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Self-avoiding walks in five or more dimensions: polymer expansion approach

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§1. Introduction and results

A self-avoiding walk of length T on the d-dimensional lattice $\mathbb{Z}^d$ is a collection of points $\omega(s) \in \mathbb{Z}^d$, $0 \leqslant s \leqslant T$, such that $\|\omega(s+1)-\omega(s)\| = 1$ and $\omega(s') \neq \omega(s'')$ for all $0 \leqslant s', s'' \leqslant T$. We denote by $\Lambda_T(d)$ the number of such paths starting from the origin. It is not hard to see that there exists $\lim_{T\to\infty} T^{-1}\log\Lambda_T(d) = \alpha(d)$. Simple estimates show that $2.6 < \alpha(2) < 2.8$ and it was even conjectured that $\alpha(2) = e$. Later the values of $\alpha(d)$, $2 \leqslant d \leqslant 6$, were calculated numerically with high accuracy (see the recent book by Madras and Slade [1]) although the exact values

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are still unknown. In particular, $2.61987 < \alpha(2) < 2.63816$, thus the 'natural' conjecture $\alpha(2) = e$ is likely to be wrong.

Two problems naturally arise in connection with self-avoiding walks. One is the asymptotics of $\Lambda_T(d)$ as $T \to \infty$. A natural conjecture is that as $T \to \infty$

$$\Lambda_T(d) \sim A(d)T^{\gamma(d)}e^{T\alpha(d)}. \tag{1.1}$$

The second is the behaviour, as $T \to \infty$, of the mean square displacement of the self-avoiding walk

$$V_T = \Lambda_T(d)^{-1} \sum_{\substack{\omega:\, \omega(0)=0,\\ |\omega|=T}} \|\omega(T)\|^2. \tag{1.2}$$

Here $|\omega|$ is the length of the walk and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Assuming that $V_T \sim D(d)T^{2\nu(d)}$, the main questions are to find the values of $\gamma(d)$ and $\nu(d)$ or at least to prove their existence.

Self-avoiding walks play an important role in many areas of physics, for example, physics of polymers, quantum field theory, and so on. That is why the main concepts were developed on the physical level; see the books by Flory [2] and by de Gennes [3]. In particular, it is generally believed that $\gamma(2) \sim 11/32$, $\gamma(3) \sim 0.162$ and $\nu(2) = 0.75$, $\nu(3) = 0.59$, $\nu(4) = 1/2$ with logarithmic corrections in the asymptotics of V_T, and $\gamma(d) = 0$, $\nu(d) = 1/2$ for $d > 4$. In other words, the answers strongly depend on dimension, as often happens in statistical mechanics.

It is clear from a general point of view that self-avoiding walks behave more and more like simple random walks as the dimension grows. The first mathematical results of this type appeared in the paper by Bridges and Spencer [4]. In that paper the so-called weakly self-avoiding walks were introduced and studied in dimensions $d \geqslant 5$. In this model every random path ω is permitted and given a statistical weight

$$W(\omega) = (2d)^{-T} \prod_{t \subset [0,T]} \left(1 - \varepsilon\delta_{\omega(l(t)),\omega(r(t))}\right), \tag{1.3}$$

where the product is taken over all intervals $t = [l(t), r(t)]$ on the time axis (that is, on $\mathbb{Z}^+$) belonging to the main time interval $[0, T]$ and, for any $x, y \in \mathbb{Z}^d$,

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \tag{1.4}$$

The small parameter $0 < \varepsilon \leqslant 1$ measures the penalty for intersection. When $\varepsilon = 1$, only strictly self-avoiding walks have non-zero statistical weight. We define the partition function (statistical sum)

$$\Xi(T) = \sum_{\omega:\, |\omega|=T,\, \omega(0)=0} W(\omega) \tag{1.5}$$

and the probability distribution

$$P_T(\omega) = \frac{W(\omega)}{\Xi(T)}. \tag{1.6}$$

The main result proved in [4] says that the mean square displacement grows linearly with T and the normalized displacements $(\omega(T) - \omega(0))/\sqrt{(T)}$ obey the central limit theorem.

Bridges and Spencer introduced a new method which is now known as the *lace expansion* method. The method was extended in a series of papers by Hara and Slade [5], [6], where the following results were obtained for $d \geqslant 5$ and $T \to \infty$:

(i) $\gamma(d) = 0$, that is, $\Lambda_T(d) \sim A(d)e^{\alpha(d)T}$;
(ii) $\nu(d) = 1/2$, that is, $V_T \sim D(d)T$;
(iii) the probability distribution of the normalized displacement $(\omega(T) - \omega(0))/\sqrt{(T)}$ converges to a Gaussian distribution.

