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AN ISING INTERFACE BETWEEN TWO WALLS: COMPETITION BETWEEN TWO TENDENCIES

J. L. LEBOWITZ*

Department of Mathematics and Physics
Rutgers University, New Brunswick
NJ 08903, USA
E-mail: Lebowitz@math.rutgers.edu

A. E. MAZEL*

International Institute of Earthquake Prediction Theory
and Mathematical Geophysics
The Russian Academy of Sciences
Moscow 113556, Russia
E-mail: Mazel@math.rutgers.edu

YU. M. SUHOV†

Statistical Laboratory
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge CB2 1SB and St John's College
Cambridge CB2 1TP, UK

Institute for Problems of Information Transmission
The Russian Academy of Sciences
GSP 4 Moscow 101447, Russia
E-mail: Y. M. Suhov@statslab.cam.ac.uk

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We consider a ferromagnetic Ising spin system, consisting of $m+1$, d-dimensional, layers with "−" boundary condition on the bottom layer and "+" on the top layer. When β is larger than $\beta_{\rm cr}$, the inverse critical temperature for the d-dimensional Ising model, the interface generated by the boundary conditions is expected to be halfway between bottom and top, for m odd, and just above or below the middle layer, for m even (each possibility with probability $\frac{1}{2}$). In this paper, we prove the above assertion under the condition that $\beta \geq \text{const} \cdot m$ and partly for $\beta > \beta_{\rm cr}$.

Keywords and phrases: ferromagnetic Ising spins, interface, entropic repulsion and attraction, contours, polymer expansions, dominant ground states, correlation inequalities.

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1. Introduction and Results

We consider a ferromagnetic Ising spin system in a lattice domain $\mathbb{D}$ of the form $\mathbb{Z}^d \times \mathbb{I}_m$ where $\mathbb{I}_m$ is the set $\{-m/2, -m/2+1, \dots, m/2\}$ and $m \geq 1$ is a fixed positive integer. A point $\boldsymbol{x}$ of $\mathbb{D}$ is represented by a pair $\boldsymbol{x} = (x, z)$ where $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ and z is integer for m even and half-integer for m odd, between $-m/2$ and $m/2$ (inclusive). A spin at site $\boldsymbol{x}$ is denoted by $\boldsymbol{\sigma}_{\boldsymbol{x}}$; it takes values ± 1. In most parts of the paper we consider the case $d = 2$, but all results can be straightforwardly extended to the case of a general $d \geq 2$.

Putting "$-$" boundary condition on the bottom layer $\partial^- \mathbb{D} = \mathbb{Z}^d \times \{-m/2 - 1\}$ and "+" on the top layer $\partial^+ \mathbb{D} = \mathbb{Z}^d \times \{m/2 + 1\}$ we consider the Gibbs ensemble, in a finite volume $\mathbf{V} = V \times \mathbb{I}_m$, where V is a finite subset of $\mathbb{Z}^d$, with 'additional' boundary condition on $\partial^\perp \mathbf{V} = \partial V \times \mathbb{I}_m$. Here, $\partial V = \{x \in \mathbb{Z}^d : x \notin V$, at least one n.n. of x is in $V\}$. The 'whole' boundary condition is denoted by $\boldsymbol{\sigma}_{\partial \mathbf{V}}$; here, $\partial \mathbf{V} = \partial^\perp \mathbf{V} \cup \partial^- \mathbf{V} \cup \partial^+ \mathbf{V}$, and $\partial^\pm \mathbf{V} = V \times \{\pm m/2 \pm 1\}$. A spin configuration in volume $\mathbf{V}$ is denoted by $\boldsymbol{\sigma}_{\mathbf{V}} = \{\boldsymbol{\sigma}_{\boldsymbol{x}} = \pm 1 \colon \boldsymbol{x} \in \mathbf{V}\}$.

The Gibbs ensemble is determined by the Hamiltonian

$$H_V(\boldsymbol{\sigma}_{\mathbf{V}} | \boldsymbol{\sigma}_{\partial \mathbf{V}}) = - \sum_{\langle \boldsymbol{x}, \boldsymbol{x}' \rangle \subset \mathbf{V}} \boldsymbol{\sigma}_{\boldsymbol{x}} \boldsymbol{\sigma}_{\boldsymbol{x}'} - \sum_{\substack{\langle \boldsymbol{x}, \boldsymbol{x}' \rangle :\, \boldsymbol{x} \in \mathbf{V} \\ \boldsymbol{x}' \in \partial \mathbf{V}}} \boldsymbol{\sigma}_{\boldsymbol{x}} \boldsymbol{\sigma}_{\boldsymbol{x}'} \,, \tag{1.1}$$

where $\langle \boldsymbol{x}, \boldsymbol{x}' \rangle$ are nearest neighbor sites, via the standard formula

$$\mathbf{Pr}_{\beta, \mathbf{V}}(\boldsymbol{\sigma}_{\mathbf{V}} | \boldsymbol{\sigma}_{\partial \mathbf{V}}) = \frac{\exp\left(-\beta H_V(\boldsymbol{\sigma}_{\mathbf{V}} | \boldsymbol{\sigma}_{\partial \mathbf{V}})\right)}{\Xi_\beta(\mathbf{V} | \boldsymbol{\sigma}_{\partial \mathbf{V}})} \,, \tag{1.2}$$

where $\Xi_\beta(\mathbf{V} | \boldsymbol{\sigma}_{\partial \mathbf{V}})$ is the partition function with boundary condition $\boldsymbol{\sigma}_{\partial \mathbf{V}}$.

A boundary condition $\boldsymbol{\sigma}_{\partial^\perp \mathbf{V}}$ is said to be *regular* if the set of the dual unit plaquettes separating the nearest neighbor sites $\boldsymbol{x}, \boldsymbol{x}' \in \partial^\perp \mathbf{V}$ with spin values of the opposite sign is connected. For any configuration $\boldsymbol{\sigma}_{\mathbf{V}}$ with regular boundary condition $\boldsymbol{\sigma}_{\partial \mathbf{V}}$ consider the complete separating surface which is a set of the unit d-dimensional plaquettes of the dual lattice $\widetilde{\mathbb{D}} = \widetilde{\mathbb{Z}^d} \times \widetilde{\mathbb{I}}_m$ (here $\widetilde{\mathbb{Z}^d} = \mathbb{Z}^d + (1/2, \dots, 1/2)$ and $\widetilde{\mathbb{I}}_m = \mathbb{I}_{m+1}$) separating neighboring sites of $\mathbf{V} \cup \partial \mathbf{V}$ occupied by spins of opposite signs. This surface is partitioned into connected components: it is clear that each connected component but one is a closed surface (in the Euclidean space $\mathbf{R}^{d+1}$). Denote by $\mathbf{V}_{\mathbf{R}^{d+1}}$ a closed domain in $\mathbf{R}^{d+1}$ given by uniting the 'standard' closed unit cubes in $\mathbf{R}^{d+1}$ centered at the sites $\boldsymbol{x} \in \mathbf{V}$. The intersection of any closed connected component of the complete separating surface with $\mathbf{V}_{\mathbf{R}^{d+1}}$ is denoted by $\omega_1(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp \mathbf{V}})$, $\omega_2(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp \mathbf{V}})$, etc, and called a bulk contour. The intersection with $\mathbf{V}_{\mathbf{R}^{d+1}}$ of the remaining connected component (more precisely, the corresponding surface in $\mathbf{R}^{d+1}$) is denoted by $\boldsymbol{\Omega}(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp \mathbf{V}})$ and called an interface (in $\mathbf{V}$).

A special role is played by the simplest regular boundary conditions $\boldsymbol{\sigma}^s_{\partial \mathbf{V}}$, $s \in \mathbb{I}_{m+1}$ determined by $\boldsymbol{\sigma}_{\boldsymbol{x}} = 1$ when $\boldsymbol{x} = (x, z)$, $x \in \partial V$ and $z > s$ and $\boldsymbol{\sigma}_{(x,z)} = -1$ when $\boldsymbol{x} = (x, z)$, $x \in \partial V$ and $z < s$.

