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Macroscopic evolution of particle systems with short- and long-range interactions

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Abstract. We consider a lattice gas with general short-range interactions and a Kac potential $J_\gamma(r)$ of range γ^{-1}, $\gamma > 0$, evolving via particles hopping to nearest-neighbour empty sites with rates which satisfy detailed balance with respect to the equilibrium measure. Scaling spacelike γ^{-1} and timelike γ^{-2}, we prove that in the limit $\gamma \to 0$ the macroscopic density profile $\rho(r, t)$ satisfies the equation

$$\frac{\partial}{\partial t}\rho(r,t) = \nabla \cdot \left[\sigma_s(\rho)\nabla\frac{\delta\mathcal{F}(\rho)}{\delta\rho(r)}\right]. \qquad (*)$$

Here $\sigma_s(\rho)$ is the mobility of the reference system, that with $J \equiv 0$, and $\mathcal{F}(\rho) = \int[f_s(\rho(r)) - \frac{1}{2}\rho(r)\int J(r-r')\rho(r')\,\mathrm{d}r\,\mathrm{d}r']$, where $f_s(\rho)$ is the (strictly convex) free energy density of the reference system. Beside a regularity condition on J, the only requirement for this result is that the reference system satisfy the hypotheses of the Varadhan–Yau theorem leading to $(*)$ for $J \equiv 0$. Therefore, $(*)$ also holds if $\mathcal{F}$ achieves its minimum on non-constant density profiles and this includes the cases in which *phase segregation* occurs. Using the same techniques we also derive hydrodynamic equations for the densities of a two-component A–B mixture with long-range repulsive interactions between A and B particles. The equations for the densities ρ_A and ρ_B are of the form $(*)$. They describe, at low temperatures, the demixing transition in which segregation takes place via vacancies, i.e. jumps to empty sites. In the limit of *very few* vacancies the problem becomes similar to phase segregation in a continuum system in the so-called *incompressible limit*.

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The state of a (one-component) macroscopic system in equilibrium can be characterized by two numbers, the temperature T $(= \beta^{-1})$ and the chemical potential λ. When T and λ correspond to a single phase (i.e. there is a unique Gibbs measure) then the particle density $\bar{\rho}(T, \lambda)$ is constant, i.e. spatially uniform. Given a microscopic dynamics for which this Gibbs measure is attractive, a disturbance in this uniform density corresponding to a profile $\rho_0(r)$, r the space coordinate, at some time t_0 is expected to relax towards the uniform density $\bar{\rho}$. In certain types of systems (when the variations in temperature and hydrodynamical flows can be neglected, see below) the relaxation of the density profile $\rho(t, r)$ will occur via the diffusion equation

$$\frac{\partial\rho}{\partial t} = \nabla \cdot [D\nabla\rho] \qquad t > t_0 \qquad (1.1)$$

where the bulk diffusion constant $D = D(\rho)$ will generally also depend on the temperature T, which is assumed to be constant and therefore is omitted in the notation.

Equation (1.1) is a continuity equation for ρ corresponding to a mass current given by Fick's law

$$j = -D(\rho)\nabla\rho. \tag{1.2}$$

To derive (1.1) from microscopic models it is most convenient to write Fick's law in its Onsagerian form,

$$j = -\sigma\nabla\lambda \tag{1.3}$$

where σ is the *conductivity*, or *mobility*, and $\lambda(\rho)$ is the local chemical potential; λ and ρ are related as in the uniform equilibrium system, i.e. we are in a situation of *local equilibrium*. Comparing (1.2) and (1.3) gives

$$D = \sigma/\chi \tag{1.4}$$

where

$$\chi(\rho) = \left(\frac{\partial\lambda}{\partial\rho}\right)^{-1} = \rho\frac{\partial\rho}{\partial p} \tag{1.5}$$

is the *compressibility* (p is the pressure). In (1.5) we have expressed the chemical potential at equilibrium as a function of the density ρ. The relation (1.4) is sometimes referred to as the *Einstein relation*, as Einstein was the first used it to relate the diffusion constant of a Brownian particle to its steady-state mobility in an external field [Sp].

Mathematically rigorous derivations of (1.2)–(1.5), with σ given by a Green–Kubo formula, have been achieved, via the use of the hydrodynamical (diffusive) scaling limit for a variety of microscopic models with fixed short (microscopic) range interactions evolving via stochastic dynamics (in which the particle number is the only conserved quantity), see [KL]†. In all of these cases the temperature T is in the uniqueness region of the phase diagram, i.e. there is a unique phase for all values of λ (or ρ).

The situation becomes much trickier when we consider temperatures where there is, for some value of λ, a coexistence of phases with two (or more) different densities, ρ_1 and ρ_2, corresponding to liquid and vapour or to fluid and solid. In such cases the macroscopic equilibrium system, with a fixed total number of particles corresponding to an average density in the interval (ρ_1, ρ_2), will not have a uniform density. Instead it will be segregated into macroscopic regions of density ρ_1 and ρ_2 with shapes minimizing the free energy of the surface between them. A non-degenerate parabolic equation is clearly not appropriate now, in fact, for any density $\rho \in (\rho_1, \rho_2)$, $\chi(\rho)$ in (1.4) will be infinite and therefore D is formally zero (unless σ is also infinite, which can be proven in some cases not to happen). Results in these directions have been proven for very particular systems (see the remarkable result in [R] where degenerate diffusion in the coexistence region is proven), but the general case of the evolution of phase domains for systems with interactions is a real challenge for the moment.

To get around this obstacle and derive macroscopic equations for a system undergoing phase segregation, some authors studied the time evolution of the macroscopic density profile in particle systems interacting via long-range (compared with the interparticle spacing) Kac potentials, [LOP, GL1]. The microscopic model considered in [GL1] is a lattice gas evolving under a particles hopping (Kawasaki exchange) dynamics which satisfies detailed balance (is

† There also exists a derivation for one system with a Hamiltonian evolution, namely a non-interacting gas of point particles moving among fixed periodic scatterers—the Sinai billiard with finite horizons [BS, Sp].

reversible) with respect to the Gibbs canonical (fixed particle number) equilibrium measure with Hamiltonian H at temperature β^{-1}. H consisted of a sum of two terms, a short-range part H_s which may be thought of as a nearest-neighbour interaction and a Kac potential H_γ characterized by a range parameter γ^{-1}, namely

$$H_\gamma = -\frac{1}{2}\sum_{x,y}\gamma^d J(\gamma|x-y|)\eta(x)\eta(y) \tag{1.6}$$

where $x, y \in \Lambda_\gamma \subset \mathbb{Z}^d$ (say a torus of diameter $[\gamma^{-1}]$) and $\eta(x) = 0, 1$ specifies the occupancy of site x. They argued that in the diffusive scaling limit, corresponding to $\gamma \to 0$ with space and time scaling like γ^{-1} and γ^{-2}, respectively, the density profile $\rho(r,t)$ would satisfy a parabolic integro-differential equation of the form,

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\sigma_s(\rho)\nabla\frac{\delta\mathcal{F}(\rho)}{\delta\rho}\right] \equiv \nabla\cdot[\sigma_s(\rho)\nabla\lambda(\rho)] \tag{1.7}$$

where

$$\mathcal{F} = \int f_s(\rho(r))\,\mathrm{d}r - \frac{1}{2}\iint J(r-r')\rho(r)\rho(r')\,\mathrm{d}r\,\mathrm{d}r' \tag{1.8}$$

f_s is the free energy density and σ_s is the mobility of the system with only short-range interactions.

Equations (1.5) and (1.7) were proven in [GL1] for the case where the short-range interactions H_s consisted only of the hard core exclusion, i.e. no more than one particle per site, in which case f_s and σ_s reduce to f_s^0 and σ_s^0, with

$$f_s^0(\rho) = \beta[\rho\log\rho + (1-\rho)\log(1-\rho)] \tag{1.9}$$

and

$$\sigma_s^0(\rho) = \beta\rho(1-\rho). \tag{1.10}$$

The validity of (1.7) and (1.8) for the case of non-trivial short-range interactions was conjectured in [GL1, section 3] for values of β^{-1} *above the critical temperature* of the reference (short-range) system, when $f_s(\rho)$ is a strictly convex function of ρ. They gave heuristic arguments for the validity of (1.7) with λ the *local* chemical potential of the system with the Kac potential, but with the mobility σ being the same as what it would be in the system without the slowly varying Kac potential. This generalized a similar conjecture by Spohn [Sp] for the case of an *external long-range interaction.* Besides the case in [GL1], the conjecture was shown to also be valid for some other special cases [BL, MM, AX]. In this paper we prove these conjectures for the general short-range interactions in the case of Ising spins, that is, in the case of a system with Hamiltonian $H^\gamma = H_s + H_\gamma$. In fact, the methods we use here extend in a straightforward manner to general systems for which a diffusion equation can be proven in the absence of H_γ. We illustrate this by deriving in section 5 integro-differential equations for a binary mixture which may undergo a demixing transition. Our results extend to the case of a system on which a weak external force is acting, characterized by the Hamiltonian

$$H_s(\eta) + \sum_x V(\gamma x)\eta(x)$$

and also to the case of a weak (of order γ) external force which is not the gradient of a potential, such as a constant force in a torus. We remark that in the true hydrodynamic limit (diffusive scaling keeping γ fixed) the relation (1.4) is expected be true with σ, D and χ computed from the Hamiltonian $H_s + H_\gamma$ [Sp].