The basic tools in the approach of [4], [5], [1] are based on methods of functional analysis. At the same time the problem is purely probabilistic. Therefore it seems worthwhile to develop a direct probabilistic or better a statistical mechanical approach leading to the results above. This is in fact the main goal of the present paper. As we shall show below, the model of self-avoiding walks can be interpreted as a one-dimensional contour model of statistical mechanics. The statistical weight of the contours decays like some power of their lengths, where the power depends on the dimension of the walk. Hence the standard techniques of equilibrium statistical mechanics and in particular *cluster or polymer expansions* (see Appendix) can be applied to this system. A similar approach, combined with a finite memory approximation and renormalization-group ideas, was in fact used in the paper by Golowich and Imbrie [7]. We believe that our method is more direct and simple.

We now describe the results of this paper in more detail. We define the partition function

$$\Xi(x, T) = \sum_{\omega:\, |\omega|=T,\, \omega(0)=0,\, \omega(T)=x} W(\omega), \tag{1.7}$$

where $W(\omega)$ is given in (1.3). (Technically it is more convenient to consider the statistical weight (1.3) with the product taken over all intervals t *strictly* belonging to $[0, T]$. This does not change the results and for $\varepsilon = 1$ it means that the self-avoiding polygons are included in the ensemble of self-avoiding walks.) The effective small parameter in our problem is ε/d. That is why our method can be used either for strictly self-avoiding walks ($\varepsilon = 1$) and d large enough or for $d \geqslant 5$ and ε small. The probability of going from the origin to the point $x \in \mathbb{Z}^d$ during the time T is

$$g(x, T) = \frac{\Xi(x, T)}{\Xi(T)}. \tag{1.8}$$

For any $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{Z}^d$ and any $k = (k^{(1)}, \ldots, k^{(d)}) \in \mathbb{R}^d$ we use the standard notations

$$x^2 = \sum_{i=1}^{d} (x^{(i)})^2, \quad \|x\| = \sqrt{x^2}, \quad k^2 = \sum_{i=1}^{d} (k^{(i)})^2, \quad \|k\| = \sqrt{k^2}, \\ (k \cdot x) = \sum_{i=1}^{d} x^{(i)}k^{(i)} \tag{1.9}$$

and define the Fourier transform of $g(x,T)$ as

$$G(k,T)=\sum_{x\in\mathbb{Z}^d} e^{i(k\cdot x)}g(x,T),\qquad k\in(-\pi,\pi]^d. \tag{1.10}$$

Let $u(d)=\min\left(2,\frac{d}{2}-\frac{5}{4}\right)$.

Theorem 1.1. *When $d\geqslant 5$ and ε/d is small enough, there exists a constant $D>0$ such that the mean square displacement V_T of $\omega(T)$ is given by*

$$V_T=\sum_{x\in\mathbb{Z}^d} x^2 g(x,T)=DT(1+R(T)), \tag{1.11}$$

where

$$|R(T)|\leqslant \text{const}\, T^{1-u(d)}\varepsilon/d. \tag{1.12}$$

Furthermore the scaling limit of the distribution of the end-point is Gaussian in the sense that

$$\lim_{s\to\infty} G\left(\frac{k}{\sqrt{s}},sT\right)=\exp\left(-\frac{DT}{2d}k^2\right). \tag{1.13}$$

Theorem 1.1 establishes the diffusive behaviour of the self-avoiding walk. In the course of its proof we exhibit, via the polymer expansion, exact expressions for $D,\Xi(T)$ and $G(k,T)$ with $k^2\leqslant \log T/T$.

The rest of the paper is organized as follows. In the next section we introduce all the necessary notations and explain the ideas of the proof, which is performed by induction on T. §3 contains our induction hypothesis and verifies its first step. §§4–6 contain the main step of the induction and §7 finishes the proof of Theorem 1.1. The Appendix collects all needed facts from the theory of polymer expansions.

§2. Laces and the idea of the proof

Following the idea of [4], we use laces to construct an appropriate representation for the quantities we are interested in. The expressions written in terms of laces then serve as an input for the polymer expansions.

We start by opening all brackets in (1.3) and representing pictorially every indicator function $\delta_{\omega(l(t)),\omega(r(t))}$ as the corresponding interval t on the time axis. The partition function (1.5) can be clearly treated as the sum over all possible collections of intervals belonging to the main interval $[0,T]$. Such a collection of intervals can be uniquely partitioned into connected components of intervals. Given a connected component of intervals one can uniquely construct a lace corresponding to this component according to the following rule.

(i) Select the longest interval starting at the leftmost point of the component; this is the first interval, $t_1=[l(t_1),r(t_1)]$, of the lace.