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$$s_{(x,z)} = \begin{cases} \dfrac{\sigma_{(x,z)} + \sigma_{(x,-z)}}{\sqrt{2}} & \text{if } z > 0 \\ \dfrac{\sigma_{(x,z)} + \varphi_{(x)}}{\sqrt{2}} & \text{if } z = 0 \end{cases} \tag{4.2}$$

and

$$t_{(x,z)} = \begin{cases} \dfrac{\sigma_{(x,z)} - \sigma_{(x,-z)}}{\sqrt{2}} & \text{if } z > 0 \\ \dfrac{\sigma_{(x,z)} - \varphi_{(x)}}{\sqrt{2}} & \text{if } z = 0. \end{cases} \tag{4.3}$$

Then (4.1) can be rewritten as

$$\begin{aligned} H_V(\boldsymbol{\sigma}_{\mathbf{V}}, \varphi_V | \boldsymbol{\sigma}_{\partial\mathbf{V}}, \varphi_{\partial V}) = &- \sum_{\langle (x,z),(x'z') \rangle \subset \mathbf{V}^*,\ zz' \neq 0} \left(s_{(x,z)} s_{(x',z')} + t_{(x,z)} t_{(x',z')} \right) \\ &- \frac{1}{2} \sum_{\langle (x,1),(x',0) \rangle \subset \mathbf{V}^*} s_{(x,1)} \left(s_{(x',0)} + t_{(x',0)} \right) \end{aligned} \tag{4.4}$$

and according to Ginibre's inequality [19] for any $\mathbf{A}, \mathbf{B} \in \mathbf{V}^*$

$$\left\langle \prod_{(x,z) \in \mathbf{A}} s_{(x,z)} \prod_{(x,z) \in \mathbf{B}} t_{(x,z)} \right\rangle_\beta^+ \geq 0. \tag{4.5}$$

Taking $\mathbf{A} = \emptyset$ and $\mathbf{B} = (0,0)$ one gets

$$\langle t_{(0,0)} \rangle_\beta^+ \geq 0. \tag{4.6}$$

The last inequality implies

$$\langle \sigma_{(0,0)} \rangle_\beta^+ \geq \langle \varphi_0 \rangle_\beta^+, \tag{4.7}$$

which proves Theorem 1.2. □

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The probability distribution $\mathbf{Pr}_{\beta,\mathbf{V}}(\,\cdot\,|\boldsymbol{\sigma}_{\partial\mathbf{V}})$ induces a probability distribution on the set of interfaces compatible with $\boldsymbol{\sigma}_{\partial\mathbf{V}}$ and we are interested in the asymptotic behavior of the random interface in the thermodynamic limit, $V \nearrow \mathbb{Z}^d$ with m kept fixed. More precisely, we consider the structure of the family of limit Gibbs states $\mathbf{Pr}_\beta(\,\cdot\,|\boldsymbol{\sigma}_{\partial\mathbf{V}})$ and we study where the "typical" interface is located for given $\mathbf{Pr}_\beta(\,\cdot\,|\boldsymbol{\sigma}_{\partial\mathbf{V}})$.

The first rigorous results in this direction were obtaned by Dobrushin [1] and Gallavotti [2]. In [1] it was shown that, for $d \geq 2$ and β large enough, in the limit $\mathbf{V} \nearrow \mathbb{Z}^d \times \mathbb{Z}^1$ (that is, $V \nearrow \mathbb{Z}^d$ and $m \to \infty$, m odd), the random interface compatible with boundary condition $\boldsymbol{\sigma}^0_{\partial\mathbf{V}}$, is asymptotically *rigid*. This means that its probability distribution is obtained by the so-called polymer expansion, about a 'ground state' represented by the flat interface that consists of the horizontal plaquettes centered at points $(x, \widetilde{z})$, with $x \in \mathbb{Z}^d$ and $\widetilde{z} = 0$. [A consequence of that fact is that the random interface possesses an exponential decay of space-correlations.] This result was later partially extended in [3] to all $\beta > \beta_{\rm cr}$, the inverse critical temperature of the d-dimensional Ising model. For $d = 1$ and β large enough [2] showed that a similar limit leads to a non-rigid interface (which is a broken line along the bonds of $\widetilde{\mathbb{Z}}^2$). The latter means that the probability, for a given lattice site x, to be 'above' or 'below' the interface line tends to 1/2.

In our situation (where $V \nearrow \mathbb{Z}^d$ while m is kept fixed), a natural conjecture is that if $\beta > \beta_{\rm cr}$, the interface becomes, for all b.c. on $\partial^\perp\mathbf{V}$, rigid around the middle plane $\widetilde{z} = 0$ for m odd, and around one of the two middle planes $\widetilde{z} = \pm 1/2$ for m even (in the last case, the interface chooses one of the planes with probability 1/2; as a result, the distribution of the interface does not have a decay of space-correlations). In this paper we prove the above assertion under the stronger condition that $\beta \geq$ const $\cdot\ m$, i.e. at very low temperature depending on m. This phenomenon of an 'irregular' behavior of an interface in an Ising ferromagnet between oppositely charged parallel planes was discussed in the physical literature, e.g., [4], and the references therein. To be precise set $\mathbf{Pr}^s_{\mathbf{V}}(\,\cdot\,) = \mathbf{Pr}_{\beta,\mathbf{V}}(\,\cdot\,|\boldsymbol{\sigma}^s_{\partial\mathbf{V}})$, $\mathbf{Pr}^{\pm}_{\mathbf{V}}(\,\cdot\,) = \mathbf{Pr}^{\pm 1/2}_{\mathbf{V}}(\,\cdot\,)$ and define a symmetry transformation

$$\boldsymbol{\sigma} \mapsto \boldsymbol{\sigma}^*, \quad \text{with } \boldsymbol{\sigma}^*_{(x,z)} = -\boldsymbol{\sigma}_{(x,-z)},\ (x,z) \in \mathbb{D}. \tag{1.3}$$

A state invariant with respect to this transformation is called symmetric.

Theorem 1.1. *There exists a value $\beta_0 = const\, m$ such that, for any $\beta > \beta_0$, the following holds.*

(i) *If m is odd, there exists a unique limit Gibbs state $\mathbf{Pr}^0 = \lim_{\mathbf{V}\to\mathbb{D}} \mathbf{Pr}^0_{\mathbf{V}}$. State $\mathbf{Pr}^0$ is ergodic and symmetric.*

(ii) *If m is even, there exist precisely two translation-periodic extremal limit Gibbs states, $\mathbf{Pr}^{\pm} = \lim_{\mathbf{V}\to\mathbb{D}} \mathbf{Pr}^{\pm}_{\mathbf{V}}$. States $\mathbf{Pr}^{\pm}$ are ergodic and taken to each other by the symmetry transformation (1.3).*

Remarks. 1. For m odd and $\beta > \beta_0$, the uniqueness of state $\mathbf{Pr}^0$ is understood in the class of *all* limit Gibbs states not just translation-periodic states such as is usually established at low temperatures, e.g., by the Pirogov–Sinai theory ([5, 6]). The stronger uniqueness property requires, in a general situation, an involved techuique from [7,8]. However, in our case the proof is simplified by using the FKG inequality.

2. Similarly, for m even, $\beta > \beta_0$ and $d = 2$ the states $\mathbf{Pr}^{\pm}$ in Theorem 1.1 are the only extremal limit Gibbs states. This may be proved by combining the method of this paper with those of [2] and [9]. For $d \geq 3$ (and m even) there exist other (non translation-periodic) states.

3. As was noted before, a problem of an interface in the three-dimensional Ising ferromagnet without constraints was studied in [1]. Our Theorem 1.1 does not follow from [1]: the presence of the constraints makes the whole picture more complicated. Another simplified version of the model under consideration, i.e. the SOS model with the constraints, was discussed in [10, 11].

4. Theorem 1.1 deals with the system where the bulk external magnetic field equals zero. Introducing a constant external magnetic field will clearly shift the separating interface in the direction of the field. More precisely, we believe that it can be shown (although we have not done so explicitly) that there exists a sequence $0 = h^{\{m/2\}-1} \leq h^{\{m/2\}} < \ldots < h^{m/2-1/2} < h^{m/2+1/2} = \infty$ such that

(i) if the value of the magnetic field $h \in \left(h^{m/2+1/2-j}, h^{m/2+3/2-j}\right), j = 1, \ldots, \left[\frac{m+1}{2}\right] + 1$, then there exists a unique limit Gibbs state that coincides with $\mathbf{Pr}^{m/2+3/2-j}$;

(ii) for $h = h^{m/2+1/2-j}$, $j = 1, \ldots, \left[\frac{m+1}{2}\right] + 1$ limit Gibbs states $\mathbf{Pr}^{m/2+1/2-j}$ and $\mathbf{Pr}^{m/2+3/2-j}$ coexist.

Here $\{\cdot\}$ and $[\cdot]$ are the fractional and integer parts respectively. The case of the negative h is obtained by symmetry.

This behavior of our model is closely related to the layering transition or Basuev phenomenon in the semi-infinite Ising model. In the last case the top layer $\partial^+\mathbb{D}$ is shifted to infinity, i.e. there exists only one wall $\partial^-\mathbb{D}$ which is fixed at the origin. The phase diagram of the model consists of the infinite number of curves $h = h_k^*(\beta)$, $k = 1, 2, \ldots$ such that on $h_k^*(\beta)$ two limit Gibbs states coexist: one with the separating interface fluctuating around the flat surface at the level k and another with the separating interface fluctuating around the flat surface at the level $k + 1$. It is a natural conjecture that the lines $h_k^*(\beta)$ exist for $\beta > \beta_{\rm cr}$ but our methods give $h_k^*(\beta)$ only for $\beta \geq \beta_k$ with $\beta_k \to \infty$ as k is growing. The proof is similar to the one presented here with minor technical modifications in treating of bulk phase. By another method based on correlation inequalities the existence of the curve $h_1^*(\beta)$ was proven in [12]. For the simplified version of the model, namely for the SOS interface fluctuating above the rigid wall in the presence of the attracting force, a detailed investigation of the low-temperature phase diagram was carried out in the papers [13], [14] and [15] where a similar phase diagram was verified with $\beta_k \leq \beta^0 < \infty$.