Finally, we note that solutions to (1.7) corresponding to interface dynamics are considered in [GL2]. There is also a review of various results concerning these models, including the case in which there is no conservation law (*Glauber* dynamics) [GLP].

Informal description of model. To be concrete, we restrict ourselves to the case $H_s(\eta) = -K\sum_{x,y:|x-y|=1}\eta(x)\eta(y)$. The Kawasaki dynamics is defined in terms of Poisson jump rates depending on the energy differences. An example of such rates is

$$c^{\gamma}_{(x,y)}(\eta) = \exp\left(-\beta\left[H^{\gamma}(\eta^{x,y}) - H^{\gamma}(\eta)\right]/2\right) \tag{1.11}$$

where x, y are nearest-neighbour sites and $\eta^{x,y}$ is the configuration in which the sites x and y exchange their occupation numbers. The microscopic current $w_{x,y}$ through the bond (x, y) is the rate at which a particle jumps from x to y minus the rate at which a particle jumps from y to x, namely

$$w^{\gamma}_{x,y}(\eta) = \left[\eta(x)c^{\gamma}_{(x,y)}(\eta) - \eta(y)c^{\gamma}_{(x,y)}(\eta)\right]. \tag{1.12}$$

Equation (1.12) will determine the form of the macroscopic current in the hydrodynamic equation.

A key ingredient in our analysis is that the dynamics with $J \neq 0$ is a *weak* perturbation of the $J \equiv 0$ dynamics. This can be seen both at the level of the rates and of the current. In particular, for the current some straightforward expansions with respect to the small parameter γ (see section 4) lead to

$$\begin{aligned} w^{\gamma}_{x,x+e_i}(\eta) = {} & c^{0}_{x,x+e_i}(\eta)\left[\eta(x) - \eta(x+e_i)\right] \\ & + \frac{\gamma\beta}{2}c^{0}_{x,x+e_i}\left[\eta(x) - \eta(x+e_i)\right]^2 (\partial_i J \star \eta)(x) + O(\gamma^2) \end{aligned} \tag{1.13}$$

where e_i is the unit vector on the lattice in the direction i, $\star$ denotes a spatial discrete convolution and the superscript 0 on c and w denotes the case in which there is no long-range force ($J \equiv 0$). Equation (1.13) indicates clearly that the dynamics with $J \neq 0$ is a weak perturbation of the reference ($J \equiv 0$) dynamics. The case studied in [GL1] corresponds to $c^0_{x,y} = 1$ in (1.13). The local equilibrium expectation of $w^0_{x,x+e_i}$ gives the macroscopic flux j for this simplified model in the form of a term due to the exclusion dynamics and one due to the mean-field force:

$$j = -\nabla\rho + \sigma(\rho)\nabla J * \rho \tag{1.14}$$

where $*$ denotes the (standard) spatial convolution, $J * \rho(r) = \int J(r - r')\rho(r')\,\mathrm{d}r'$, and the mobility $\sigma(\rho) = \beta\rho(1-\rho)$ is just the expectation of $(\beta/2)\left[\eta(x) - \eta(x+e_i)\right]^2$.

The naive extension of the above argument to the general case, i.e. just applying local equilibrium to (1.13), would give a macroscopic current of the form

$$j = -D\nabla\rho + \tilde{\sigma}(\rho)\nabla J * \rho \tag{1.15}$$

where D is, as before, the diffusion coefficient found in [VY], but $\tilde{\sigma}$ would not satisfy the Einstein relation (1.4). This apparent contradiction is explained by the fact that the microscopic current of the reference system in the general case, i.e. the first term in the right-hand side of (1.13), is not a lattice gradient of a function of the configuration, i.e. there is no function h such that $w^0_{x,y}(\eta) = h(\tau_y\eta) - h(\tau_x\eta)$, where τ_x is the translation in configuration space, unless $K = 0$ (i.e. $H_s \equiv 0$). The non-gradient nature of the dynamics is responsible for the presence in (1.15) of a third term which, combined with the second term, gives the correct expression for the macroscopic current.

The main ingredient in our derivation is the recent work of Varadhan and Yau [VY] who proved (1.1) for a lattice gas with general short-range interactions at small β. This is a major achievement since previous derivations all required that the dynamics be either of gradient type, or that the invariant measure be of product type, i.e. independent occupation values at different sites (see [KL] for a complete treatment and extended bibliography on this topic). The class of non-gradient models with invariant measures of product type includes the n-colour simple exclusion process, which is a standard simple exclusion process but each particle is coloured with one of n possible colours. Exchanges only occur between occupied and empty sites, so that two sites with different colours cannot exchange. Our arguments do apply to this case too. This can be viewed as a multi-species system and the effect of long-range interactions is of interest in its own right and we discuss it in section 5 of this paper, limiting ourselves to the $n = 2$ case, corresponding to a binary alloy in which exchanges take place only through vacancies.

The paper is organized as follows. In section 2 we give the precise definition of our model and we state the main results. In section 3 we recall some important results of [VY] that we need for our proof. Section 4 contains the proofs which are based on the control of the Radon–Nykodim derivative of the full process with respect to the reference process and on the [GPV]–[V] method. Finally, in section 5 we consider the example, that we just mentioned, of a dynamics with two conservation laws, for which we prove a result analogous to that in section 2. Moreover, in this case we obtain a stronger convergence result, since uniqueness of the weak solution for the limit equation does hold, under suitable hypotheses.

2. The model and the main result

We work in the discrete torus of dimension d and diameter γ^{-1}, $\gamma > 0$, that we denote by Λ_γ. Associated with Λ_γ there is a natural notion of Λ_γ^*, the set of (non-directed) bonds, i.e. the pairs of nearest-neighbour points of Λ_γ. The configuration space on Λ_γ is $\{0, 1\}^{\Lambda_\gamma} \equiv \Omega_{\Lambda_\gamma}$. All of these spaces are endowed with the discrete topology: when later we will deal with $\{0, 1\}^\Lambda$, Λ countable, we will use the product topology.

2.1. The Gibbsian reference measure

Let F be a local isotropic function from $\Omega_{\mathbb{Z}^d}$ to $\mathbb{R}$. By isotropic we mean that $F(\theta\eta) = F(\eta)$ for every reflection θ with respect of an axis as well as any lattice rotation θ. For any $x \in \mathbb{Z}^d$ set $F_x(\eta) = F(\tau_x\eta)$, where τ_x is the translation of $x \in \mathbb{Z}^d$ in $\Omega_{\mathbb{Z}^d}$:

$$\tau_x\eta(y) = \eta(x + y) \qquad \text{for every } y \in \mathbb{Z}^d.$$

And τ will also be the translation operator acting on functions F of the configuration: $\tau_x F(\eta) = F(\tau_x\eta)$.

Let us consider the formal Hamiltonian

$$H(\eta) = \sum_x F_x(\eta). \tag{2.1}$$

It is well know that if the inverse temperature κ is sufficiently small, given any chemical potential λ, there exists a unique extremal Gibbs measure μ^ρ, $\rho \in [0, 1]$, which satisfies $\rho = \mathbb{E}^{\mu^\rho}[\eta(x)]$ for every $x \in \mathbb{Z}^d$. We will always work in this *uniqueness* regime: this also guarantees that μ^ρ is translation invariant and that it has some mixing properties. However, in some proofs in [VY] the authors require a stronger mixing condition (*exponential mixing condition*, [VY, assumption A]), given in terms of finite-volume grand-canonical measures. Since we are using this assumption only indirectly we do not give it explicitly: we just stress that our results are proven only if κ is smaller then a certain $\kappa_0 > 0$, which depends on the dimension d and on the interaction F. In some cases it can be shown that $\kappa_0 = \kappa_c$, the inverse of the critical temperature of the reference system.

We will deal with the F-interaction also in the case of Λ_γ: for γ sufficiently small we can keep the very same infinite-volume definitions by lifting Λ_γ to the whole of $\mathbb{Z}^d$ in the natural way. In this way the notion of translation τ_x is unchanged. It is, however, often more natural to look at the periodic case as a finite-volume case, and that is what we will do. The same applies to every local function on $\Omega_{\mathbb{Z}^d}$, which will be viewed as a function on Ω_{Λ_γ} without notational changes.

2.2. *The equilibrium measure for the full system*

We consider the probability measure on Ω_{Λ_γ} defined by

$$\mu_\gamma^{\beta,\kappa,\lambda}(\eta) = \frac{1}{Z_\gamma(\beta,\kappa,\lambda)} \exp\left\{-\kappa \sum_x F_x - \frac{\beta}{2}\sum_{x,y} J_\gamma(x,y)\eta(x)\eta(y) + \lambda \sum_x \eta(x)\right\} \tag{2.2}$$

$$\beta > 0 \qquad \gamma > 0 \qquad \lambda \in \mathbb{R}$$

where $J_\gamma(x,y) = \gamma^d J(\gamma(x-y))$, $J \in C^2(\mathbb{T}^d) \to \mathbb{R}$ ($\mathbb{T}^d$ is the d-dimensional torus of diameter 1) such that

$$J(r) = J(-r) \qquad \int_{\mathbb{T}^d} J(r)\,\mathrm{d}r = 1 \tag{2.3}$$

and $Z_\gamma(\beta,\kappa,\lambda)$ is the normalization factor.