(ii) Then consider all intervals of this connected component that contain $r(t_1)$ internally. Find among them those with the rightmost end-point. Choose from these the longest one. This is the second element, $t_2=[l(t_2),r(t_2)]$, of the lace.

(iii) Repeat until the rightmost end of the component is reached.

The above rule produces a *lace*, which can be characterized abstractly as a collection $L=\{t_1,\dots,t_{n(L)}\}$ of $n(L)$ time intervals $t_i=[l(t_i),r(t_i)]$ such that $l(t_i)<l(t_{i+1})\leqslant r(t_i)<r(t_{i+1})$ for $i=1,\dots,n(L)-1$ and $t_i\cap t_{i'}=\varnothing$ for $i'\neq i\pm 1$. We use the notations

$$I(L)=\bigcup_{i=1}^{n(L)} t_i(L);\quad I_1(L)=[l(t_1),l(t_2)];\quad I_{2n(L)-1}(L)=[r(t_{n(L)-1}),r(t_{n(L)})];$$

$I_m(L)=[l(t_{m/2+1}),r(t_{m/2})]$ for even m;

$I_m(L)=[r(t_{m/2-1/2}),l(t_{m/2+3/2})]$ for odd $m\neq 1,2n(L)-1$.

Obviously $I(L)=\bigcup_{m=1}^{2n(L)-1} I_m(L)$ (see Fig. 1).

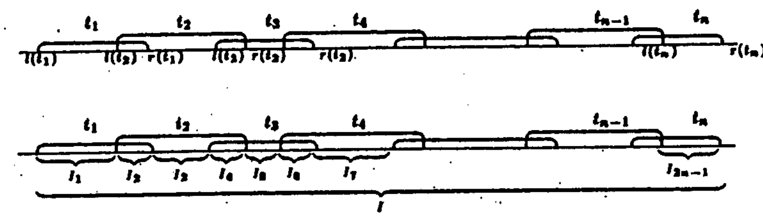


Figure 1. Lace and corresponding time intervals

We say that an interval t is *compatible* with the lace $L=\{t_i\}$ if $t\subset I(L)$ and for some i either $t\subset t_i$ or $t\subset (t_i\cup t_{i+1})\setminus t_{i-1}$. We write $t\prec L$ if t is compatible with L (see Fig. 2 and Fig. 3).

Figure 2. Intervals (⏟) compatible with lace

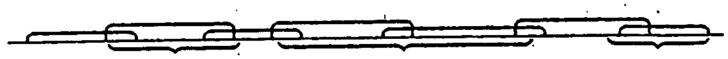


Figure 3. Intervals (⏟) non compatible with lace

The meaning of this compatibility condition is the following. Take a connected component of intervals which contains the lace L and an arbitrary number of intervals compatible with L. Then the lace constructed from such a connected component coincides with L.

With every lace $L=\{t_i\}$ we associate the *lace partition function*

$$\Xi(L)=(2d)^{-T}\sum_{\substack{\omega:\ \omega(l(I(L)))=0,\\ |\omega|=|I(L)|}}\prod_{i=1}^{n(L)}\left(-\varepsilon\delta_{\omega(l(t_i(L))),\omega(r(t_i(L)))}\right)\prod_{t\prec L}\left(1-\varepsilon\delta_{\omega(l(t)),\omega(r(t))}\right) \tag{2.1}$$

and the statistical weight of the lace

$$W(L) = \Xi(L)\left(\prod_{m=1}^{2n(L)-1} \Xi(I_m(L))\right)^{-1}, \tag{2.2}$$

where we put $\Xi(I) = \Xi(|I|)$ for any time interval $I = [l(I), r(I)]$ with length $|I| = r(I) - l(I)$.

The partition function (1.5) can be written in the form

$$\begin{aligned}\Xi(T) = \Xi([0,T]) &= \sum_{\{L_j\}^{\text{ext}}\subseteq[0,T]} \prod_j \Xi(L_j) \\ &= \sum_{\{L_j\}^{\text{ext}}\subseteq[0,T]} \prod_j W(L_j) \prod_{m=1}^{2n(L_j)-1} \Xi(I_m(L_j)),\end{aligned} \tag{2.3}$$

where the sum is taken over all collections of *mutually external* laces from the interval $[0,T]$: mutually external means that $I(L_{j'})\cap I(L_{j''}) = \varnothing$ for any $L_{j'}, L_{j''} \in \{L_j\}$. Whenever we write $\{L_j\} \subseteq I$ we mean that for every j either $n(L_j) \geqslant 2$ and $I(L_j) \subseteq I$ or $n(L_j) = 1$ and $I(L_j) \subset I$. Iterating (2.3), we come to the final expression

$$\Xi(T) = \sum_{\{L_j\}\subseteq[0,T]} \prod_j W(L_j), \tag{2.4}$$

where the sum is now taken over all *compatible collections* of laces: a collection $\{L_j\}$ is called compatible if for any $L_{j'}, L_{j''} \in \{L_j\}$ either $L_{j'}, L_{j''}$ are mutually external or $L_{j'} \subseteq I_m(L_{j''})$ for some m. The right-hand side of (2.4) is a typical cluster or contour representation of the partition function $\Xi(T)$ with laces L playing the role of contours and $\Xi(L)$ being the contour partition function (see [8], [9], and the Appendix).