As was already noted a natural conjecture is that the value β_0 in Theorem 1.1 does not depend on m. To support this conjecture we are able to prove the following theorem.

Remark. All estimates used in the proof of Lemma 3.2 are rather rough and by making them more accurate one can prove this lemma (and hence Theorem 1.1) for $\beta >$ const $\log m$. To get the estimate for β not depending on m one faces the principal difficulty of the polymer expansion method. Namely, trying to compare two different polymer series we took (calculated "by hand") contributions given by some particular polymers (unit spikes) and verified that these contributions dominate the series constructed from the absolute values of the statistical weights of all other polymers. It is not hard to see that for given β one can always find m so large that the series of absolute values will dominate the contribution of the unit spikes. Hence to improve the result up to the β not depending on m one should compare two different polymer series term by term taking into account the sign of the statistical weights of the polymers which seems to be a very hard task.

Now Lemma 3.2 allows us to prove the existence of state $\mathbf{Pr}^0$ in Theorem 1.1 (i) and of states $\mathbf{Pr}^{\pm}$ in Theorem 1.1 (ii), and their uniqueness in the class of translation-periodic Gibbs states, by using a general theory of dominant ground states (see [16]). Reference [16] contains a general theorem (Theorem B [16]) describing low-temperature phase diagrams for the wide class of models. The key condition of this theorem, namely generalized Peierls condition (2.9) of [16], is verified by Lemma 3.2. Other two conditions of Theorem B [16]: the retouch property and Condition $\mathcal{L}$ are obviously true for our model which makes the application of Theorem B [16] straightforward. In fact a simpler Theorem A [16] is also applicable to our model. What is not covered by the theorems of [16] is the uniqueness of $\mathbf{Pr}^0$ in the class of all Gibbs states. The additional argument is the following.

It follows from [16] that for m odd and $\beta \geq \beta_0$, the translation-periodic limit Gibbs states $\mathbf{Pr}^{\pm(m/2+1/2)}$ corresponding to the boundary conditions $\boldsymbol{\sigma}^{\pm(m/2+1/2)}$ coincide with each other and with $\mathbf{Pr}^0$. On the other hand, the FKG inequality guarantees that *any* limit Gibbs state is between $\mathbf{Pr}^{+(m/2+1/2)}$ and $\mathbf{Pr}^{-(m/2+1/2)}$ (see [17]). This completes the proof of Theorem 1.1 (i). □

4. Proof of Theorem 1.2

Similarly to [3] consider model (1.1) in the volume $\mathbf{V} = V \times \mathbb{I}_m$, where V is a d-dimensional cube of linear size $2N + 1$, with the boundary condition $\boldsymbol{\sigma}^{-1/2}$ on $\partial\mathbf{V}$. Take also an independent d-dimensional Ising model in the volume V with "+" boundary condition on ∂V. We denote by φ_x the spin variable of this model. The Hamiltonian of the joint system is given by

$$H_V(\boldsymbol{\sigma}_\mathbf{V}, \varphi_V | \boldsymbol{\sigma}_{\partial\mathbf{V}}, \varphi_{\partial V}) = -\sum_{\langle \mathbf{x},\mathbf{x}'\rangle \subset \mathbf{V}} \sigma_\mathbf{x}\sigma_{\mathbf{x}'} - \sum_{\substack{\langle \mathbf{x},\mathbf{x}'\rangle:\, \mathbf{x}\in\mathbf{V},\\ \mathbf{x}'\in\partial\mathbf{V}}} \sigma_\mathbf{x}\sigma_{\mathbf{x}'} - \sum_{\langle x,x'\rangle \subset V} \varphi_x\varphi_{x'} - \sum_{\substack{\langle x,x'\rangle:\, x\in V\\ x'\in\partial V}} \varphi_x\varphi_{x'}. \tag{4.1}$$

Set $\mathbf{V}^* = \{\mathbf{x} = (x, z) \in \mathbf{V} :\ z \geq 0\}$ and introduce for $\mathbf{x} \in \mathbf{V}^*$ new variables of the statistical weights of all labeled polymers $(\varrho^s, L(\varrho^s))$ passing through a given point and having $L(\varrho^s) \equiv 0$ and $||\varrho^s||^{\text{ver}} \geq n$ is less than

$$S_n \leq \sum_{i=n}^{\infty}\sum_{t=0}^{\infty}(i+t)c_7^{(i+t)}e^{-2(\beta-c_2-c_6)i}e^{-(\beta-c_6)t} \leq \exp(-2(\beta-c_8)n), \tag{3.31}$$

where c_8 is a sufficiently large absolute constant, $\beta > c_8$ and we used (3.2).

The contribution to the right-hand side of (3.27) coming from non-canceled labeled polymers which are not shorter than $(m+5)/2 - |s|$ (and have at least $4(m+5)/2 - 4|s|$ vertical plaquettes) does not exceed

$$|V| \sum_{h=(m+5)/2-|s|}^{\infty} \exp(-8(\beta-c_8)h). \tag{3.32}$$

Non-canceled labeled polymers which have the height $(m+3)/2 - |s|$ and are not counted in (3.29) have at least $4(m+3)/2 - 4|s| + 1$ vertical plaquettes. Thus the absolute value of their contribution to (3.27) does not exceed

$$|V| \exp(-2(\beta-c_8)(4(m+3)/2 - 4|s| + 1). \tag{3.33}$$

If the height, h, of non-canceled labeled polymer is less than $(m+3)/2-|s|$ then at least one small generalized cylinder ε_l from this polymer has the corresponding label $L(\varepsilon_l) = 1$ and enters the statistical weight of the labeled polymer with the factor $w_1(\varepsilon_l)$ satisfying

$$\begin{aligned}|w_1(\varepsilon_l)| &\leq 3|w(\varepsilon_l)|\,|\text{Int}(\varepsilon_l)|e^{-8\beta\big((m+3)/2-\max(|E(\varepsilon_l)|,|I_j(\varepsilon_l)|)\big)}\\ &\leq |w(\varepsilon_l)|e^{-8\beta\big((m+3)/2-h\big)+\beta/4}.\end{aligned} \tag{3.34}$$

For other small generalized cylinders $\varepsilon_{l'}$ having $L(\varepsilon_{l'}) = 1$ we use even rougher estimates

$$|w_1(\varepsilon_l)| \leq 3|w(\varepsilon_l)|\,|\text{Int}(\varepsilon_l)|e^{-8\beta\big((m+3)/2-\max(|E(\varepsilon_l)|,|I_j(\varepsilon_l)|)\big)} \leq |w(\varepsilon_l)|e^{-\beta/4}. \tag{3.35}$$

Applying (3.31) one easily gets that the absolute value of the contribution to (3.27) given by non-canceled polymers of the just described type does not exceed

$$|V| \sum_{h=1}^{\infty} \exp(-8(\beta-c_8)h) \exp\big(-8\beta((m+3)/2-h)) + \beta/4\big). \tag{3.36}$$

Combining (3.29), (3.32), (3.33) and (3.36) and taking c_4 such that for $\beta > c_4 m$

$$c_8\, 8(m+5)/2 \leq \beta/3$$

one gets (3.17). □

Theorem 1.2. *For $\beta > \beta_{\text{cr}}$ and m even there exist at least two different limit Gibbs states $\mathbf{Pr}^+$ and $\mathbf{Pr}^-$ taken to each other by the symmetry transformation (1.3). The spontaneous magnetization $\langle\boldsymbol{\sigma}_{(0,0)}\rangle_\beta^+$ is not less than the spontaneous magnetization of the d-dimensional Ising model.*

To conclude this section, we comment on the statements made and their proofs. Pictorially speaking, the mechanism behind our Theorem 1.1 is an entropic 'repulsion' from the constraint surfaces $\partial^{\pm}\mathbb{D}$: the interface tends to stay in the middle of $\mathbb{D}$ where it has the most freedom for random fluctuations. This is similar to the case of the SOS model considered in [11]. However, unlike the SOS model, we now have to deal with a 'bulk' phase living between the interface and the constraint surfaces $\partial^{\pm}\mathbb{D}$. The presence of the bulk phase creates a sort of entropic 'attraction' towards $\partial^{\pm}\mathbb{D}$, approximately of the same size as the entropic repulsion. In physical terms, an interface forbids the *contours* of the bulk phase (bulk contours) to exist in its vicinity. The closer the interface is to a constraint surface, the more space is available for the bulk contours: this is the source of an entropic attraction. The balance is quite delicate, but in the end the repulsion prevails. Unfortunately we are able to proof this only for $\beta \geq$ const $\cdot\, m$.