2.3. *The dynamics*

This will be introduced, for the reference system, both in the case $\Lambda = \Lambda_\gamma$ and in the case $\Lambda = \mathbb{Z}^d$. Let $\Phi \in C^2(\mathbb{R};\mathbb{R}^+)$ be of the form†

$$\begin{aligned} &\Phi(E) = \exp(-E/2)\phi(E), \\ &\phi : \mathbb{R} \to \mathbb{R}^+ \qquad \phi(E) = \phi(-E) \qquad \text{for every } E \in \mathbb{R} \text{ and } \phi(0) = 1. \end{aligned} \tag{2.4}$$

For $b = (x,y) \in \Lambda^*$ we define

$$c_b^0(\eta) = \Phi(\kappa\Delta_b H(\eta)) \tag{2.5}$$

where $\Delta_b H(\eta) = H(\eta^b) - H(\eta)$ and

$$\eta^b(z) = \begin{cases} \eta(x) & \text{if} \quad z = y, \\ \eta(y) & \text{if} \quad z = x, \\ \eta(z) & \text{otherwise.} \end{cases} \tag{2.6}$$

In the infinite-volume case H is not well defined, but $\Delta_b H$ is taken, by definition, to be equal to $\lim_{R\to\infty} \Delta_b \sum_{|x|\leqslant R} F_x(\eta)$. To make the notation a bit lighter, if $b = (x,y)$ appears as a subscript, we will often drop the brackets. Given $f : \Omega_\Lambda \to \mathbb{R}$ for $b = (x,y)$ we set

$$\mathcal{L}_b^0 = c_b^0(\eta)\left[f(\eta^b) - f(\eta)\right]. \tag{2.7}$$

A Markov pregenerator is then defined by setting

$$\mathcal{L}^0 f(\eta) = \sum_{b\in\Lambda^*} \mathcal{L}_b^0 f(\eta) \tag{2.8}$$

† This is equivalent to the more customary [Sp, p 163] detailed balance condition $\Phi(E) = \Phi(-E)\exp(-E)$, $E \in \mathbb{R}$.

where f is assumed to be local in the case $\Lambda = \mathbb{Z}^d$. If $\Lambda = \Lambda_\gamma$, $\mathcal{L}^0$ is actually a generator and it is easy to construct a unique process in Skorohod space $\{\eta_t\}_{t\in\mathbb{R}^+} \in D(\mathbb{R}^+; \Omega_{\Lambda_\gamma})$ associated with it, once an initial condition is given. The law of this process will be denoted by $P^{\gamma,0}$ or $P^{\gamma,0}_{\mu_\gamma}$ if there is the need to stress the initial condition μ_γ. It can be immediately verified that, by (2.4), $\mathcal{L}^0$ viewed as an operator in $L^2(\mu_\gamma^{0,\kappa,\lambda})$ is self-adjoint for every $\lambda \in \mathbb{R}$. In the infinite-volume setting the construction of the process is more delicate. We refer to [Li] for this construction: there the process $\{\eta_t\}_{t\in\mathbb{R}^+} \in D(\mathbb{R}^+; \Omega_{\mathbb{Z}^d})$ associated with $\mathcal{L}^0$ is constructed. We also remark that, for every $\rho \in [0, 1]$, $\mathcal{L}^0$ can be extended to a self-adjoint operator on $L^2(\mu^\rho)$. The law of the process $\{\eta_t\}_{t\in\mathbb{R}^+}$ will be denoted by P^0. We will sometimes stress the chosen initial condition, say $\mu \in \mathcal{P}_1(\Omega_\Lambda)$, by writing P^0_μ. Here we used $\mathcal{P}_1(\cdot)$ to denote the probability measures on $\cdot$.

The full dynamics is considered only in the case $\Lambda = \Lambda_\gamma$. We define

$$c_b^\gamma(\eta) = \Phi\left(\Delta_b\left[\kappa\sum_x F_x - \beta\sum_{x,y} J_\gamma(x,y)\eta(x)\eta(y)\right]\right) \tag{2.9}$$

and

$$\mathcal{L}_\gamma f(\eta) = \sum_{b\in\Lambda^*} c_b^\gamma(\eta)\left[f(\eta^b) - f(\eta)\right]. \tag{2.10}$$

As before, associated with the finite-dimensional operator $\mathcal{L}_\gamma$ there is a process with trajectories in $D(\mathbb{R}^+; \Omega_{\Lambda_\gamma})$. $\mathcal{L}_\gamma$ is self-adjoint in $L^2(\mu_\gamma^{\beta,\kappa,\lambda})$ for every $\lambda \in \mathbb{R}^+$. The law of this process will be denoted by $P^\gamma = P^\gamma_{\mu_\gamma}$, where $\mu_\gamma \in \mathcal{P}(\Omega_{\Lambda_\gamma})$ is the initial condition.

2.4. *The main result*

The compressibility χ for the system is defined as

$$\chi(\rho) = \sum_{x\in\mathbb{Z}^d} \mathrm{cov}^{\mu^\rho}(\eta(0), \eta(x)) \tag{2.11}$$

in terms of the local interaction alone. The corresponding diffusion matrix D can be expressed via a variational formula. It is the ρ-dependent symmetric matrix defined by

$$\langle v, Dv\rangle_{\mathbb{R}^d} = \frac{1}{2\chi(\rho)}\inf_g \mathbb{E}^{\mu^\rho}\left[\sum_{i=1}^d c_{e_i}^0(\eta)\left(v_i(\eta(e_i)-\eta(0)) - \sum_{x\in\mathbb{Z}^d}\Delta_{(0,e_i)}g(\tau_x\eta)\right)^2\right] \tag{2.12}$$

for every $v \in \mathbb{R}^d$. In (2.12) e_i denotes the unit vector in the i direction, $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is the scalar product in $\mathbb{R}^d$ and the infimum is taken over all local functions g. We will take the freedom of using both the notation v_i and the notation v_{e_i}. Moreover, for e a unit vector, $v_e = \pm v_{e_i}$ if $e = \pm e_i$.

A number of facts are known about D: first of all it is a continuous function of ρ, cf [VY], and there exists a constant c such that in the sense of matrices

$$\frac{I}{c\chi(\rho)} \leqslant D(\rho) \leqslant cI \tag{2.13}$$

where I is the $d\times d$ identity matrix. While the upper bound is an immediate consequence of (2.12), the lower bound is much more subtle and it has been established in [SY]. In [VY, lemma 8.3] it is shown moreover that in our isotropic set up, $D(\rho)$ is a multiple of I. We will also keep the notation $D(\rho)$ to denote the scalar proportionality factor between I and the matrix $D(\rho)$.

We are going to consider initial particle configurations associated with a density profile $\rho_0 : \mathbb{T} \to [0, 1]$, in the following sense: if we define the empirical density field

$$\nu^\gamma(t, x) = \gamma^d \sum_{y \in \Lambda} \delta(x - \gamma y)\eta_{\gamma^{-2}t}(y) \tag{2.14}$$

we require that for any smooth test function G from $\mathbb{T}^d$ to $\mathbb{R}$ and $\delta > 0$

$$\lim_{\gamma \to 0} P^\gamma_{\mu_\gamma} \left(\left| \int_{\mathbb{T}^d} \nu^\gamma(x, 0) G(x)\, dx - \int_{\mathbb{T}^d} \rho_0(x) G(x)\, dx \right| > \delta \right) = 0. \tag{2.15}$$

Denote by Q^γ the law of the process $\{\nu^\gamma(t)\}_{t \in [0,T]}$ on the space $D\big([0, T], \mathcal{M}\big)$, induced by $P^\gamma_{\mu_\gamma}$, where $\mathcal{M} = \mathcal{M}(\mathbb{T}^d)$ is the space of non-negative measures on the torus with total mass bounded by 1: this space is endowed with the weak topology weakened by the continuous functions. We also consider the subspace $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{T}^d)$ of $\mathcal{M}$ consisting of absolutely continuous measures with density bounded above by one. Our notation for the gradient in d-dimensional Euclidean space is ∂.

Theorem 2.1. *Consider an initial datum satisfying (2.15). Then, the sequence of probability measures Q^γ is tight and all its limit points Q are concentrated on absolutely continuous paths whose densities $\rho \in C^0([0, T]; \mathcal{M}_1(\mathbb{T}^d)) \cap L^2\big([0, T], H_1(\mathbb{T}^d)\big)$ are weak solutions of the equation*

$$\begin{aligned} \partial_t \rho &= \partial \left\{ D(\rho) \left[\partial\rho - \beta\chi(\rho)(\partial J * \rho) \right] \right\} \\ \rho(0, \cdot) &= \rho_0(\cdot). \end{aligned} \tag{2.16}$$

Moreover, if the diffusion coefficient D is Lipschitz continuous, then Q^γ converges and the limit point Q is the unique weak solution of (2.16).

Throughout the text Φ, J, κ and β will be considered fixed. We will often be interested in finding estimates which are uniform on the configuration η or on the history of the process $\{\eta_t\}_{t \geqslant 0}$. We therefore introduce the notation $o_u(\cdot)$ and $O_u(\cdot)$ in the standard sense of $o(\cdot)$ and $O(\cdot)$, but uniformly with respect to the configuration or to the history of the process.

3. The fluctuation–dissipation equation

In this section we recall the fundamental result proven in [VY]: the approximate decomposition of the current in a gradient term and a fluctuation term.