To derive an appropriate expression for $\Xi(x,T)$ we introduce a *space-time lace* $\mathcal{L}$ as a collection $\mathcal{L} = (\{t_l\}, \{x_m\})$, where $L(\mathcal{L}) = \{t_l\}$ is a lace and $\{x_m\}$ is a family of space displacements satisfying the relations

$$\begin{aligned}&x_1 + x_2 = 0;\\ &x_{2s} + x_{2s+1} + x_{2s+2} = 0, \quad s = 1,2,\dots,n(\mathcal{L}) - 1;\\ &x_{2n(\mathcal{L})-2} + x_{2n(\mathcal{L})-1} = 0.\end{aligned} \tag{2.5}$$

We observe that only the trajectories corresponding to the displacements satisfying (2.5) are compatible with the lace L, that is, they make a non-zero contribution to (2.1).

We put $x(\mathcal{L}) = \sum_{m=1}^{2n(\mathcal{L})-1} x_m(\mathcal{L})$; the notations $n(\mathcal{L})$, $L(\mathcal{L})$, $t_l(\mathcal{L})$, $I_m(\mathcal{L})$, $x_m(\mathcal{L}) = x(I_m(\mathcal{L}))$, $I(\mathcal{L})$ are self-explanatory.

The *space-time lace partition function* is given by

$$\Xi(\mathcal{L}) = (2d)^{-T}(-\varepsilon)^{n(\mathcal{L})} \sum_{\substack{\omega:\ \omega(l(I(\mathcal{L})))=0,\ |\omega|=|I(\mathcal{L})|\\ \omega(r(I_m(\mathcal{L})))-\omega(l(I_m(\mathcal{L})))=x_m(\mathcal{L})}} \prod_{l\prec L(\mathcal{L})} \left(1 - \varepsilon\delta_{\omega(l(l)),\omega(r(l))}\right). \tag{2.6}$$

In contrast with (2.1) we have omitted the first product, because in view of (2.5) all the δ in this product are identically 1. Now $\Xi(x,T)$ can be written as

$$\Xi(x,T) = \sum_{\{\mathcal{L}_j\}^{\text{ext}}\subseteq[0,T]} p\Big(x - \sum_j x(\mathcal{L}_j),\ T - \sum_j |I(\mathcal{L}_j)|\Big) \prod_j \Xi(\mathcal{L}_j), \tag{2.7}$$

where $p(x,T)$ is the probability of the standard random walk going from the origin to the point $x \in \mathbf{Z}^d$ during the time T.

With every lace L there is associated a random variable ξ_L having the distribution

$$\Pr\{\xi_L = x\} = \Xi(L)^{-1} \sum_{\mathcal{L}:\ L(\mathcal{L})=L,\ x(\mathcal{L})=x} \Xi(\mathcal{L}) \tag{2.8}$$

and we denote by V_L the corresponding mean square displacement

$$V_L = \sum_{x\in\mathbf{Z}^d} x^2 \Pr\{\xi_L = x\}. \tag{2.9}$$

We prove Theorem 1.1 by induction on the size of the system. Let $q(s) = \max_{x\in\mathbf{Z}^d} g(x,s)$, $V_t = \sum_{x\in\mathbf{Z}^d} x^2 g(x,t)$ and $u = \min\left(2, \frac{d}{2} - \frac{5}{4}\right)$. In the induction hypothesis (see §3) we suppose that for all $t < T$

$$\sum_{s=1}^{t} s^u q(s) \leqslant h_1 d^{-1} \tag{2.10}$$

and

$$(1 - h_1\varepsilon d^{-1})t \leqslant V_t \leqslant (1 + h_1\varepsilon d^{-1})t. \tag{2.11}$$

Here and below $h_1, h_2, \dots$ denote positive absolute constants. Using (2.10) and the property of 'repulsion', which is characteristic for the self-avoiding walk, it is not hard to verify that

$$\sum_{L:\ l(I(L))=0,\ |I(L)|\leqslant T} |I(L)|\,|W(L)| e^{2n(L)\varepsilon d^{-1}} \leqslant \varepsilon d^{-1}. \tag{2.12}$$

