beta,h) = \lim_{\Lambda\to\infty} \Psi(\beta,h;\Lambda,b_\Lambda),$$

and of the correlation functions

$$<\sigma_A> (\beta,h) = \lim_{\Lambda\to\infty} <\sigma_A> (\beta,h;\Lambda,b_\Lambda);$$

for references see[3].

(i) $\Psi(\beta,h)$ exists (independent of b_Λ) and is continuous in β and h for all real β and h.

(ii) Ψ is analytic in the complex h-plane for Re h $\neq$ 0 and $\beta \geqq 0$.

(iii) Ψ is real analytic in β for $\beta \geqq 0$ when Re h $\neq$ 0.

(iv) There exists a $\beta' > 0$ such that for $\beta \leqq \beta'$, $\Psi(\beta,h)$ is also real analytic in β and h at h = 0.

(v) The thermodynamic limit of the correlation functions $<\sigma_A> (\beta,h)$, exist (independent of b_Λ), are translation invariant, having some clustering property and are continuous in h at all those values of $\beta \geqq 0$, and of h at which $\Psi(\beta,h)$ is differentiable with respect to h.

(vi) The regions of β and h, mentioned in (ii)-(iv), at which Ψ is known to be analytic are also regions of analyticity for the $<\sigma_A> (\beta,h)$ with the same analyticity properties as $\Psi(\beta,h)$.

Thus, if we define the reciprocal critical temperature β_c as the value of β above which the spontaneous magnetization, which is a non-decreasing function of β, is positive;

$$m^*(\beta) \equiv \lim_{h\to 0+} m(\beta,h) \begin{cases} = 0, & \beta < \beta_c, \\ > 0, & \beta > \beta_c, \end{cases}$$

where

$$m(\beta,h) \equiv \partial\Psi(\beta,h)/\partial h,$$

we have, at $h = 0$, uniqueness and continuity of the correlation functons and differentiability of $\Psi(\beta,h)$ (in h) for $\beta < \beta_c$ and analyticity in h and Re β for $\beta \leqq \beta' < \beta_c$ (for $h \neq 0$ the analyticity holds of course for all β). For $\beta > \beta_c$ the correlation $<\sigma_A>(\beta,h)$ are discontinuous at $h = 0$ and the limit $\Lambda \to \infty$ of $<\sigma_A>(\beta,h=0;\Lambda,b_\Lambda)$ will depend on the boundary condition b_Λ (at least for $|A|$ add).

There are still many questions, however, about the analytic nature of $\Psi(\beta,h)$ and of the $<\sigma_A>$ at $h = 0$ and $\beta' < \beta < \beta_c$ as well as about their limits as $h \to 0$ for $\beta > \beta_c$.

We shall now quote some general results which when combined with the explicit results of Onsager for the two dimensional Ising system with nearest neighbor interactions answer some, but unfortunately not all, of these equestions for this system.

Let

$$U_2(i,j,\beta,h;\Lambda b_\Lambda) \equiv <\sigma_i\sigma_j> - <\sigma_i><\sigma_j>, \quad i,j\varepsilon\Lambda$$

be the two spin Ursell function. We have proven the following[3].

(vii) A uniform bound on U_2 of the form

$$U_2(i,j,\beta,h;\Lambda,b_\Lambda) \leq K \exp[-\kappa\, r_{ij}], \qquad (5)$$

where r_{ij} is the distance between the sites i and j and K and K and κ are positive constants independent of Λ, implies that $\Psi(\beta,h)$ and $\langle\sigma_\Lambda\rangle(\beta,h)$ are infinitely differentiable with respect to β and h in any region of the Re h, Re $\beta \geq 0$ plane in which the bound (5) holds uniformly.