Let us recall the definition of current in the general J-dependent case. The current is defined for every x, every unit vector e and every η as

$$w^\gamma_{x,x+e}(\eta) = c^\gamma_{x,x+e}(\eta)[\eta(x) - \eta(x + e)]. \tag{3.1}$$

It has the property that

$$\mathcal{L}_\gamma \eta(x) = -\sum_{j=1}^{d} \left[w^\gamma_{x,x+e_j}(\eta) - w^\gamma_{x-e_j,x}(\eta) \right]. \tag{3.2}$$

The analogous definitions in the case of the unperturbed dynamics generated by $\mathcal{L}^0$ are just obtained by setting $\gamma = 0$.

We follow very closely [VY]. For $\zeta \in \Omega_{\mathbb{Z}^d}$ or $\zeta : \mathbb{Z}^d \longrightarrow \mathbb{R}$ and $\ell \in \mathbb{R}^+$ we set

$$\mathrm{Av}_\ell \zeta(x) = \frac{1}{(2\ell + 1)^d} \sum_{y:|y-x| \leqslant \ell} \zeta(y). \tag{3.3}$$

We also set $\Lambda'_\ell = \{x \in \mathbb{Z}^d : |x| \leqslant \ell\}$. We define the σ-algebra $\mathcal{F}_{x,s}$ generated by $\{\mathrm{Av}_s\eta(x)\} \cup \{\eta(y) : |y-x| > s\}$ and the space $\mathcal{G}$ of local functions $h : \Omega_{\mathbb{Z}^d} \to \mathbb{R}$ with the property $\mathbb{E}^\mu[h|\mathcal{F}_s] = 0$ for some s. To any $h \in \mathcal{G}$ and any $\eta \in \Omega_{\mathbb{Z}^d}$ we associate an element (still denoted by h) of $\Omega_{\mathbb{Z}^d}$ by setting $h(x) = \tau_x h(\eta)$. We call $\Omega_{\mathcal{G}}$ the subset of $\Omega_{\mathbb{Z}^d}$ obtained from $\mathcal{G}$ with this procedure. We then introduce the finite-volume variance

$$V_\ell(h,\rho,\xi) = \ell^d \langle \mathrm{Av}_{\ell_1} h, (-\mathcal{L}^0_{\Lambda'_\ell})^{-1} \mathrm{Av}_{\ell_1} h\rangle_{\mu_{\Lambda'_\ell,\rho,\xi}} \tag{3.4}$$

where $\ell_1 = \ell - \sqrt{\ell}$, $\mathcal{L}^0_\Lambda = \sum_{b\in\Lambda^*} \mathcal{L}^0_b$ for any finite set $\Lambda \subset \mathbb{Z}^d$ and $\mu_{\Lambda'_\ell,\rho,\xi}$ is the canonical Gibbs measure with interaction F on Λ'_ℓ with boundary condition $\xi \in \Omega_{\mathbb{Z}^d}$ and density $\rho \in [0,1]$. Two observations are in order for canonical measures. The first is that we view $\mu_{\Lambda'_\ell,\rho,\xi}$ as an element of the probability measures over $\Omega_{\mathbb{Z}^d}$: the extension is made in the natural way obtaining a measure concentrated on $\{\eta : \eta(y) = \xi(y)$ for every $y \in \Lambda'^c_\ell\}$. Second, if ρ is not an integer multiple of $1/|\Lambda'_\ell|$, the density of the canonical Gibbs measure is taken to be $\max\{k/|\Lambda'_\ell| : k \in \mathbb{Z}, k \leqslant \rho|\Lambda'_\ell|\}$. We observe then that if $x \in \Lambda'_{\ell_1}$, $h(x)$ depends only on $\eta(y)$, $y \in \Lambda'_\ell$, for sufficiently large ℓ. Moreover, $V_\ell(h,\rho,\xi)$ is an $\mathcal{F}_\ell$-measurable function of $\xi \in \Omega$. Given μ^ρ, the infinite-volume Gibbs measure with density ρ, we define $V : \Omega_{\mathcal{G}} \times [0,1] \to \mathbb{R}^+$ by

$$V(h,\rho) = \limsup_{\ell\to\infty} \int (V_\ell(h,\rho,\xi)\,\mathrm{d}\mu^\rho(\xi)). \tag{3.5}$$

Finally, we extend this definition to every local function h by setting $h_k(x) = h(x) - \mathbb{E}^{\mu_{\Lambda'_\ell,\rho,\xi}}[h(x)|\mathcal{F}_{x,k}]$ and taking the limit

$$\limsup_{k\to\infty} V(h_k,\rho). \tag{3.6}$$

Below we use the notation

$$\nabla_{e_j}\eta(x) = \eta(x+e_j) - \eta(x). \tag{3.7}$$

We are now ready to state the *fluctuation–dissipation* theorem [VY, theorem 3.4].

Theorem 3.1. *The symmetric, in fact diagonal, density-dependent matrix D defined in (2.12) coincides with the following Green–Kubo formula:*

$$\chi(\rho)D_{i,j}(\rho) = \delta_{i,j}\frac{1}{2}\mathbb{E}^{\mu^\rho}\left[c^0_{(0,e_j)}\right] - \int_0^\infty \sum_x \mathbb{E}^{\mu^\rho}\left[w^0_{0,e_i}\exp\left(\mathcal{L}^0 t\right)\tau_x w^0_{0,e_j}\right]\mathrm{d}t \tag{3.8}$$

where χ is the compressibility defined in (2.11) and $\delta_{i,j}$ is the Kronecker delta. Then for any $\alpha \in \mathbb{R}^d$

$$\inf_{h\in\mathcal{G}^d} V\left(\sum_{j=1}^d \alpha_j\left[w^0_{0,e_j}(\eta) + (D(\rho)\nabla\eta(0))_j + \mathcal{L}^0 h_j(\eta)\right], \rho\right) = 0. \tag{3.9}$$

Moreover, for any $\delta > 0$ there exists $g^\delta : [0,1] \times \Omega_{\mathbb{Z}^d} \to \mathbb{R}^d$, $g^\delta(\rho,\cdot) \in \mathcal{G}^d$ for every ρ and $g^\delta(\cdot,\eta)$ is smooth for every η, such that

$$\sup_{\rho\in[0,1]} V\left(\sum_j \alpha_j\left[w^0_{0,e_j}(\eta) + (D(\rho)\nabla\eta(0))_j + \mathcal{L}^0 g^\delta_j\right], \rho\right) \leqslant \delta. \tag{3.10}$$

We observe that, by polarization, from the bilinear functional $V(\cdot,\rho)$ we can define a scalar product (covariance) that will be denoted by $\langle\!\langle f, g\rangle\!\rangle(\rho)$, defined for f and g local function on $\{0,1\}^{\mathbb{Z}^d}$. This covariance is carefully analysed in section 8 of [VY]. We collect here two properties that will be crucial for us. First, formula (8.7) in [VY] tells us that

$$\langle\!\langle w^0_{0,e_i}, w^0_{0,e_j}\rangle\!\rangle(\rho) = \tfrac{1}{2}\delta_{i,j}E^{\mu^\rho}\left[c_{0,e_j}(\eta)(\nabla_{e_j}\eta(0))^2\right]. \tag{3.11}$$

Moreover, formula (8.13) in [VY] tells us that

$$\langle\!\langle w^0_{0,e_i}, \nabla_{e_j}\eta(0)\rangle\!\rangle(\rho) = \delta_{i,j}\chi(\rho). \tag{3.12}$$

4. Proof of theorem 2.1

4.1. Preliminary lemmas

We will repeatedly need the expansion (in powers of γ) of the jump rates: we take advantage of the fact that an exchange of two particles changes the long-range energy of $O_u(\gamma)$. We will use the following notation for discrete convolution:

$$(f\star\eta)(x) = \gamma^d\sum_{z\in\Lambda_\gamma} f(\gamma(x-z))\,\eta(z) \tag{4.1}$$

where f is a function from $\mathbb{T}^d$ to $\mathbb{R}$ and $x\in\Lambda_\gamma$.

Lemma 4.1. *For every Φ and J, there exists C such that for every $b\in\Lambda^*_\gamma$, every $\eta\in\Omega_{\Lambda_\gamma}$ and every $\gamma\in(0,1)$*

$$\left|\Delta_b\left(\sum_{x,y} J_\gamma(x,y)\eta(x)\eta(y)\right)\right| \leqslant C\gamma \tag{4.2}$$

and

$$\left|c^\gamma_b(\eta) - c^0_b(\eta) - \gamma\beta\Phi'(\kappa\Delta_b H(\eta))\,[\eta(x+e)-\eta(x)]\,e\cdot(\partial J\star\eta)(x)\right| \leqslant C\gamma^2 \tag{4.3}$$

where $b=(x,x+e)$.

Proof. The proof follows immediately from the expansion in Taylor series of Φ (2.4). □

A first application of lemma 4.1 is in proving that the concept of a superexponentially small event coincides for perturbed and unperturbed processes. As usual, β and κ are fixed, but recall that we denote by $P^{\gamma,0}$ the law of the process in the box Λ_γ with $\beta=0$, that is the reference process in the periodic box.