This immediately implies the absolute convergence of the polymer expansion for $\log\Xi(t)$ with $t \leqslant T$. Similarly it is possible to deduce from (2.11) that for laces L with $|I(L)| \leqslant T$

$$V_L \leqslant |I(L)|. \tag{2.13}$$

Using the control over the Gibbs distribution

$$\mu_T(\{L_j\}) = \Xi(T)^{-1} \sum_{\{L_i\}\subseteq[0,T]:\ \{L_i\}\supseteq\{L_j\}} \prod_i W(L_i), \tag{2.14}$$

given by the polymer expansion, one can derive an exact expression for the mean square displacement

$$V_T = \sum_{x\in\mathbf{Z}^d} x^2 g(x,T) = T - \sum_{L\subseteq[0,T]} (|I(L)| - V_L)\,\mu_T(L^{\text{ext}}), \tag{2.15}$$

where $\mu_T(L^{ext}) = \Xi(T)^{-1}\cdot\Xi(L)\cdot\Xi([0,l(I(L))-1])\cdot\Xi([r(I(L))+1,T])$ is the probability (correlation function) that the lace L is external in the ensemble of laces inside the interval $[0,T]$. The fact that $|\mu_T(L^{ext})| \leqslant |W(L)|e^{2n(L)\epsilon d^{-1}}$, and the substitution of (2.12), (2.13) in (2.15), easily lead to (2.11) when $t = T$.

To obtain (2.10) when $t = T$, we study the Fourier transform of $g(x,t)$ and obtain the bound

$$(2\pi)^{-d}\int_{[-\pi,\pi]^d}\sum_{t=1}^{T} t^{u}|G(k,t)|\,dk \leqslant h_1 d^{-1}, \tag{2.16}$$

which clearly implies (2.10).

The way to establish (2.16) again lies in the polymer expansion, but now for the partition function

$$\Xi(k,T) = \sum_{x\in Z^d} e^{i(k\cdot x)}\Xi(x,T), \qquad k\in[-\pi,\pi]^d. \tag{2.17}$$

According to (2.7)

$$\Xi(k,T) = C(k)^T \sum_{\{L_j\}^{ext}\subseteq[0,T]}\prod_j \frac{\Xi(k,L_j)}{C(k)^{|I(L_j)|}}, \tag{2.18}$$

where

$$\Xi(k,L) = \sum_{\mathcal{L}:\ L(\mathcal{L})=L}\Xi(\mathcal{L})e^{i(k\cdot x(\mathcal{L}))} \tag{2.19}$$

and

$$C(k) = \frac{1}{d}\sum_{i=1}^{d}\cos k^{(i)}, \qquad k=(k^{(1)},\dots,k^{(d)}), \tag{2.20}$$

denotes the Fourier transform of a single step of the standard random walk. Obviously

$$G(k,L) = \frac{\Xi(k,L)}{\Xi(L)} \tag{2.21}$$

is a characteristic function of the random variable ξ_L. Therefore, defining

$$\begin{aligned} W(k,L) &= \Xi(k,L)\left(\prod_{m=1}^{2n(L)-1}\Xi(k,I_m(L))\right)^{-1} \\ &= W(L)G(k,L)\left(\prod_{m=1}^{2n(L)-1}G(k,I_m(L))\right)^{-1}, \end{aligned} \tag{2.22}$$

one can rewrite (2.18) as

$$\frac{\Xi(k,T)}{C(k)^T} = \sum_{\{L_j\}^{ext}\subseteq[0,T]}\prod_j W(k,L_j)\prod_{m=1}^{2n(L_j)-1}\frac{\Xi(k,I_m(L_j))}{C(k)^{|I_m(L_j)|}}. \tag{2.23}$$

Iterating (2.23), we get the final expansion

$$\Xi(k,T) = C(k)^T\sum_{\{L_j\}\subseteq[0,T]}\prod_j W(k,L_j), \tag{2.24}$$

where the sum is taken over all compatible collections of laces (see (2.4)).