(viii)

$$U_2(i,j,\beta,h;\Lambda,b_o) \leqq U_2(i,j,\beta,0;\Lambda,b_o) \leqq U_2(i,j,\overline{\beta},0;\overline{\Lambda},b_o)$$

$$\leqq U_2(i,j,\overline{\beta},0;\overline{\Lambda},b_p) = \langle\sigma_i\sigma_j\rangle(\overline{\beta},0;\overline{\Lambda},b_p) \qquad (6)$$

for $\beta \leqq \overline{\beta}$ and $\Lambda \subset \overline{\Lambda}$. Here b_o indicates free boundaries, $\overline{\sigma}_j = 0$ in (1), and b_p indicates "periodic" boundary conditions. This includes cylindrical boundary conditions which are periodic in some directions and free in others. "Screw" boundary conditions are also included. For all these boundary conditions r_{ij} is defined with the proper "modulo". The inequalities in (6) follow from Griffiths type inequalities[2] using the fact that $\langle\sigma_i\rangle\ (\beta,0;\Lambda,b_p) = 0$.

It follows from (6) that (5) with $b_\Lambda = b_o$, will hold for all values of h and all $\beta' \leqq \beta$ whenever

$$\lim_{\Lambda\to\infty} \langle\sigma_i\sigma_j\rangle\ (\beta,h=0;\Lambda,b_p) \leqq K \exp[-\kappa\, r_{ij}],\ K < \infty,\ \kappa > 0. \qquad (7)$$

An inequality of the form (7) has been established by Onsager for the two-dimensional square lattice with nearest neighbor interactions for $\beta < \beta_o$; β_o is the (reciprocal) Onsager temperature defined by the relation $\sinh(2\beta_o J_1)\sinh(2\beta_o J_2) = 1$, with J_1 and J_2 the "horizontal" and "vertical" nearest neighbor interaction. It is a direct consequence of the expression for $\langle\sigma_i\sigma_j\rangle$ in terms of the eigenvalues and eigenvectors of the transfer matrix. The value of κ in (7) is essentially equal to $\ln(1+\Delta)$ with Δ proportional to the "gap" in the spectrum of the transfer matrix which is positive, $\Delta > 0$ for $\beta < \beta_o$[1].

We have thus established that for this two-dimensional system $\Psi(\beta,h)$ and $\langle\sigma_A\rangle(\beta,h)$ are C^∞ in h for $\beta < \beta_o \leqq \beta_c$. We also know[4], that the formula for the "long range order spontaneous magnetization", $m^*(\beta) = [1 - (\sinh \beta/\beta_o)^{-4}]^{1/8}$ for $\beta \geqq \beta_o$, calculated by Onsager and Yang[2], is a lower bound for $m^*(\beta)$. Thus $m^*(\beta) > 0$ for $\beta > \beta_o$ which implies $\beta_o \geqq \beta_c$. Hence $\beta_o = \beta_c$ for this system.

Using now the fact that β_o is the center of the duality symmetry of this system, for $h = 0$, and the Onsager result that $\langle\sigma_i\sigma_j\rangle(\beta,h=0) - [m^*(\beta)]^2$ decays exponentially for $\beta > \beta_c$ it is possible to prove[4] that $\Psi(\beta,h)$ and $\langle\sigma_A\rangle(\beta,h)$ are infinitely differentiable in β and h as $h \to 0$ for $\beta > \beta_c$.

This still leaves open the question of whether $\Psi(\beta,h)$ and $\langle\sigma_A\rangle(\beta,h)$ are a) analytic at $h = 0$ for $\beta < \beta_c$ and b) have an analytic continuation across $h = 0$ for $\beta > \beta_c$? It is generally believed that a) is true and only a rigorous proof is missing while there is real doubt about b) which relates to the question of the existence of metastable states which are the analytic continuation of equilibrium states at a first order phase transition.

To prove a) it would be sufficient to show that

$\Psi(\beta,h;\Lambda,b_\Lambda)$ is analytic for $\beta < \beta_c$ in some fixed neighborhood of $h = 0$ for all sufficiently large Λ and some b_Λ. It was shown recently by Lebowtiz and Penrose[5] that when this is true for periodic boundary conditions and finite range ferromagnetic pair interactions then we also get a bound of type (5) on all the Ursell functions. It seems reasonable to expect this to be the case for general Ising spin systems with translation invariant interactions; analyticity and exponential decay of correlations for $\beta < \beta_c$.

I hope that there will be a definitive answer to these questions when we gather again to celebrate Lars' eightieth birthday.

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