Lemma 4.2. *Let T be a fixed positive number, $\Lambda=\Lambda_\gamma$ and let $\mathcal{B}_{\gamma,q}$ be a bounded functional of the process $\{\eta_s\}_{s\in[0,T\gamma^{-2}]}$ which depends on $\gamma>0$ and on a vector parameter q. Furthermore, let $\mu=\mu^{0,\kappa,\lambda}_\gamma$ (recall (2.2)) and let μ' be any probability measure on Ω_{Λ_γ}. If for every $p\in\mathbb{R}$*

$$\lim_q\limsup_{\gamma\to0}\gamma^d\log E^{\gamma,0}_\mu\left(\exp\left(p\gamma^{-d}\mathcal{B}_{\gamma,q}\right)\right) = 0 \tag{4.4}$$

then the same is true with $E^{\gamma,0}_\mu$ replaced by $E^\gamma_{\mu'}$. Analogously if $\{E_{\gamma,q}\}_{\gamma,q}$ is a family of $\mathcal{F}_{T\gamma^{-2}}$ measurable events which are superexponentially small under $P^{\gamma,0}_\mu$, that is

$$\lim_q\limsup_{\gamma\to0}\gamma^d\log P^{\gamma,0}_\mu\left(E_{\gamma,q}\right) = -\infty \tag{4.5}$$

then they are also superexponentially small under $P^\gamma_{\mu'}$.

Proof. First we observe that, since for some $c = c(F, \kappa, \lambda) \in \mathbb{R}$ we have that $\sup_{\mu',\eta}(\mathrm{d}\mu'/\mathrm{d}\mu)(\eta) \leqslant \mathrm{e}^{c|\Lambda_\gamma|}$, it is sufficient to prove the statements in the case $\mu' = \mu$. We therefore omit the subscript μ in this proof.

For $b = (x, y)$ and $t \geqslant 0$, let $N_b(t)$ denote the number of jumps between x and y in the time span $[0, t]$. The Radon–Nikodym derivative of P^γ with respect to $P^{\gamma,0}$, both restricted to $\mathcal{F}_{t\gamma^{-2}}$ will be denoted by M_t and it is given by

$$M_t = \exp\left(-\int_0^{t\gamma^{-2}} \sum_b \left[c_b^\gamma(\eta_s) - c_b^0(\eta_s)\right] \mathrm{d}s + \int_0^{t\gamma^{-2}} \sum_b \log\left(\frac{c_b^\gamma(\eta_{s-})}{c_b^0(\eta_{s-})}\right) \mathrm{d}N_b(s)\right) \tag{4.6}$$

cf [KL, proposition 2.6, appendix 1]. With respect to the measure $P^{\gamma,0}$ and the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, the process $\{M_t\}_{t\geqslant 0}$ is a martingale. Therefore, for $p > 1$, $\{M_t^p\}_{t\geqslant 0}$ is a submartingale. If we define

$$\begin{aligned} A_t = &-\int_0^{t\gamma^{-2}} pM_s^p \sum_b \left[c_b^\gamma(\eta_s) - c_b^0(\eta_s)\right] \mathrm{d}s \\ &+ \int_0^{t\gamma^{-2}} M_s^p \sum_b c_b^0(\eta_s)\left[\exp\left[p\log\left(\frac{c_b^\gamma(\eta_s)}{c_b^0(\eta_s)}\right]\right) - 1\right] \mathrm{d}s \end{aligned} \tag{4.7}$$

we have that $\tilde{M}_t = M_t^p - A_t$ is a martingale with $\tilde{M}_0 = 1$. We now expand the expression in the right-hand side of (4.7) by taking advantage of the fact that $(c_b^\gamma(\eta_s)/c_b^0(\eta_s)) - 1 = O_u(\gamma)$. Precisely, by lemma 4.1,

$$\exp\left(p\log\left(\frac{c_b^\gamma(\eta_s)}{c_b^0(\eta_s)}\right)\right) - 1 = p\left(\frac{c_b^\gamma(\eta_s)}{c_b^0(\eta_s)} - 1\right) + \frac{p(p-1)}{2}\left(\frac{c_b^\gamma(\eta_s)}{c_b^0(\eta_s)} - 1\right)^2 + O_u(\gamma^3) \tag{4.8}$$

where we have assumed p bounded, say $p \leqslant 2$, for the last term. Therefore, we obtain that there exists C such that

$$\begin{aligned} A_t &= \int_0^{t\gamma^{-2}} M_s^p \sum_b \left[c_b^0(\eta_s)\frac{p(p-1)}{2}\left(\frac{c_b^\gamma(\eta_s)}{c_b^0(\eta_s)} - 1\right)^2 + O_u(\gamma^3)\right] \mathrm{d}s \\ &\leqslant C(p(p-1)\gamma^{-d+2} + \gamma^{-d+3}) \int_0^t M_s^p \, \mathrm{d}s \end{aligned} \tag{4.9}$$

where in the last step we have used the fact that $c_b^0(\eta)$ is uniformly bounded as well as the positivity of M_t. By taking the expectation of this last expression, recalling that $E^{\gamma,0}[\tilde{M}_t] = 1$, by Gronwall's lemma we obtain

$$E^{\gamma,0}\left[M_t^p\right] \leqslant \exp\left(Ct\left(p(p-1)\gamma^{-d} + \gamma^{-d+1}\right)\right). \tag{4.10}$$

This suffices for our purposes since, by applying the Hölder inequality, we obtain that for every $p > 1$ and $q = p/(p-1)$

$$\begin{aligned} &\limsup_q \limsup_{\gamma\to 0} \gamma^d \log E^\gamma\left(\exp(p\gamma^{-d}\mathcal{B}_{\gamma,q}\right) \\ &\qquad \leqslant \limsup_{\gamma\to 0} \frac{\gamma^d}{p} \log E^{\gamma,0}\left(M_T^p\right) + \lim_q \limsup_{\gamma\to 0} \frac{\gamma^d}{q} \log E^{\gamma,0}\left(\exp(q\gamma^{-d}\mathcal{B}_{\gamma,q}\right) \\ &\qquad \leqslant Ct(p-1). \end{aligned} \tag{4.11}$$

By letting $p \searrow 1$ the first statement is proven. The proof of the second statement runs in the same way. □

If h is a local function, we define $\tilde{h}(\rho) = \mathbb{E}^{\mu^\rho}[h]$ for $\rho \in [0, 1]$. Here is the first application of lemma 4.2.

Lemma 4.3 (Replacement lemma). *Let h be a local function and*

$$\mathcal{B}_{b\gamma^{-1}}(\eta) = \left| \frac{1}{(2b\gamma^{-1}+1)^d} \sum_{|y|\leqslant b\gamma^{-1}} \left[h(\tau_y \eta) - \tilde{h}\left(\mathrm{Av}_{[b\gamma^{-1}]}\eta(0)\right) \right] \right| \tag{4.12}$$

for $b > 0$. Then, for any $\delta > 0$

$$\limsup_{b\to 0} \lim_{\varepsilon\to 0} P^\gamma \left[\int_0^T \gamma^d \sum_x \mathcal{B}_{b\gamma^{-1}}\left(\tau_x \eta_{\gamma^{-2}t}\right) \mathrm{d}t \geqslant \delta \right] = 0. \tag{4.13}$$

Proof. This goes through the by now classical one- and two-block estimates. These can be found in [VY] (lemma 5.2 and theorem 6.2) for the unperturbed process and these estimates are superexponential. The extension is therefore just lemma 4.2. □

We conclude this subsection with a computation that is very relevant for us to identify the limit equation.

Lemma 4.4. *For any bounded local function $f : \Omega_{\mathbb{Z}^d} \to \mathbb{R}$ we have that*

$$\mathbb{E}^{\mu^\rho}\left[\Phi'(\kappa\Delta_{0,e}H)\nabla_e\eta(0)\Delta_{0,e}f(\eta)\right] = -\tfrac{1}{2}\mathbb{E}^{\mu^\rho}\left[\Phi(\kappa\Delta_{0,e}H)\nabla_e\eta(0)\Delta_{0,e}f(\eta)\right]. \tag{4.14}$$

Proof. By differentiating both sides of (2.4), we reduce (4.14) to proving that

$$E^{\mu^\rho}\left[\exp\left(-\kappa\Delta_{0,e}H/2\right)\phi'(\kappa\Delta_{0,e}H)\left(\eta(e)-\eta(0)\right)\Delta_{0,e}f(\eta)\right] = 0. \tag{4.15}$$

Let us now approximate μ^ρ with a sequence of finite-volume grand-canonical Gibbs measures on Ω_{Λ_γ}. The result follows because

$$\begin{aligned}
&\sum_{\eta\in\Omega_{\Lambda_\gamma}} \mathrm{e}^{-\kappa\Delta_{0,e}H(\eta)/2}\phi'(\kappa\Delta_{0,e}H(\eta))\left(\eta(e)-\eta(0)\right)\Delta_{0,e}f(\eta)\mathrm{e}^{-\kappa H(\eta)}\mathrm{e}^{\lambda\sum_x\eta(x)} \\
&\quad = \sum_{\eta\in\Omega_{\Lambda_\gamma}} \left(\eta(e)-\eta(0)\right)\Delta_{0,e}f(\eta)\mathrm{e}^{-\kappa[H(\eta)+H(\eta^{0,e})]/2}\phi'(\kappa\Delta_{0,e}H(\eta))\mathrm{e}^{\lambda\sum_x\eta(x)} \\
&\quad = \sum_{\eta\in\Omega_{\Lambda_\gamma}} \left(\eta(e)-\eta(0)\right)\Delta_{0,e}f(\eta)\mathrm{e}^{-\kappa[H(\eta^{0,e})+H(\eta)]/2}\phi'(-\kappa\Delta_{0,e}H(\eta))\mathrm{e}^{\lambda\sum_x\eta(x)}
\end{aligned} \tag{4.16}$$

where in the last step we have used the fact that $(\eta(e)-\eta(0))\Delta_{0,e}f(\eta)$ is invariant under the transformation $\eta \longrightarrow \eta^{0,e}$. Recall now that ϕ' is odd and the proof of (4.14) is complete. □

4.2. Tightness and energy estimate

We now go to the set up of theorem 2.1. Recall that Q^γ is the law of the empirical process $\{\nu^\gamma(t)\}_{t\in[0,T]}$, cf (2.14), on the Skorohod space $D([0,T],\mathcal{M})$.