The main tool used in the proof of Theorem 1.1 is a polymer expansion technique combined with the theory of dominant ground states (cf. [16]). The whole argument resembles the one given in [11] but is technically different in view of the presence of the bulk phase. The uniqueness statement of Theorem 1.1(i) and Theorem 1.2 are proven via correlation inequalities similar to the ones used in [17] and [3].

2. The Reduction to Statistics of Generalized Interfaces

From now on we concentrate on the case $d = 2$; as a rule, the modifications needed for the general case $d \geq 2$ are immediate, and we omit them from the paper. Given a configuration $\boldsymbol{\sigma}_{\mathbf{V}}$ and boundary condition $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$ we denote by $|\boldsymbol{\Omega}(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})|$ and $|\boldsymbol{\omega}_i(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})|$ the total number of the plaquettes in $\boldsymbol{\Omega}(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})$ and $\boldsymbol{\omega}_i(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})$, respectively. Then for the regular boundary condition $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$

$$H_{\mathbf{V}}(\boldsymbol{\sigma}_{\mathbf{V}}|\boldsymbol{\sigma}_{\partial\mathbf{V}}) = 2\Big(\big|\boldsymbol{\Omega}(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})\big| + \sum_i \big|\boldsymbol{\omega}_i(\boldsymbol{\sigma}_{\mathbf{V}}, \boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})\big|\Big) - \sharp(\mathbf{V}), \tag{2.1}$$

where the term $\sharp(\mathbf{V}) = \sharp\big\{\langle x, x'\rangle : x' \in \mathbf{V}\big\}$ does not depend on $\boldsymbol{\sigma}_{\mathbf{V}}$, $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$ and may be omitted. Furthermore, any collection $\Big\{\{\boldsymbol{\omega}_i\}, \boldsymbol{\Omega}\Big\}$ of disjoint plaquette surfaces in $\mathbf{V}_{\mathbf{R}^3}$, where

(i) each $\boldsymbol{\omega}_i$ is closed and connected and belongs to $\mathbf{V}_{\mathbf{R}^3}$,

(ii) $\boldsymbol{\Omega}$ is connected and has a connected boundary $\partial\boldsymbol{\Omega}$ that belongs to $\partial^\perp\mathbf{V}_{\mathbf{R}^3}$,

uniquely determines a configuration $\boldsymbol{\sigma}_{\mathbf{V}}$ with a regular boundary condition $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$. Here, $\partial^\perp\mathbf{V}_{\mathbf{R}^3}$ is a 'vertical' cylinder surface in $\mathbf{R}^3$ which is the vertical part of the boundary $\partial\mathbf{V}_{\mathbf{R}^3}$. It is worth noting that $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$ is determined by $\boldsymbol{\Omega}$ only. Any collection $\Big\{\{\boldsymbol{\omega}_i\}, \boldsymbol{\Omega}\Big\}$ with the properties listed in this paragraph is called *compatible.*

The one-to-one correspondence, between the compatible collections $\{\{\omega_i\},\mathbf{\Omega}\}$ in $\mathbf{V}_{\mathbb{R}^3}$ and the configurations $\boldsymbol{\sigma}_{\mathbf{V}}$ with regular boundary conditions $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$, together with formula (2.1), enables us to represent the partition function

$$\Xi(\mathbf{V}|\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}})=\sum_{\boldsymbol{\sigma}_{\mathbf{V}}}\exp\Big[-\beta H_{\mathbf{V}}\left(\boldsymbol{\sigma}_{\mathbf{V}}|\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}\right)-\beta\sharp(\mathbf{V})\Big] \tag{2.2}$$

in the following form

$$\Xi\left(\mathbf{V}|\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}\right)=\sum_{\{\{\omega_i\},\mathbf{\Omega}\}\subset\mathbf{V}_{\mathbb{R}^3}} w(\mathbf{\Omega})\prod_i w(\omega_i). \tag{2.3}$$

Here, $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$ is a regular boundary condition and the sum is extended to the compatible collections $\{\{\omega_i\},\mathbf{\Omega}\}$, with interface $\mathbf{\Omega}$ corresponding to $\boldsymbol{\sigma}_{\partial^\perp\mathbf{V}}$. The statistical weight $w(\cdot)$ is given by

$$w(\mathbf{\Omega})=\exp\left(-2\beta|\mathbf{\Omega}|\right),\quad w(\omega)=\exp\left(-2\beta|\omega|\right). \tag{2.4}$$

Note that it is invariant with respect to space translations of $\mathbf{\Omega}$ and ω.

The strategy of the proof of Theorem 1.1 is as follows. First, we construct a polymer expansion, for the partition function $\Xi^0(\mathbf{V})=\Xi(\mathbf{V}|\boldsymbol{\sigma}^0_{\partial^\perp\mathbf{V}})$, for m odd, and for the partition functions $\Xi^\pm(\mathbf{V})=\Xi(\mathbf{V}|\boldsymbol{\sigma}^{\pm1/2}_{\partial^\perp\mathbf{V}})$, for m even. The existence of such an expansion immediately implies, via a standard argument, the existence of the limit Gibbs states $\mathbf{Pr}^0$ and $\mathbf{Pr}^\pm$ mentioned in Theorem 1.1. The proof of the unqueness requires some additional standard constructions based again on the polymer expansion (see [6, 16]) and the FKG inequality (see [17]).

To derive the polymer expansion we proceed in several steps. To begin with we write the partition functions $\Xi^0(\mathbf{V})$ and $\Xi^\pm(\mathbf{V})$ in the form

$$\Xi^\bullet(\mathbf{V})=\Xi^{\mathrm{b}}(\mathbf{V})\frac{\Xi^\bullet(\mathbf{V})}{\Xi^{\mathrm{b}}(\mathbf{V})}, \tag{2.5}$$

where $\Xi^\bullet(\mathbf{V})$ denotes any of $\Xi^0(\mathbf{V})$ or $\Xi^\pm(\mathbf{V})$ and $\Xi^{\mathrm{b}}(\mathbf{V})$ is a partition function for an ensemble of bulk contours:

$$\Xi^{\mathrm{b}}(\mathbf{V})=\sum_{\{\omega_i\}\subset\mathbf{V}}\prod_i w(\omega_i). \tag{2.6}$$

A well-known fact (see, e.g., [18]) is that, for $\beta\ge\beta_1$ where β_1 does not depend on m, the logarithm of the partition function $\Xi^{\mathrm{b}}(\mathbf{V})$ admits a polymer expansion

$$\log\,\Xi^{\mathrm{b}}(\mathbf{V})=\sum_{\pi\subset\mathbf{V}} w(\pi). \tag{2.7}$$

Here, the sum $\sum_{\pi\subset\mathbf{V}}$ is over the so-called *polymers*, i.e. collections $(\omega_1,\dots,\omega_s)$ of bulk contours (possibly, with repetitions), in $\mathbf{V}$, such that the union $\bigcup_{j=1}^{s}\omega_j$ forms

Labeled polymers have a nice property that for $L(\varepsilon)\equiv 0$ (i.e. $L(\varepsilon_l)=0$ for all $\varepsilon_l\in\varrho^s$) the corresponding statistical weight $w\big((\varrho^s,L(\varrho^s))\big)$ is invariant under the vertical shifts of ϱ^s. Indeed, define $h^+(\varrho^s)=\max_l h^+(\varepsilon_l)$, $h^-(\varrho^s)=\min_l h^-(\varepsilon_l)$ and suppose that $h^+(\varrho^s)+(s-s')\le(m+1)/2$, $h^-(\varrho^s)+(s-s')\ge-(m+1)/2$. Then the vertical shift on the vector $(0,0,s-s')$ maps ϱ^s into $\varrho^{s'}$ and for $L(\varrho^s)=L(\varrho^{s'})\equiv 0$ clearly $w\big((\varrho^s,L(\varrho^s))\big)=w\big((\varrho^{s'},L(\varrho^{s'}))\big)$.

Suppose for the definiteness that $\bullet=0$ (the cases $\bullet=1/2$ and $\bullet=-1/2$ can be considered in the similar way). We need to estimate

$$\frac{Z^{s/\mathrm{b}}(V)}{Z^{\bullet/\mathrm{b}}(V)}=\exp\Bigg(\sum_{(\varrho^s,L(\varrho^s))\subset V} r(\varrho^s)w\big((\varrho^s,L(\varrho^s))\big) - \sum_{(\varrho^\bullet,L(\varrho^\bullet))\subset V} r(\varrho^s)w\big((\varrho^\bullet,L(\varrho^\bullet))\big)\Bigg). \tag{3.27}$$

The correspondence between ϱ^s and $\varrho^\bullet$ given by the vertical shift on the vector $(0,0,\bullet-s)$ allows us to cancel some number of common terms in the difference above.