Given a value of k we now partition all laces L onto *k-small* and *k-large* ones. We say that the lace L is k-small if $\dfrac{\log|I_m(L)|}{h_2|I_m(L)|} \geqslant -\log|C(k)|$ for all $m=1,\dots,2n(L)-1$ and we say that L is k-large if there exists m such that $\dfrac{\log|I_m(L)|}{h_2|I_m(L)|} < -\log|C(k)|$. Here h_2 is some large absolute constant. For our purposes it is enough to put $h_2 = 15$. A *restricted partition function* is defined by

$$Z(k,T) = C(k)^T\sum_{\{L_j\}^s\subseteq[0,T]}\prod_j W(k,L_j), \tag{2.25}$$

where the sum is taken over all compatible collections of k-small laces only. In fact, we include in our induction hypothesis an additional assumption that

$$\Xi(t)C(k)^{(1+h_3\epsilon d^{-1})t} \leqslant Z(k,t) \leqslant \Xi(t)C(k)^{(1-h_3\epsilon d^{-1})t} \tag{2.26}$$

for any $t < T$ and k satisfying $C(k)\geqslant 0$. Substituting (2.26) into (2.22), we observe that the statistical weight $W(k,L)$ for k-small laces remains small enough to have an absolutely convergent polymer expansion for $\log Z(k,T)$. In turn this polymer expansion allows us to obtain (2.26) for $Z(k,t)$.

Finally, the estimate (2.16) can be obtained from (2.10), (2.26) and the representation

$$\Xi(k,T) = \sum_{\{L_j\}^{l,ext}\subseteq[0,T]}\prod_r Z\left(k, I_r(\{L_j\}^{l,ext})\right)\prod_j \Xi(k,L_j), \tag{2.27}$$

where the sum is taken over all collections of mutually external k-large laces and intervals $I_r(\{L_j\}^{l,ext})$ from the complement $[0,T]\setminus(\cup_j I(L_j))$. This completes the induction step.

In the region

$$-\log C(k) \leqslant \frac{\log T}{h_2 T} \tag{2.28}$$

all laces are k-small. Hence $Z(k,T)\equiv\Xi(k,T)$, and we have the polymer expansion for $\log\Xi(k,T)$. Since $-\log C\left(\dfrac{k}{\sqrt{s}}\right) \leqslant \dfrac{\log(sT)}{h_2 sT}$ for s large enough, the polymer expansion for $\log\Xi\left(\dfrac{k}{\sqrt{s}}, sT\right)$ implies Theorem 1.1 with

$$D = 1 - \sum_{L:\ l(I(L))=0}\left(|I(L)| - V_L\right)\mu_\infty(L^{ext}). \tag{2.29}$$

Here the correlation function $\mu_\infty(L^{ext})$ in the infinite time interval $(-\infty,\infty)$ is a well-defined quantity because the polymer expansion for $\log\Xi(T)$ is absolutely convergent for any T.

§3. Induction hypothesis and initial step

We formulate the induction hypothesis as a set of assumptions, which are supposed to be true for all $t \leqslant T-1$:

(i) for $q(s) = \max_{x \in \mathbb{Z}^d} g(x, s)$ and $u = u(d) = \min\left(2, \frac{d}{2} - \frac{5}{4}\right)$ we have

$$\sum_{s=1}^{t} s^u q(s) \leqslant h_1 d^{-1}, \quad h_1 > 0; \tag{3.1}$$

(ii) the mean square displacement of the self-avoiding walk $V_t = \sum_{x \in \mathbb{Z}^d} x^2 g(x, t)$ satisfies the estimate

$$(1-\beta)\, t \leqslant V_t \leqslant (1+\beta)\, t, \tag{3.2}$$

where $\beta = 3h_1 \varepsilon d^{-1}$ is the main small parameter of our calculations;

(iii) given k such that $C(k) \geqslant 0$, the reduced partition function $Z(k, t)$ satisfies the inequalities

$$\Xi(t) C(k)^{(1+h_3\beta)t} \leqslant Z(k,t) \leqslant \Xi(t) C(k)^{(1-h_3\beta)t}, \quad h_3 > 0. \tag{3.3}$$

Note that both h_1 and h_3 are assumed to be independent of d and in the definition of u an arbitrary number from the interval $(1, \frac{3}{2})$ can be taken instead of $\frac{5}{4}$.

We begin our induction with the evident direct calculations

$$\Xi(1) = \Xi(2) = 1, \quad q(1) = q(2) = \frac{1}{2d}, \quad Z(k,1) = C(k), \quad Z(k,2) = C(k)^2, \tag{3.4}$$

which establish (3.1)–(3.3) for $t = 1, 2$. In the next three sections we establish (3.1)–(3.3) for $t = T$.