Lemma 4.5. *The sequence $\{Q^\gamma\}_{\gamma>0}$ is tight and every limit point Q is concentrated on absolutely continuous paths whose densities belong to*

$$C^0([0,T];\mathcal{M}_1(\mathbb{T}^d)) \cap L^2([0,T],H_1(\mathbb{T}^d)).$$

Proof. This is standard. One way to prove tightness in $D([0, T], \mathcal{M})$ is to repeat the proof in [VY, section 4]: lemma 4.1 in [VY] depends only on the uniform boundedness of the jump rates and lemma 4.2, still in [VY], is easily upgraded to our situation via lemma 4.2. The limit points actually lie in a smaller space: by the exclusion rule it can be immediately verified that the limit is in $\mathcal{M}_1$ for every t and, since every jump produces a discontinuity $O_u(\gamma^d)$, we can substitute D with C^0. The *energy estimate*, that is the existence of a constant C such that any limit point Q satisfies

$$\mathbb{E}^Q\left[\int_0^T \int_{\mathbb{T}^d} (\nabla\rho(s, r))\, dr\, ds\right] < C \tag{4.17}$$

requires a more sophisticated argument. Once again most of the work has been done in [VY], section 5: it is sufficient to upgrade (5.25) and (5.28)–(5.30) in [VY] to our situation. One way would be to obtain a volume order estimate on the relative entropy of P^γ with respect to P^0, with both measures restricted to $[0, T\gamma^{-2}]$, but we choose to stick to L^p estimates on the Radon–Nicodym derivative M_T of the process. We will be as close as possible to the notation of lemma 4.2 and, like in its proof, we will omit the dependence on the initial condition: the only difference is that we will not need any extra parameter q and we will therefore omit the subscript q from the notation. We choose, with $G \in C^1(\mathbb{R}^+, \mathbb{T}^d; \mathbb{R})$ and $C_1 > 0$,

$$\mathcal{B}_{\gamma,q} = \mathcal{B}_\gamma = \gamma^{d-1}\int_0^T \sum_x G(t, x\gamma)\nabla_e\eta_{t\gamma^{-2}}(x)\, dt - \frac{C_1}{2}\gamma^d\int_0^T \sum_x |G(t, x/N)|^2\, dt. \tag{4.18}$$

In the proof of lemma 3.2 in [VY] it is shown that for some C_1 (4.4) holds for $p = 1$, that is

$$\lim_{\gamma\to 0} \gamma^d \log E^{\gamma,0}\left[\exp\left(\gamma^{-d}\mathcal{B}_\gamma\right)\right] = 0. \tag{4.19}$$

Using (4.19) we will show that there exists C_2 such that

$$\limsup_{\gamma\to 0} E^\gamma\left[\mathcal{B}_\gamma\right] \leqslant C_2. \tag{4.20}$$

We refer the reader to [VY, section 5] for a proof of the fact that (4.20), with the choice (4.18), implies (4.17) with $C = \max(C_1, C_2)$. Here we prove (4.20); it is just an application of Jensen's inequality and Cauchy–Schwarz:

$$\begin{aligned} E^\gamma\left[\mathcal{B}_\gamma\right] &\leqslant 2\gamma^d \log E^{\gamma,0}\left[M_T \exp\left(\frac{\gamma^{-d}}{2}\mathcal{B}_\gamma\right)\right] \\ &\leqslant \gamma^d \log E^{\gamma,0}\left[M_T^2\right] + \gamma^d \log E^{\gamma,0}\left[\exp\left(\gamma^{-d}\mathcal{B}_\gamma\right)\right] \end{aligned} \tag{4.21}$$

and, recalling (4.19), (4.20) follows from (4.11) with $p = 2$. □

4.3. *The (partial) identification of the limit*

By the definition of the process, for every $G \in C^1(\mathbb{T}^d; \mathbb{R})$ we can write

$$\begin{aligned} &\gamma^d \sum_x G(\gamma x)\eta_{t\gamma^{-2}}(x) - \gamma^d \sum_x G(\gamma x)\eta_0(x) \\ &\quad = \gamma^d \int_0^{t\gamma^{-2}} \sum_{x,e} \left[G(\gamma(x+e)) - G(\gamma x)\right] w^\gamma_{x,x+e}(\eta_s)\, ds + M^G_\gamma(t) \end{aligned} \tag{4.22}$$

in which $M^G_\gamma = \{M^G_\gamma(t)\}_{t\in\mathbb{R}^+}$ is a P^γ-martingale with respect to the filtration associated with $\{\eta_t\}_{t\in\mathbb{R}^+}$. The quadratic variation of M^G_γ is easily computed and estimated: if we set

$X_\gamma(\eta) = \gamma^d \sum_x G(\gamma x)\eta(x)$ it can be immediately seen that there exists a constant $c = c(d, G)$ such that for every $\gamma > 0$

$$\langle M_\gamma^G, M_\gamma^G\rangle(t) = \int_0^{t\gamma^{-2}} \left(\mathcal{L}_\gamma X_\gamma^2 - 2X_\gamma \mathcal{L}_\gamma X_\gamma\right)(\eta_s)\,\mathrm{d}s \leqslant c\gamma^d \tag{4.23}$$

and thus, by Doob's inequality, for every $T > 0$ and every $\delta > 0$

$$\lim_{\gamma\to 0} P^\gamma\left(\sup_{t\in[0,T]} \left|M_\gamma^G(t)\right| > \delta\right) = 0. \tag{4.24}$$

We now split the current:

$$w^\gamma = [w^0] + [w^\gamma - w^0]. \tag{4.25}$$

The non-gradient difficulties come from the first term. Below we use the convention that $\sum_e$ is the sum over the unit vectors $\{e_j\}_{j=1,2,\ldots,d}$.

We first have a quick look at the *easy* second term. By lemma 4.1, for every x and every unit vector e

$$[w^\gamma - w^0]_{x,x+e}(\eta) = -\beta\gamma\Phi'\left(\Delta_{x,x+e}H(\eta)\right)[\nabla_e\eta]^2(x)e\cdot(\partial J \star \eta)(x) + O_u(\gamma^2). \tag{4.26}$$

If we define

$$\begin{aligned}\mathcal{R}_{b,\gamma}\left(\{\eta_s\}_{s\geqslant 0}\right) = \Bigg| &\int_0^{\gamma^{-2}T} \gamma^d \sum_{x,e}\left[G(\gamma(x+e)) - G(\gamma x)\right]\left[w^\gamma_{x,x+e}(\eta_t) - w^0_{x,x+e}(\eta_t)\right]\mathrm{d}t \\ &+\beta\int_0^{\gamma^{-2}T}\gamma^{d+2}\sum_{x,e}(\partial_e G)(\gamma x)(\partial_e J\star\eta_t)(x)\frac{1}{(2b\gamma^{-1}+1)^d} \\ &\times \sum_{|y|\leqslant b\gamma^{-1}} h_e\left(\tau_{x+y}\eta_t\right)\mathrm{d}t\Bigg|\end{aligned} \tag{4.27}$$

where $h_e(\eta) = \Phi'(\Delta_{0,e}H(\eta))(\eta(e) - \eta(0))^2$, and therefore

$$\tilde{h}_e(\rho) = \mathbb{E}^{\mu^\rho}\left[\Phi'\left(\kappa\Delta_{0,e}H(\eta)\right)(\eta(e) - \eta(0))^2\right]. \tag{4.28}$$

Note that, by applying lemma 4.4, with $f \equiv 1$, we can immediately rewrite

$$\tilde{h}_e(\rho) = -\tfrac{1}{2}\mathbb{E}^{\mu^\rho}\left[c^0_{0,e}(\eta)(\eta(e) - \eta(0))^2\right]. \tag{4.29}$$

By the smoothness of G and J it can be immediately obtained that

$$\lim_{b\to 0}\limsup_{\gamma\to 0}\sup_{\{\eta_s\}_{s\geqslant 0}} \mathcal{R}_{b,\gamma}\left(\{\eta_s\}_{s\geqslant 0}\right) = 0. \tag{4.30}$$

This, together with lemma 4.3 applied to the spatial average of h, immediately implies that

$$\begin{aligned}\Bigg| &\int_0^{\gamma^{-2}T}\gamma^d\sum_{x,e}\left[G(\gamma(x+e)) - G(\gamma x)\right]\left[w^\gamma_{x,x+e}(\eta_t) - w^0_{x,x+e}(\eta_t)\right]\mathrm{d}t \\ &- \int_0^T\int_{\mathbb{T}^d}\partial G(r)(\partial J\star\nu^\gamma)(t,r)\tilde{h}\left(\mathbf{1}_{(-b,b)^d}\star\nu^\gamma)(t,r)\right)\mathrm{d}r\,\mathrm{d}t\Bigg|\end{aligned} \tag{4.31}$$

tends to zero in P^γ-probability as $\gamma \to 0$ and then $b \to 0$.