Among non-canceled $(\varrho^\bullet,L(\varrho^\bullet))$ a special role is played by labeled polymers consisting of a single small generalized cylinder $\varepsilon=\big(\mathbf{\Omega}^\varepsilon,\{\pi_i^\varepsilon\}\big)$ such that $E(\varepsilon)=0$, $L(\varepsilon)=0$, ε has an empty collection $\{\pi_i^\varepsilon\}$ and $\mathbf{\Omega}^\varepsilon$ is a unit spike of height $h(\varepsilon)\ge(m+3)/2-|s|$. The last means that $\mathbf{\Omega}^\varepsilon$ can be constructed as follows. Take a vertical cylinder in $\mathbb{R}^3$ of height $h(\varepsilon)$ with the vertical projection on $\mathbb{Z}^2$ being a unit plaquette. Put the horizontal surfaces of this cylinder at levels $h^-(\varepsilon)$ and $h^+(\varepsilon)$. Then $\mathbf{\Omega}^\varepsilon$ contains all the plaquettes of this cylinder except the horizontal plaquette at level 0 (remember that either $h^+(\varepsilon)=0$ or $h^-(\varepsilon)=0$). For the polymers consisting from a single generalized cylinder the corresponding combinatorial coefficient is equal to 1 (see [18]). Hence the polymers $(\varrho^\bullet,L(\varrho^\bullet))$, just constructed have

$$r(\varrho^\bullet)w\big((\varrho^\bullet,L(\varrho^\bullet))\big)=\mathrm{w}(\varepsilon)=w(\varepsilon)=e^{-8\beta h(\varepsilon)} \tag{3.28}$$

and the total contribution to (3.24) coming from such polymers is equal

$$-|V|\sum_{t=(m+3)/2-|s|}^{(m+3)/2} e^{-8\beta t} \tag{3.29}$$

Now we will show that for $\beta>$ const m the absolute value of the contribution coming from the rest of non canceled polymers in both sums in (3.27) is less than

$$|V|e^{-8\beta((m+3)/2-|s|)-\beta/3} \tag{3.30}$$

and this will imply Lemma 3.2.

Observe that the number of connected sets consisting of n vertical and t horizontal plaquettes and passing through a given point is less than $(n+t)c_7^{(n+t)}$ with $c_7>0$ depending on the dimension only. Hence the sum, S_n, of the absolute values for all ε with $|\bar{\varepsilon}| < N$ and for all V with $|V| < N$. Take any ε with $|\bar{\varepsilon}| = N$. In view of (3.8)

$$|\mathrm{Int}(\varepsilon)| \le (c_3 m)^2. \tag{3.19}$$

Hence there exists an absolute constant c_5 such that for $\beta > c_5 \log m$

$$|\mathrm{Int}(\varepsilon)| e^{-8\beta\left((m+3)/2 - \max(|E(\varepsilon)|, |I_j(\varepsilon)|)\right)} \le 1. \tag{3.20}$$

As $|\mathrm{Int}^*_j(\varepsilon)| < N$ one can use bound (3.17) to estimate the product in (3.12). Together with the elementary inequality

$$e^x \le 1 + 2|x| \quad \text{for} \quad |x| \le 1 \tag{3.21}$$

and (3.2) this reproduces for ε bound (3.16).

Take now the volume V with $|V| = N$. To reproduce (3.17) we use a polymer expansion for $\log Z^{s/\mathrm{b}}(V)$, $s \in \mathbb{I}_{m+1}$. As $|\varepsilon| < N$ for any ε contributing to $Z^{s/\mathrm{b}}(V)$ bounds, (3.16) and (3.2) imply the convergence of the following polymer expansion (see [18]).

$$\log Z^{s/\mathrm{b}}(V) = \sum_{\varrho^s \subset V} r(\varrho^s)\mathrm{w}(\varrho^s), \tag{3.22}$$

where

(i) a polymer ϱ^s is a collection $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ (possibly, with repetitions), of small generalized cylinders in V, such that the union $\bar{\varrho}^s = \cup_{l=1}^k \bar{\varepsilon}_l$ forms a connected set and $E(\varepsilon_l) = s$ for any ε_l from the collection;

(ii) $r(\cdot)$ is a combinatorial coefficient (see [18]) satisfying the estimate

$$|r(\varrho^s)| \le \exp(c_6 ||\varrho^s||), \quad c_6 > 0, \quad ||\varrho^s|| = \sum_{l=1}^k ||\varepsilon_l||; \tag{3.23}$$

(iii) the statistical weight of the polymer is equal to

$$\mathrm{w}(\varrho^s) = \prod_{l=1}^k \mathrm{w}(\varepsilon_l). \tag{3.24}$$

Define a function $L(\varrho^s)$ which assigns to every $\varepsilon_l \in \varrho^s$ a label $L(\varepsilon)$ taking the values 0 and 1. The pair $\left(\varrho^s, L(\varrho^s)\right)$ is called a *labeled polymer*. Substituting the representation $\mathrm{w}(\varepsilon) = \left(w_0(\varepsilon) + w_1(\varepsilon)\right)$ in (3.24) and opening all brackets one can rewrite (3.22) as the sum over labeled polymers

$$\log Z^{s/\mathrm{b}}(V) = \sum_{(\varrho^s, L(\varrho^s)) \subset V} r(\varrho^s) w\left((\varrho^s, L(\varrho^s))\right) \tag{3.25}$$

where

$$w\left((\varrho^s, L(\varrho^s))\right) = \prod_{l=1}^k w_{L(\varepsilon_l)}(\varepsilon_l). \tag{3.26}$$

a connected set of plaquettes. The statistical weight, $\boldsymbol{w}(\pi)$, of a polymer π, is invariant with respect to space translations and satisfies the bound

$$|\boldsymbol{w}(\pi)| \le \exp\left(-2(\beta - c_1)|\pi|\right), \tag{2.8}$$

where $|\pi|$ is the total number of the plaquettes in π: $|\pi| = \sum_{j=1}^{s} |\omega_j|$, and $c_1 > 0$ is a constant.*

We now write representation (2.3), for $\Xi^\bullet(\mathbf{V})$, in the form

$$\Xi^\bullet(\mathbf{V}) = \sum_{\boldsymbol{\Omega} \subset \mathbf{V}}^{\bullet} \boldsymbol{w}(\boldsymbol{\Omega}) \Xi^{\mathrm{b}}\left(\mathbf{V}^{\mathrm{up}}(\boldsymbol{\Omega})\right) \Xi^{\mathrm{b}}\left(\mathbf{V}^{\mathrm{lo}}(\boldsymbol{\Omega})\right). \tag{2.9}$$

Here, the sum $\sum_{\boldsymbol{\Omega} \subset \mathbf{V}}^{\bullet}$ is over all interfaces $\boldsymbol{\Omega}$ compatible with the corresponding boundary condition ($\sigma^0_{\partial\perp\mathbf{V}}$ for m odd and $\sigma^{\pm1/2}_{\partial\perp\mathbf{V}}$ for m even). The sub-volume $\mathbf{V}^{\mathrm{up}}(\boldsymbol{\Omega}) \subset \mathbf{V}$ consists of the sites $\boldsymbol{x}$ lying above $\boldsymbol{\Omega}$, at least distance one apart (which means that the sites adjacent to $\boldsymbol{\Omega}$ are not included). Similarly, $\mathbf{V}^{\mathrm{lo}}(\boldsymbol{\Omega}) \subset \mathbf{V}$ consists of the sites $\boldsymbol{x}$ lying below $\boldsymbol{\Omega}$, again at least distance one apart.

Using representation (2.7) and a similar representation for $\log \Xi^{\mathrm{b}}(\mathbf{V}^{\mathrm{up/lo}})$, we can write the ratio $\dfrac{\Xi^\bullet(\mathbf{V})}{\Xi^{\mathrm{b}}(\mathbf{V})}$ as

$$\begin{aligned}
\frac{\Xi^\bullet(\mathbf{V})}{\Xi^{\mathrm{b}}(\mathbf{V})} &= \sum_{\boldsymbol{\Omega} \subset \mathbf{V}}^{\bullet} \boldsymbol{w}(\boldsymbol{\Omega}) \exp\left[- \sum_{\pi:\ \pi \cap \boldsymbol{\Omega} \neq \emptyset,\ \pi \in \mathbf{V}} \boldsymbol{w}(\pi) \right] \\
&= \sum_{\boldsymbol{\Omega} \subset \mathbf{V}}^{\bullet} \boldsymbol{w}(\boldsymbol{\Omega}) \prod_{\pi:\ \pi \cap \boldsymbol{\Omega} \neq \emptyset,\ \pi \in \mathbf{V}} \left(1 + \boldsymbol{W}(\pi)\right) \\
&= \sum_{\{\boldsymbol{\Omega}, \{\pi_i\}\} \subset \mathbf{V}}^{\bullet} w(\boldsymbol{\Omega}) \prod_i \boldsymbol{W}(\pi_i) \\
&= \sum_{\boldsymbol{\Gamma} \subset \mathbf{V}}^{\bullet} w(\boldsymbol{\Gamma}).
\end{aligned} \tag{2.10}$$

Here, $\boldsymbol{\Gamma}$ is a collection $\{\boldsymbol{\Omega}, \{\pi_i\}\}$ where $\boldsymbol{\Omega}$ is an interface in $\mathbf{V}$ compatible with the corresponding boundary condition and $\{\pi_i\}$ is a set of bulk polymers in $\mathbf{V}$ which intersect $\boldsymbol{\Omega}$. Below we call $\boldsymbol{\Gamma}$ a *generalized interface*. The statistical weights $\boldsymbol{W}(\pi)$ and $w(\Gamma)$ are defined as

$$\boldsymbol{W}(\pi) = e^{-\boldsymbol{w}(\pi)} - 1 \le \exp(-2(\beta - c_2)|\pi|) \tag{2.11}$$

and

$$w(\boldsymbol{\Gamma}) = w(\boldsymbol{\Omega}) \prod_i W(\pi_i). \tag{2.12}$$

*All constants c_j used here do not depend on m.