§4. Induction step: estimation of $|W(L)|$ and $|W(L)|V_L$

Given a lace $L = \{t_i\}$, we partition the family $\{I_m(L)\}$ into subfamilies $\{I'_{m'_j}(L)\}$ and $\{I''_{m''_j}(L)\}$ according to the rule:

(i) the interval $I_{m^*}(L)$ giving $\max_{1 \leqslant m \leqslant 2n(L)-1} |I_m(L)|$ is included in $\{I'_{m'_j}(L)\}$;

(ii) from every pair

$(I_1(L),\ I_2(L)),\ (I_3(L),\ I_4(L)), \ldots,$
$(I_{m^*-1}(L),\ I_{m^*+1}(L)), \ldots, (I_{2n(L)-2}(L),\ I_{2n(L)-1}(L))$ if m^* is even,

or

$(I_1(L),\ I_2(L)),\ (I_3(L),\ I_4(L)), \ldots, (I_{m^*-2}(L),\ I_{m^*-1}(L)),$
$(I_{m^*+1}(L),\ I_{m^*+2}(L)), \ldots, (I_{2n(L)-2}(L),\ I_{2n(L)-1}(L))$ if m^* is odd,

the interval with the maximal length is included in $\{I'_{m'_j}(L)\}$, while the interval with the minimal length is included in $\{I''_{m''_j}(L)\}$.

Clearly the numbers of intervals in $\{I'_{m'_j}(L)\}$ and $\{I''_{m''_j}(L)\}$ are $n(L)$ and $n(L)-1$ respectively. Note also that some $I''_{m''_j}(L)$ may have zero length, while always $|I'_{m'_j}(L)| \geqslant 1$. Let

$$M(L) = \varepsilon^{n(L)} \prod_{j=1}^{n(L)} q\left(\left|I'_{m'_j}(L)\right|\right). \tag{4.1}$$

Lemma 4.1. *The statistical weight of the lace $L = \{t_i\}$ can be estimated by*

$$|W(L)| \leqslant M(L). \tag{4.2}$$

Proof. Take any space-time lace $\mathcal{L} = (\{t_i\}, \{x_m\})$ with $L(\mathcal{L}) = L$. A trajectory ω making a non-zero contribution to $\Xi(\mathcal{L})$ passes through the points $y_i = \omega(l(t_i)) = \omega(r(t_i))$ in the following order:

$$\begin{aligned}
&y_1,\\
&y_{s+1},\ y_s, \qquad s = 1, 2, \ldots, n(L)-1,\\
&y_{n(L)}
\end{aligned}$$

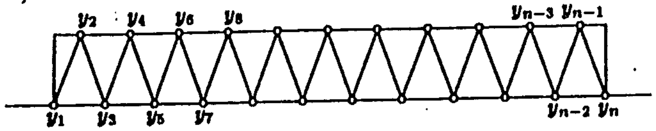
and

$$\begin{aligned}
x_1 &= y_2 - y_1;\\
x_{2n(L)-1} &= y_{n(L)} - y_{n(L)-1};\\
x_m &= y_{m/2} - y_{m/2+1} && \text{for even } m;\\
x_m &= y_{m/2+3/2} - y_{m/2-1/2} && \text{for odd } m \neq 1,\ 2n(L)-1.
\end{aligned}$$

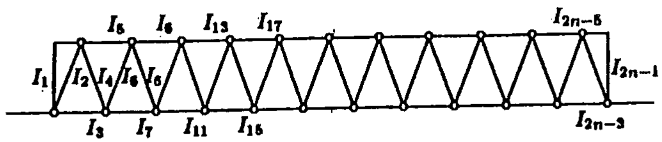
Consider an abstract graph $\Gamma(L)$ with $n(L)$ vertices labelled by y_i, and $2n(L)-1$ edges labelled by $I_m(L)$, such that the edge labelled by $I_m(L)$ joins the vertices labelled by

y_2 and y_1	for $m = 1$;
$y_{n(L)}$ and $y_{n(L)-1}$	for $m = 2n(L)-1$;
$y_{m/2}$ and $y_{m/2+1}$	for even m;
$y_{m/2+3/2}$ and $y_{m/2-1/2}$	for odd $m \neq 1,\ 2n(L)-1$.

The graph $\Gamma(L)$ represents schematically the space structure of the trajectory ω see Fig. 4).



Labeling of vertices of $\Gamma(L)$



Labeling of edges of $\Gamma(L)$

Figure 4. Graph $\Gamma(L)$

According to our partiton rule, if we delete from $\Gamma(L)$ all edges labelled by he intervals from $\{I'_{m'_j}(L)\}$, then the remaining edges (that is, edges labelled by he intervals from $\{I''_{m''_j}(L)\}$) form a connected tree-like subgraph $\Gamma^{(2)}(L)$ of $\Gamma(L)$ ontaining all the vertices of $\Gamma(L)$.