Now we turn to the non-gradient term. Given a, b, δ and $\delta_1 > 0$, we define

$$\mathcal{B}_{a,b,\delta} = \left\{ \sup_{t\in[0,T]} \left| \gamma^{d+1} \int_0^{\gamma^{-2}t} \sum_x \sum_e \partial_e G(\gamma x) w^0_{x,x+e}(\eta_t)\, \mathrm{d}t \right.\right.$$
$$+\gamma^{d+1} \int_0^{\gamma^{-2}T} (2b)^{-1} \sum_{x,e,e'} \left[\eta_s(x + b\gamma^{-1}e) - \eta_s(x - b\gamma^{-1}e)\right]$$
$$\times D_{e,e'}\left(\mathrm{Av}_{[a\gamma^{-1}]}\eta_s(x)\right) \partial_e G(\gamma x)$$
$$\left.\left. +\gamma^{d+1} \int_0^{\gamma^{-2}T} \sum_{x,e} \partial_e G(\gamma x)\mathcal{L}^0 g^\delta_e(\tau_x\eta_t) \right| \geqslant \delta_1 \right\}. \tag{4.32}$$

The following result follows from theorem 3.3 in [VY] and lemma 4.2, with the only observation that in the statement in [VY] the term $\mathcal{L}^0 g^\delta$ does not appear: in their case it is irrelevant (see section 3 of [VY]).

Lemma 4.6. *For every* $\delta_1 > 0$

$$\lim_{a\to 0} \limsup_{b\to 0} \limsup_{\delta\to 0} \limsup_{\gamma\to 0} P^\gamma\left(\mathcal{B}_{a,b,\delta}\right) = 0. \tag{4.33}$$

We now observe that in (4.32) we can replace $\mathcal{L}^0 g^\delta$ with $(\mathcal{L}^0 - \mathcal{L}_\gamma)g^\delta$ and lemma 4.6 still holds since

$$\gamma^{d+1} \int_0^{\gamma^{-2}T} \sum_{x,e} \partial_e G(\gamma x)\mathcal{L}_\gamma g^\delta_e(\tau_x\eta_t)$$
$$= \gamma\left[\gamma^d \sum_{x,e} \partial_e G(\gamma x)\left(g^\delta_e\left(\tau_x\eta_{\gamma^{-2}T}\right) - g^\delta_e(\tau_x\eta_0)\right)\right] + \gamma\tilde{M}_\gamma(T) + E_\gamma(T) \tag{4.34}$$

where $\{\tilde{M}^\gamma(t)\}_{t\geqslant 0}$ with respect to the natural filtration is a P^γ-martingale and $E_\gamma(T)$ is the error we made by considering g^δ independent of $\{\mathrm{Av}_{\ell_\delta}\eta(x)\}_x$, ℓ_δ the range of g^δ. The first term in the second line of (4.34) is $O_u(\gamma)$, the martingale $\tilde{M}_\gamma$ has a quadratic variation $O(\gamma^d)$ and $\sup_{t\in[0,T]}|E_\gamma(t)|$ tends to zero in probability as $\gamma \to 0$ and $\delta \to 0$. The last claim follows from the argument in [VY] (formulae (3.26) and (3.27)) and lemma 4.2.

We are therefore left with evaluating

$$\gamma \int_0^{\gamma^{-2}T} \gamma^d \sum_{x,e} \partial_e G(\gamma x)\left[\mathcal{L}^0 - \mathcal{L}_\gamma\right] g^\delta_e(\tau_x\eta_t)\, \mathrm{d}t. \tag{4.35}$$

By using lemma 4.1 to express $\mathcal{L}^0 - \mathcal{L}_\gamma$, recalling that g^δ_e is a local function and repeating exactly the same steps, i.e. *smearing* and using the replacement lemma, as in (4.27)–(4.31), we obtain that, up to a negligible error term, this term can be replaced by

$$\beta \sum_{e,e'} \int_0^T \int_{\mathbb{T}^d} \partial_e G(r)\theta^\delta_{e,e'}\left(\chi_{[-\delta,\delta]^d} * \nu^\gamma(t,r)\right) \partial_{e'} J * \nu^\gamma(t,r)\, \mathrm{d}r\, \mathrm{d}t \tag{4.36}$$

where

$$\theta^\delta_{e,e'}(\rho) = E^{\mu^\rho}\left[\sum_z \left(\eta(z+e') - \eta(z)\right)\Phi'\left(\kappa\Delta_{z,z+e'}H_0(\eta)\right)\left[g^\delta_e(\eta^{z,z+e'}) - g^\delta_e(\eta)\right]\right]. \tag{4.37}$$

Let us define $\theta(\rho) = \limsup_{\delta\to 0}\theta^\delta(\rho)$ for every $\rho \in [0, 1]$. By lemma 4.5 and collecting the results (4.22), (4.24), (4.30) and lemma 4.6, under the hypothesis that θ^δ converges uniformly as $\delta \to 0$, we obtain that every limit of the sequence $\{\nu^\gamma\}_{\gamma>0}$ concentrates on trajectories ρ which solve weakly the partial differential equation (PDE)

$$\begin{aligned} \partial_t \rho &= \sum_{i,j} \partial_i \left\{ D_{i,j}(\rho)\partial_j \rho - \beta\left(\delta_{i,j}\tilde{h}_j(\rho) + \theta_{i,j}(\rho)\right)(\partial_j J * \rho) \right\} \\ \rho(0,\cdot) &= \rho_0(\cdot). \end{aligned} \tag{4.38}$$

We are therefore left with showing that

Lemma 4.7. *The sequence θ^δ converges uniformly and for every $\rho \in [0, 1]$, every i and every $j = 1, 2, \ldots, d$ we have that*

$$\delta_{i,j}\tilde{h}_j(\rho) + \theta_{i,j}(\rho) = \chi(\rho) D_{i,j}(\rho). \tag{4.39}$$

Proof. We divide this algebraic computation into steps.

Step 1. We start by rewriting θ in a more convenient form, using lemma 4.4 with $f = g^\delta_{e_j}$. We obtain that

$$\theta^\delta_{i,j}(\rho) = -\frac{1}{2}\mathbb{E}^{\mu^\rho}\left[\sum_z c_{z,z+e_j}(\eta)\left(\eta(z+e_j) - \eta(z)\right)\Delta_{z,z+e_j} g^\delta_{e_i}(\eta)\right]. \tag{4.40}$$

Step 2. Reversibility and summation by parts on (4.40) imply

$$\theta^\delta_{i,j}(\rho) = \mathbb{E}^{\mu^\rho}\left[\sum_z c_{z,z+e_j}(\eta)\left(\eta(z+e_j) - \eta(z)\right) g^\delta_{e_i}(\tau_z\eta)\right] = -\langle w^0_{0,e_j}, g^\delta_{e_i}\rangle_0(\rho) \tag{4.41}$$

where in the last step we used the notation

$$\langle f, g\rangle_0(\rho) = \sum_x \mathbb{E}^{\mu^\rho}\left[f(\eta) g(\tau_x\eta)\right] \tag{4.42}$$

for f and g local functions. It is not too difficult to show [VY, section 8] that

$$\langle f, g\rangle_0 = -\langle\langle f, \mathcal{L}_0 g\rangle\rangle \tag{4.43}$$

and we recall that the *covariance* $\langle\langle \cdot, \cdot\rangle\rangle$ is defined right after theorem 3.1. Therefore,

$$\theta^\delta_{i,j} = \langle\langle w^0_{0,e_j}, \mathcal{L}_0 g^\delta_{e_i}\rangle\rangle. \tag{4.44}$$

Step 3. We now use the decomposition induced by (3.9) to express $\mathcal{L}_0 g^\delta_{e_j}$ obtaining that $\theta^\delta_{i,j}$ is equal to

$$\langle\langle w^0_{0,e_j}, \mathcal{L}_0 g^\delta_{e_i}\rangle\rangle(\rho) = \langle\langle w^0_{0,e_i}, w^0_{0,e_i}\rangle\rangle(\rho) - \langle\langle w^0_{0,e_j}, (D\nabla\eta)_{e_i}(0)\rangle\rangle(\rho) + R(\rho,\delta) \tag{4.45}$$

where $\lim_{\delta\to 0}\sup_\rho |R(\rho,\delta)| = 0$.

Completing the proof of (4.39) is now just a matter of applying (3.11), note the cancellation with the term $\tilde{h}$, cf (4.29), and (3.12) and the proof is complete. □

Lemma 4.7 leads us to the weak formulation of the PDE (2.16), that is that every limit measure Q is concentrated on trajectories $\rho \in C^0([0,T]; \mathcal{M}_1(\mathbb{T}^d)) \cap L^2\left([0,T], H_1(\mathbb{T}^d)\right)$ which solve

$$\begin{aligned} &\int_{\mathbb{T}^d} G(r)\rho(T,r)\,\mathrm{d}r - \int_{\mathbb{T}^d} G(r)\rho(0,r)\,\mathrm{d}r \\ &\qquad = \int_0^T\int_{\mathbb{T}^d} \partial G(r)\left(D(\rho(t,r))\chi(\rho(t,r))\partial J * \rho - D(\rho)\partial\rho(t,r)\right)\mathrm{d}r\,\mathrm{d}t \end{aligned} \tag{4.46}$$

for every $G \in C^1(\mathbb{T}^d)$ and every $T > 0$. Therefore, the proof of theorem 2.1 is complete modulo discussing the uniqueness issue. This is considered in the next subsection.