Returning to (2.5), one can see that the problem is reduced to constructing an expansion for the partition function

$$\Xi^{\bullet/\mathrm{b}}(\mathbf{V}) = \sum_{\mathbf{\Gamma}\subset\mathbf{V}}^{\bullet} w(\mathbf{\Gamma})\,. \tag{2.13}$$

To construct this new polymer expansion one needs a detailed knowledge of the geometry of a generalized interface. Following an idea going back to [1], we treat the generalized interfaces as an ensemble of more elementary geometrical objects called cylinders (=walls in [1]). The cylinders describe deviations from a flat surface.

Observe that all flat interfaces have the same (and maximally possible) statistical weight $\exp(-2\beta|V|)$. However, they differ in the total weight carried by the cylinder ensembles built around them: the flat surfaces at height 0, for m odd, and around $\pm 1/2$, for m even, are *dominant* (see [16]). To verify this fact, we partition the cylinders into small and large ones. Comparing the free energies of the ensembles of the small cylinders, we first check that the above surfaces dominate the others at the level of the small cylinders, and then apply the dominant ground states theory to show that the large cylinders do not destroy this picture.

3. Statistics of Generalized Interface

We begin with studying the geometry of a generalized interface $\mathbf{\Gamma} = \left\{\mathbf{\Omega}, \{\pi_i\}\right\}$. It is convenient to extract from $\mathbf{\Gamma}$ the so-called *free* plaquettes. A plaquette κ from $\mathbf{\Omega}\bigcup\left(\cup_i \pi_i\right)$ is called free (in $\mathbf{\Gamma}$) if

(i) *κ is horizontal (i.e., parallel to $\mathbb{Z}^2$),*

(ii) *there is no other horizontal plaquette, from $\mathbf{\Omega}\bigcup\left(\cup_i \pi_i\right)$, which is projected to κ.*

It is plain that $\kappa \in \mathbf{\Omega}$ and the number of free plaquettes is not greater than $|V|$, the area of the base of $\mathbf{V}$ (recall, $\mathbf{V} = V\times\mathbb{I}_m$). The set of non-free plaquettes (vertical or horizontal) from $\mathbf{\Omega}\bigcup\left(\cup_i \pi_i\right)$ is partitioned into connected components denoted by $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r$. Plaquettes from $\mathbf{\Gamma}$ may enter the set $\widetilde{\gamma}_j$ with multiplicities (the interface $\mathbf{\Omega}$ plus some contour(s) from some polymer(s) π_i may pass through the same plaquette). We denote by $n_{\widetilde{\gamma}_i}$ the positive integer function whose value $n_{\widetilde{\gamma}_i}(\kappa)$ indicates the multiplicity assigned to a plaquette $\kappa\in\widetilde{\gamma}_i$. Observe that if a plaquette of a bulk polymer π belongs to $\widetilde{\gamma}$ then the whole π belongs to $\widetilde{\gamma}$. This allows us to define a *generalized cylinder* as a collection $\gamma = \left(\mathbf{\Omega}^\gamma, \{\pi_j^\gamma\}\right)$, where $\mathbf{\Omega}^\gamma = \mathbf{\Omega}\cap\widetilde{\gamma}$ and π_j^γ are the bulk polymers from $\widetilde{\gamma}$. The statistical weight $w(\gamma)$ of a generalized cylinder γ is denoted

$$w(\gamma) = \prod_j \boldsymbol{W}(\pi_j^\gamma)e^{-2\beta|\mathbf{\Omega}^\gamma|}e^{2\beta|\mathbf{\Omega}^\gamma_{\mathrm{pr}}|}\,, \tag{3.1}$$

where $|\mathbf{\Omega}^\gamma_{\mathrm{pr}}|$ denotes the area of the vertical projection $\mathbf{\Omega}^\gamma_{\mathrm{pr}}$ (on $\mathbb{Z}^2$) of $\mathbf{\Omega}^\gamma$. Note that $\boldsymbol{W}(\pi_j^\gamma)$ may take negative values, and, for $\beta\ge 4c_2$,

$$|w(\gamma)| \le \exp\left(-2(\beta-c_2)||\gamma||^{\mathrm{ver}} - \beta||\gamma||^{\mathrm{hor}}\right) \le \exp(-\beta||\gamma||)\,, \tag{3.2}$$

Then

$$\begin{aligned}\Xi^{s/\mathrm{b}}(V) &= \exp(-2\beta|V|)\sum_{\{[\varepsilon_i],\{\nu_j\}\}^e\subset V}^{s}\prod_i w(\varepsilon_i)\prod_k \Xi^{I_k(\varepsilon_i)/\mathrm{b}}(\mathrm{Int}_k^*\varepsilon_i)\\ &\qquad\times\prod_j w(\nu_j)\prod_l \Xi^{I_l(\nu_j)/\mathrm{b}}(\mathrm{Int}_l^*\nu_j)\\ &= \exp(-2\beta|V|)\sum_{\{[\varepsilon_i],\{\nu_j\}\}^e\subset V}^{s}\prod_i \mathrm{w}(\varepsilon_i)\prod_k \Xi^{E(\varepsilon_i)/\mathrm{b}}(\mathrm{Int}_k^*\varepsilon_i)\\ &\qquad\times\prod_j w(\nu_j)\prod_l \Xi^{I_l(\nu_j)/\mathrm{b}}(\mathrm{Int}_l^*\nu_j)\end{aligned} \tag{3.14}$$

where the sum $\sum^{s}_{\{[\varepsilon_i],\{\nu_j\}\}^e\subset V}$ is taken over all strictly compatible collections of mutually external generalized cylinders (small and large) belonging to $\mathbf{V} = V\times\mathbb{I}_m$. As $\mathrm{Int}_k^*\varepsilon_i$, $\mathrm{Int}_l^*\nu_j\subset V$ one can substitute in the right-hand side of (3.14) the expressions for $\Xi^{E_k(\varepsilon_i)/\mathrm{b}}(\mathrm{Int}_k^*\varepsilon_i)$ and $\Xi^{I_l(\nu_j)/\mathrm{b}}(\mathrm{Int}_l^*\nu_j)$ coming from (3.13). After regrouping terms this gives Lemma (3.1). □

Define the *restricted partition functions* (see [16]) as

$$Z^{s/\mathrm{b}}(V) = \sum_{[\varepsilon_i]\subset V}\prod_i \mathrm{w}(\varepsilon_i)\,, \tag{3.15}$$

where the sum extends over weakly compatible collections of small generalized cylinders $[\varepsilon_i]$ with $E(\varepsilon_i) = s$. Set $w_0(\varepsilon) = w(\varepsilon)$ and $w_1(\varepsilon) = \mathrm{w}(\varepsilon) - w(\varepsilon)$. The key role in the proof of Theorem 1.1 is played by Lemma 3.2 below which verifies that $\boldsymbol{\sigma}^0$ (for odd m) and $\boldsymbol{\sigma}^\pm$ (for even m) are the *dominant ground states.*

Lemma 3.2. *There exist a constant $c_4 > 0$ such that for $\beta > c_4 m$*

$$|w_1(\varepsilon)| \le 3|w(\varepsilon)|\,|\mathrm{Int}(\varepsilon)|e^{-8\beta\left((m+3)/2-\max(|E(\varepsilon)|,|I_j(\varepsilon)|)\right)} \tag{3.16}$$

and for any $s\in\mathbb{I}_{m+1}$, $s\neq\bullet$

$$\exp\left(-\frac{3}{2}|V|e^{-8\beta((m+3)/2-|s|)}\right) \le \frac{Z^{s/\mathrm{b}}(V)}{Z^{\bullet/\mathrm{b}}(V)} \le \exp\left(-\frac{1}{2}|V|e^{-8\beta((m+3)/2-|s|)}\right)\,, \tag{3.17}$$

where for m odd $\bullet = 0$ and for m even either $\bullet = 1/2$ or $\bullet = -1/2$.

Proof. According to definition (3.8) all small cylinders imbedded into the interior of given small cylinder are also small. Hence for any small cylinder ε

$$\Xi^{E(\varepsilon)/\mathrm{b}}(\mathrm{Int}_j^*\varepsilon) = Z^{E(\varepsilon)/\mathrm{b}}(\mathrm{Int}_j^*\varepsilon),\quad \Xi^{I_j(\varepsilon)/\mathrm{b}}(\mathrm{Int}_j^*\varepsilon) = Z^{I_j(\varepsilon)/\mathrm{b}}(\mathrm{Int}_j^*\varepsilon). \tag{3.18}$$

Now we proceed by induction in volume. Clearly (3.16) is valid for ε with $|\bar\varepsilon| = 1$ and (3.17) is valid for V with $|V| = 1$. Suppose that (3.16) and (3.17) were verified

The whole collection is called mixed-compatible if any pair of distinct generalized cylinders γ', γ'' from the collection, not separated by any third generalized cylinder from the same collection, is mixed-compatible. We use the notation $[\,\cdot\,\}$ for the mixed compatible collections of generalized cylinders.

Observe that for the partition function $\Xi^s(\mathbf{V}) = \Xi\left(\mathbf{V}|\boldsymbol{\sigma}^s_{\partial^\perp \mathbf{V}}\right)$ (see (2.3)), with $s = \pm 1, \ldots, \pm(m+1)/2$, for m odd and $s = \pm 3/2, \ldots, \pm(m+1)/2$, for m even, one can obtain a representation analogous to (3.7): $\Xi\left(\mathbf{V}|\boldsymbol{\sigma}^s_{\partial^\perp \mathbf{V}}\right) = \Xi^s(\mathbf{V}) = \Xi^{\mathrm{b}}(\mathbf{V})\Xi^{s/\mathrm{b}}(V)$, where

$$\Xi^{s/\mathrm{b}}(V) = \exp\left(-2\beta|V|\right) \sum_{\{\gamma_i\}\subset V}^{s} \prod_i w(\gamma_i)\,. \tag{3.10}$$

Here, the sum $\sum^{s}_{\{\gamma_i\}\subset V}$ is over the strongly compatible collections of generalized cylinders γ_i, with $\bar{\gamma}_i \subset V$, such that, $E(\gamma_i) \equiv s$ for all external generalized cylinders γ_i. Furthermore, for $\Xi^{s/\mathrm{b}}(\mathbf{V})$, one can write down a formula analogous to (3.9):

$$\Xi^{s/\mathrm{b}}(V) = \exp\left(-2\beta|V|\right) \sum_{\{\{\varepsilon_i\},\{\nu_j\}\}\subset V}^{s} \prod_i w(\varepsilon_i) \prod_j w(\nu_j)\,. \tag{3.11}$$

Here, the same definition of small and large contours is used as before (see (3.8)).

The modified statistical weight, $\mathrm{w}(\varepsilon)$, of a small generalized cylinder ε, is given by

$$\mathrm{w}(\varepsilon) = w(\varepsilon) \prod_j \frac{\Xi^{I_j(\varepsilon)/\mathrm{b}}(\,\mathrm{Int}^*_j(\varepsilon))}{\Xi^{E(\varepsilon)/\mathrm{b}}(\,\mathrm{Int}^*_j(\varepsilon))}\,, \tag{3.12}$$

with $w(\varepsilon)$ defined as in (3.1) and $\mathrm{Int}^*_j(\varepsilon)$ being $\mathrm{Int}_j(\varepsilon)$ without plaquettes adjacent to $\bar{\varepsilon}$.

Lemma 3.1. *For any finite* $\mathbf{V} = V \times \mathbb{I}_m$ *and any* $s \in \mathbb{I}_{m+1}$, *the following formula holds*

$$\Xi^{s/\mathrm{b}}(V) = \exp\left(-2\beta|V|\right) \sum_{[[\varepsilon_i],\{\nu_j\}\}\subset V}^{s} \prod_i w(\varepsilon_i) \prod_j w(\nu_j)\,. \tag{3.13}$$

Here, and below the sum $\sum^{s}_{\{\{\varepsilon_i\},\{\nu_j\}]\subset V}$ *is over the mixed-compatible collections of small and large generalized cylinders* ε_i *and* ν_j, *with the projections* $\bar{\varepsilon_j}, \bar{\nu_j} \subset V$, *such that, for any external generalized cylinder* γ *(small or large) from the collection* $E(\gamma) = s$.

Proof. We proceed by induction in volume V. For V with $|V| = 1$ the lemma is obvious. Suppose that it was verified for all $V' \subset V$ and consider the volume V.

where

$$||\gamma||^{\mathrm{ver}} = \sum_{\substack{\kappa\in\widetilde{\gamma} \\ \kappa \text{ is vertical}}} n_{\tilde{\gamma}}(\kappa), \quad ||\gamma||^{\mathrm{hor}} = \sum_{\substack{\kappa\in\widetilde{\gamma} \\ \kappa \text{ is horizontal}}} n_{\tilde{\gamma}}(\kappa)\,,$$

$$||\gamma|| = \sum_{\kappa\in\widetilde{\gamma}} n_{\tilde{\gamma}}(\kappa) = |\boldsymbol{\Omega}^\gamma| + \sum_j |\pi^\gamma_j|. \tag{3.3}$$

To obtain bound (3.2), we use bounds (2.4) and (2.11). Indeed, denote by $k = k(\kappa)$ the total number of horizontal plaquettes from $\widetilde{\gamma}$ projected onto $\kappa \in \boldsymbol{\Omega}^\gamma_{\mathrm{pr}}$. Then $k \geq 3$ and the contribution of these plaquettes is at most $-2(k-1)(\beta - c_2) - 2\beta$ as at least one of them comes from $\boldsymbol{\Omega}^\gamma$. After adding $(+2\beta)$ we still have a contribution to the exponent, at most $-k\beta$ if $\beta \geq 4c_2$. Summing over κ and adding the contribution of the vertical plaquettes of $\widetilde{\gamma}$ (at most $-2(\beta - c_2)$ per plaquette) leads to (3.2).

Using (3.1), we re-write $\Xi^{\bullet/\mathrm{b}}(\mathbf{V})$ in the form

$$\Xi^{\bullet/\mathrm{b}}(\mathbf{V}) = \exp\left(-2\beta|V|\right) \sum_{\boldsymbol{\Gamma}\subset\mathbf{V}}^{\bullet} \prod_{\gamma\in\boldsymbol{\Gamma}} w(\gamma)\,. \tag{3.4}$$

Note that, by definition, the generalized interface $\boldsymbol{\Gamma}$ containing no generalized cylinders gives the contribution 1 to the sum in (3.4). [It is plain that such $\boldsymbol{\Gamma}$ is unique and consists of a flat interface $\boldsymbol{\Omega}$ and no bulk polymers π_i].

Given a generalized cylinder γ from a generalized interface $\boldsymbol{\Gamma} = \left\{\boldsymbol{\Omega}, \{\pi_i\}\right\}$, the vertical projection $\bar{\gamma}$ is a connected set of plaquettes and bonds of the dual lattice $\widetilde{\mathbb{Z}}^2$. Consider the plaquettes of $\widetilde{\mathbb{Z}}^2$ which do not belong to $\bar{\gamma}$: a pair of such plaquettes is called $\bar{\gamma}$-connected if they have a common bond that does not belong to $\bar{\gamma}$. The whole set of the plaquettes that do not belong to $\bar{\gamma}$ is partitioned into $\bar{\gamma}$-connected components; among these components there exists a unique one that is infinite. This infinite component is called the exterior of γ and denoted $\mathrm{Ext}\gamma$. The remaining connected components (if any) are denoted $\mathrm{Int}_1\gamma$, $\mathrm{Int}_2\gamma$, etc; we call them the interior components of γ. The union $\mathrm{Int}\gamma = \bigcup_j \mathrm{Int}_j\gamma$ is called the interior of γ.

The next observation is that, for each component $\mathrm{Ext}\gamma$, $\mathrm{Int}_1\gamma$, $\mathrm{Int}_2\gamma$, ..., the set of the free plaquettes of interface $\boldsymbol{\Omega}$, adjacent to $\widetilde{\gamma}$ and projected into this component, is placed at the same vertical level (depending on the component). We denote these levels, respectively, by $E(\gamma)$, $I_1(\gamma)$, $I_2(\gamma)$, etc.

For each generalized cylinder γ we define $h^+(\gamma) = \max\left(E(\gamma), I_j(\gamma)\right)$ and $h^-(\gamma) = \min\left(E(\gamma), I_j(\gamma)\right)$. The difference $h(\gamma) = h^+(\gamma) - h^-(\gamma)$ is called a height of the generalized cylinder.