4.4. Uniqueness

A proof of uniqueness is available if D is a uniformly Lipschitz continuous function. This follows from the control of the H^{-1} norm. In [GL2] this result is proven under an additional condition on the time derivative, in the sense of distributions, of ρ. This condition can be removed if one replaces the kernel of the H^{-1} norm with a smoothened kernel, as is done in [KL, appendix A]. Note that the weak formulation (4.46) is in this set up equivalent to the formulation that we obtain by multiplying both sides of (2.16) by $\tilde{G} \in C^1(\mathbb{R}^+ \times \mathbb{T}^d; \mathbb{R})$ and formally integrating by parts.

5. Multispecies systems

The scheme of the proof of sections 3 and 4 can be applied to several other systems. Of particular interest for applications is the case of systems with several types of particles. One (apparently) simple system in this class is the n-colour exclusion process: take a simple exclusion process and distinguish the particles by painting them with n colours (n is kept fix). The hydrodynamics of this system has been explored in [Q]: here we would like to consider the case in which the particles interact via long-range potentials that distinguish between colours.

From the technical viewpoint this case has essentially already been considered in [QRV], where a large-deviation principle for an n-colour exclusion system is proven as a step in proving a process large deviation for the simple exclusion. The lower bound of the large deviations depends on the standard change of measure argument which entails studying the n-colour system driven by a weak external force. This is also our case: the weak driving force is, however, configuration dependent, but since it depends on the configuration only via an empirical average, the changes with respect to [QRV] are minimal. We will therefore simply state the result and make some observations on the proof.

To simplify the notation and the statement of the result we restrict our attention to the case of two species (A and B) and to the case in which A and B interact with each other, but A does not interact with any A and similarly for B particles.

5.1. The A–B model

As in the previous sections Λ_γ will denote the lattice torus with diameter $[\gamma^{-1}]$. We are looking at a d-dimensional system of A and B particles evolving via a Kawasaki dynamics with the Kac Hamiltonian

$$H_\gamma(\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_\gamma} J_\gamma(x, y)\eta^A(x)\eta^B(y) \tag{5.1}$$

where as usual $\gamma > 0$ and η^A and η^B are elements of $\{0, 1\}^{\Lambda_\gamma}$ with the hard core restriction that there can be at most one particle per site

$$\eta^A + \eta^B \leqslant 1. \tag{5.2}$$

$J_\gamma(x, y) = \gamma^d J(\gamma(x - y))$, where J is a smooth function from the d-dimensional unit torus to $\mathbb{R}$. The particle configuration can alternatively be described by

$$\eta \in \{0, A, B\}^{\Lambda_\gamma} \tag{5.3}$$

and we will identify η with (η^A, η^B). The dynamics which conserves both A and B particles (or interchanging with) are specified by A and B particles hopping to nearest-neighbour empty sites at a rate $Dc_b^{\gamma,AB}(\eta)$ but no direct exchanges between A and B particles are permitted. This models the physical situation of polymers in a fluid considered in [EP, DG]. The generator of the dynamics is

$$L_\gamma f(\eta) = D \sum_{b=(x,y)\in\Lambda_\gamma^*} c_b^{\gamma,AB}(\eta)\left[(\eta^A(x)-\eta^B(y))^2+(\eta^B(x)-\eta^A(y))^2\right]\left[f(\eta^{x,y})-f(\eta)\right] \tag{5.4}$$

where $D > 0$, f is a real-valued function defined in $\{0, A, B\}^{\Lambda_\gamma}$ and $\eta^{x,y}$ is η with the occupation of the sites x and y exchanged. As before, the rates $c_b^{\gamma,AB}$ are such that the dynamics satisfies the detailed balance condition with respect to the Gibbs measure with the Hamiltonian (5.1) at inverse temperature $\beta > 0$:

$$c_{(x,y)}^{\gamma,AB}(\eta) = \Phi\left(\beta\left[H_\gamma(\eta^{x,y})-H_\gamma(\eta)\right]\right) \tag{5.5}$$

Φ as in (2.4). To denote the process we keep the same notation as in previous sections.

5.2. *The hydrodynamics of the A–B model*

For the hydrodynamic limit we look at the empirical measure

$$\nu_\alpha^\gamma(t,x) = \gamma^d \sum_{y\in\Lambda_\gamma} \delta(x-\gamma y)\eta_{\gamma^{-2}t}^\alpha(y) \tag{5.6}$$

for $\alpha \in \{A, B\}$, $x \in \mathbb{T}^d$ and $\eta^\alpha(x, s)$ specifies the presence or absence of an α-particle on site x at time s.

For the initial datum we assume $\nu_\alpha^\gamma(0,\cdot)$ to be close to $\rho^\alpha(\cdot)$ in the sense of (2.15). In this case we have to make further assumptions on ρ^α, precisely that there exists $\delta > 0$ such that for every x and every α

$$\delta \leqslant \rho^\alpha(x) \leqslant 1-\delta \tag{5.7}$$

and that ρ^α is differentiable with bounded derivatives

$$\sup_{x,\alpha} |\partial\rho^\alpha(x)| < \infty. \tag{5.8}$$

For the moment let us set $J = 0$. In [Q] it is shown that ν_α^γ converges *weakly in probability*, that is in the sense of (2.15) for every $t \geqslant 0$, to $\rho^\alpha : \mathbb{R}^+ \times \mathbb{T}^d \to [0, 1]$ and the pair (ρ^A, ρ^B) solves

$$\partial_t \begin{pmatrix}\rho^A\\ \rho^B\end{pmatrix} = \partial\cdot\left[\begin{pmatrix} \dfrac{\rho^B}{\rho}D_s + \dfrac{\rho^A}{\rho}D & \dfrac{\rho^A}{\rho}(D-D_s) \\ \dfrac{\rho^B}{\rho}(D-D_s) & \dfrac{\rho^A}{\rho}D_s + \dfrac{\rho^B}{\rho}D \end{pmatrix} \partial\begin{pmatrix}\rho^A\\ \rho^B\end{pmatrix}\right] \tag{5.9}$$

with the initial condition $(\rho^A(0,\cdot), \rho^B(0,\cdot)) = (\rho^A(\cdot), \rho^B(\cdot))$. In (5.9) $\rho = \rho^A + \rho^B$ and $D_s = D_s(\rho)$ is the self-diffusion coefficient. The expression for the diffusion matrix D in (5.9) can be derived from elementary considerations on the microscopic system from which it is derived [LS]. We observe that, as expected, the evolution equation for ρ is simply

$$\partial_t\rho = D\Delta\rho. \tag{5.10}$$

This follows from the observation that if we ignore the distinction between A and B particles then, in the absence of interactions, we are just dealing with the one-component simple symmetric exclusion process [Sp].

We can rewrite the system (5.9) as

$$\partial_t \underline{\rho} = \nabla \cdot [\mathcal{D} \nabla \underline{\rho}] = \nabla \cdot \left[M \nabla \frac{\delta \mathcal{F}_0}{\delta \underline{\rho}} \right] \tag{5.11}$$

in which $\underline{\rho} = (\rho^A, \rho^B)^t$, and

$$\mathcal{F}_0(\rho^A, \rho^B) = -\frac{1}{\beta} \int s(\rho^A, \rho^B)\, \mathrm{d}x \tag{5.12}$$

with

$$s(\rho^A, \rho^B) = -\left[\rho^A \log \rho^A + \rho^B \log \rho^B + (1 - \rho^A - \rho^B) \log(1 - \rho^A - \rho^B)\right] \tag{5.13}$$

and

$$M = -\beta \mathcal{D} (\mathbf{Hess}(s))^{-1}. \tag{5.14}$$

Explicitly,

$$\mathbf{Hess}(s)^{-1} = \begin{pmatrix} -\rho^A(1-\rho^A) & \rho^A \rho^B \\ \rho^A \rho^B & -\rho^B(1-\rho^B) \end{pmatrix} \tag{5.15}$$

and

$$M = \beta \begin{pmatrix} D_s \dfrac{\rho^A \rho^B}{\rho} + D \dfrac{(\rho^A)^2 (1-\rho)}{\rho} & -D_s \dfrac{\rho^A \rho^B}{\rho} + D \dfrac{\rho^A \rho^B (1-\rho)}{\rho} \\ D \dfrac{\rho^B \rho^A (1-\rho)}{\rho} - D_s \dfrac{\rho^A \rho^B}{\rho} & D \dfrac{(\rho^B)^2 (1-\rho)}{\rho} + D_s \dfrac{\rho^A \rho^B}{\rho} \end{pmatrix}. \tag{5.16}$$

We now claim that, in the case in which J is not zero, the limit equation (5.11) has to be changed by replacing $\mathcal{F}_0$ with

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \iint J(x - x') \rho^A(x) \rho^B(x')\, \mathrm{d}x\, \mathrm{d}x'. \tag{5.17}$$

Proposition 5.1. *Set $d \geqslant 3$. For every $t \geqslant 0$ the empirical field $(\nu_a^\gamma(t, \cdot), \nu_B^\gamma(t, \cdot))$ converges weakly in probability, i.e. in the sense of (2.15), to the unique weak solution of*

$$\partial_t \underline{\rho} = \nabla \cdot \left[\mathcal{D} \partial \underline{\rho} + M \partial J * \underline{\rho}\right]. \tag{5.18}$$

The restriction to $d \geqslant 3$ is due to the fact that only under this restriction is D_s known to be a Lipschitz function and does uniqueness of the weak solution hold.

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