Given a pair γ', γ'' of generalized cylinders from $\boldsymbol{\Gamma}$, we say that γ' and γ'' are not separated by a third generalized cylinder γ from $\boldsymbol{\Gamma}$ if $\bar{\gamma}'$ and $\bar{\gamma}''$ belong to a single connected component among $\mathrm{Ext}\gamma$, $\mathrm{Int}_1\gamma$, $\mathrm{Int}_2\gamma$, etc.

We need, for further use, a concept of weak compatibility of generalized cylinders. Two generalized cylinders, γ, $\gamma' \in \mathbb{D}$, are called *weakly compatible* if their vertical projections $\bar{\gamma}$ and $\bar{\gamma}'$ do not intersect and $E(\gamma)$=$E(\gamma')$. A collection of generalized cylinders is called weakly compatible if each pair from the collection is weakly compatible.

An unconstrained partition function, $\Xi^\infty(V)$, is defined as

$$\Xi^\infty(V) = \exp(-2\beta|V|) \sum_{[\gamma_i]\subset V} \prod_i w(\gamma_i) \tag{3.5}$$

where the sum $\sum_{[\gamma_i]\subset V}$ is over the weakly compatible collections of generalized cylinders $\gamma_i \in V \times \mathbb{I}_\infty$ with fixed $E(\gamma_i) = E$. As in (3.4), the empty collection gives the contribution 1. Pictorially speaking, the partition function $\Xi^\infty(V)$ corresponds to a model in an infinite vertical 'strip' $V \times \mathbb{I}_\infty$. Indeed, consider a collection $\{\gamma_i'\}$ which is obtained from $[\gamma_i]$ by changing $E(\gamma_i)$, $I_k(\gamma_i)$ onto $E(\gamma_i)+t$, $I_k(\gamma_i)+t$ where $t = \sum_{\gamma_{i_j}:\, \mathrm{Int}_l\gamma_{i_j} \ni \bar\gamma_i} I_l(\gamma_{i_j})$. Pictorially this corresponds to the vertical shifting of γ_i on the vector $(0,0,t)$. Because of the absence of constraints any such $\{\gamma_i'\}$ can be uniquely completed to the generalized interface in $V \times \mathbb{I}_\infty$. Moreover, the correspondence between generalized interfaces in $V \times \mathbb{I}_\infty$ and weakly compatible collections of generalized cylinders in V is one-to-one. Bound (3.2) guarantees the convergence of the polymer expansion for $\log \Xi^\infty(V)$ (see [18]) which leads, in a straightforward way, to Dobrushin's result [1] on the rigidity of the interface in the three-dimensional Ising model. Unfortunately, for partition function (3.4), the polymer expansion cannot be written directly in terms of weakly compatible generalized cylinders. The condition $\mathbf{\Gamma} \subset \mathbb{D}$ leads to a more complicated, non-local compatibility rule that is discussed below.

Observe that any two generalized cylinders γ', γ'' from $\mathbf{\Gamma} \subset \mathbb{D}$, not separated by a third generalized cylinder from $\mathbf{\Gamma}$, satisfy the following *strong compatibility* condition:

(i) $\bar\gamma' \cap \bar\gamma'' = \emptyset$,

(ii) $h^+(\gamma'), h^+(\gamma'') \le (m+1)/2$, and $h^-(\gamma'), h^-(\gamma'') \ge -(m+1)/2$,

(iii) either $\bar\gamma' \subset \mathrm{Int}_{j''}\gamma''$ and $E(\gamma') = I_{j''}(\gamma'')$ for some j''
or $\gamma'' \subset \mathrm{Int}_{j'}\gamma'$ and $E(\gamma'') = I_{j'}(\gamma')$ for some j'
or $\mathrm{Int}\gamma' \cap \mathrm{Int}\gamma'' = \emptyset$ and $E(\gamma') = E(\gamma'')$.

Conversely, any collection of generalized cylinders $\{\gamma_i\}$, with $\bar\gamma_i \subset V$, such that, for any pair $\gamma_{i'}$, $\gamma_{i''}$ that is not separated by a third generalized cylinder from $\{\gamma_i\}$, $\gamma_{i'}$ and $\gamma_{i''}$ are strong compatible in the above sense, determines, in a unique way, a generalized interface $\mathbf{\Gamma} \subset \mathbf{V}$. Such a collection is called strongly compatible.

In terms of strongly compatible collections of generalized cylinders, formula (3.4) may be written as

$$\Xi^{\bullet/\mathrm{b}}(\mathbf{V}) = \Xi^{\bullet/\mathrm{b}}(V) = \exp(-2\beta|V|) \sum_{\{\gamma_i\}\subset V}^{\bullet} \prod_i w(\gamma_i); \tag{3.6}$$

the sum $\sum^\bullet_{\{\gamma_i\}\subset V}$ is here over the strongly compatible collections of generalized cylinders γ_i, with $\bar\gamma_i \subset V$, such that, for any external γ_i (i.e. for any γ_i for which $\bar\gamma_i$ does not belong to Int $\gamma_{i'}$, for any other generalized cylinder $\gamma_{i'}$ from the collection), $E(\gamma_i)$ takes the same value, and

$$E(\gamma_i) \equiv \begin{cases} 0, & \text{for } m \text{ odd} \\ +\frac{1}{2} \text{ or } -\frac{1}{2}, & \text{for } m \text{ even (depending on the boundary condition } \sigma^{\pm 1/2}_{\partial^\perp \mathbf{V}}). \end{cases}$$

The non-locality of the strong compatibility condition is expressed by conditions (ii) and (iii) above.

Substituting (3.6) into (2.5), we obtain the following representation for $\Xi^\bullet(\mathbf{V})$:

$$\Xi^\bullet(\mathbf{V}) = \Xi^{\mathrm{b}}(\mathbf{V}) \exp(-2\beta|V|) \sum_{\{\gamma_i\}\subset V}^{\bullet} \prod_i w(\gamma_i). \tag{3.7}$$

Recall that for $\log \Xi^{\mathrm{b}}(\mathbf{V})$ we already have a polymer expansion (3.1). The factor $\exp(-2\beta|V|)$ does not affect the analysis of $\log \Xi^\bullet(\mathbf{V})$. The main problem is now to analyse the last partition function $\sum^\bullet_{\{\gamma_i\}\subset V} \prod_i w(\gamma_i)$.

For this purpose we partition the generalized cylinders into two classes: 'small' and 'large' cylinders, according to the following rule. A generalized cylinder γ is called *small* if with fixed absolute constant $c_3 > 1$

$$\mathrm{diam}\ \bar\gamma \le c_3 m, \tag{3.8}$$

otherwise γ is called *large*. From now on we denote small generalized cylinders by ε, ε', etc, and large ones by ν, ν', etc. In terms of small and large generalized cylinders, the partition function $\Xi^{\bullet/\mathrm{b}}(\mathbf{V})$ is re-written as

$$\Xi^{\bullet/\mathrm{b}}(V) = \exp(-2\beta|V|) \sum_{\{\{\varepsilon_i\},\{\nu_j\}\}\subset V}^{\bullet} \prod_i w(\varepsilon_i) \prod_j w(\nu_j). \tag{3.9}$$

Now we will replace the rule of strong compatibility and the statistical weights of small generalized cylinders by new ones in such a way that partition function $\Xi^{\bullet/\mathrm{b}}(\mathbf{V})$ is preserved. The advantage is that the new compatibility rule between the short generalized cylinders will be local.

Fix a collection of small and large generalized cylinders. A pair of distinct generalized cylinders (small or large), γ' and γ'', from the collection, not separated by any third generalized cylinder from the same collection, is called *mixed-compatible* if

(i) their vertical projections, $\bar\gamma'$ and $\bar\gamma''$, do not intersect: $\bar\gamma' \cap \bar\gamma'' = \emptyset$,

(ii) the heights $h^\pm(\gamma')$, $h^\pm(\gamma'')$ obey $h^+(\gamma'), h^+(\gamma'') \le (m+1)/2$ and $h^-(\gamma'), h^-(\gamma'') \ge -(m+1)/2$,

(iii.1) in the case both γ' and γ'' are large cylinders,
either $\bar\gamma' \subset \mathrm{Int}_{j''}\gamma''$ and $E(\gamma') = I_{j''}(\gamma'')$ for some j'',
or $\bar\gamma'' \subset \mathrm{Int}_{j'}(\gamma')$ and $E(\gamma'') = I_{j'}(\gamma')$ for some j',
or $\mathrm{Int}\gamma' \cap \mathrm{Int}\gamma'' = \emptyset$ and $E(\gamma') = E(\gamma'')$,

(iii.2) in the case γ' is large and γ'' small,
either $\gamma'' \subset \mathrm{Int}_{j'}\gamma'$ and $E(\gamma'') = I_{j'}(\gamma')$ for some j'
or $\mathrm{Int}\gamma' \cap \mathrm{Int}\gamma'' = \emptyset$ and $E(\gamma') = E(\gamma'')$,

(iii.3) in the case both γ' and γ'' are small cylinders, $E(\gamma') = E(\gamma'')$.