Clearly $0 \leqslant (1 - \varepsilon\delta_{\omega(l(t)),\omega(r(t))}) \leqslant 1$. Hence for any $t \prec L$ the absolute value of he right-hand side of (2.1) only increases if we exclude from the second product ll $t \prec L$ having $l(t)$ and $r(t)$ not in the same $I_m(L)$. On the other hand, the ight-hand side of (2.1) modified in such a way coincides with

$$(-\varepsilon)^{n(L)} \sum_{\mathcal{L}:\ L(\mathcal{L})=L} \prod_{m=1}^{2n(L)-1} \Xi\left(x_m(\mathcal{L}), |I_m(\mathcal{L})|\right). \tag{4.3}$$

Thus, by the definition of $q(t)$,

$$\begin{aligned} |W(L)| &\leqslant \varepsilon^{n(L)} \sum_{\mathcal{L}:\ L(\mathcal{L})=L} \prod_{m=1}^{2n(L)-1} g\left(x_m(\mathcal{L}), |I_m(\mathcal{L})|\right) \\ &\leqslant \varepsilon^{n(L)} \prod_{j=1}^{n(L)} q\left(|I'_{m'_j}(L)|\right) \sum_{\mathcal{L}:\ L(\mathcal{L})=L} \prod_{j=1}^{n(L)-1} g\left(x_{m''_j}(\mathcal{L}), |I''_{m''_j}(\mathcal{L})|\right) \\ &= M(L) \prod_{j=1}^{n(L)-1} \left(\sum_{x_{m''_j}\in \mathbb{Z}^d} g\left(x_{m''_j}, |I''_{m''_j}(L)|\right)\right) = M(L), \end{aligned} \tag{4.4}$$

where in the last two equalities we have used the facts that the displacements $x(I''_{m''_j}(\mathcal{L}))$ are independent (tree-like structure of $\Gamma^{(2)}(L)$) and $g(x,t)$ is a probability distribution.

Lemma 4.2. *For any lace $L = \{t_i\}$ the mean square displacement of its end-point V_L satisfies the bound*

$$|W(L)V_L| \leqslant M(L) \sum_{j=1}^{n(L)-1} V_{I''_{m''_j}}. \tag{4.5}$$

Proof. By definition

$$W(L)V_L = \left(\prod_{m=1}^{2n(L)-1} \Xi(I_m(L))\right)^{-1} \sum_{\mathcal{L}:\ L(\mathcal{L})=L} x(\mathcal{L})^2\, \Xi(\mathcal{L}), \tag{4.6}$$

and cancelling in (2.6) all factors corresponding to $t \prec L$ with $l(t)$ and $r(t)$ not in the same $I_m(L)$, we come to the estimate

$$|W(L)V_L| \leqslant \varepsilon^{n(L)} \sum_{\mathcal{L}:\ L(\mathcal{L})=L} x(\mathcal{L})^2 \prod_{m=1}^{2n(L)-1} g\left(x_m(\mathcal{L}), |I_m(\mathcal{L})|\right). \tag{4.7}$$

According to our partition rule (the structure of $\Gamma^{(2)}(L)$) we can find a family of constants $\{a_1, \ldots, a_{n(L)-1}\}$ taking the values 0 or 1 such that

$$x(\mathcal{L}) = \sum_{j=1}^{n(L)-1} a_j x\left(I''_{m''_j}(\mathcal{L})\right). \tag{4.8}$$

Because of the independence of $x(I''_{m''_j}(\mathcal{L}))$,

$$\begin{aligned} |W(L)V_L| &\leqslant \varepsilon^{n(L)} \prod_{j=1}^{n(L)} q\left(|I'_{m'_j}(L)|\right) \sum_{\mathcal{L}:\ L(\mathcal{L})=L} x(\mathcal{L})^2 \prod_{j=1}^{n(L)-1} g\left(x_{m''_j}(\mathcal{L}), |I''_{m''_j}(\mathcal{L})|\right) \\ &\leqslant M(L) \sum_{j=1}^{n(L)-1} a_j \sum_{x_{m''_j}\in \mathbb{Z}^d} x^2_{m''_j} g\left(x_{m''_j}, |I''_{m''_j}(L)|\right) \\ &\leqslant M(L) \sum_{j=1}^{n(L)-1} V_{I''_{m''_j}(L)}, \end{aligned} \tag{4.9}$$

which proves the lemma.

§5. Induction step: convergence of polymer expansions for $\log \Xi(T)$ and $\log Z(k,T)$

In this section we apply the general theory of polymer expansions (see Appendix) to the particular case of lace models. We consider the models defined on the interval $[0,T]$. So the corresponding laces have $|I_m(L)| \leqslant T-1$, which allows us to exploit the induction assumptions.

emma 5.1. *To every lace $L = \{t_i\}$ we assign the statistical weight*

$$w(L) = 2\,(2n(L)-1)^2 \prod_{j=1}^{n(L)} \left|I'_{m'_j}(L)\right|^{u-1} M(L). \tag{5.1}$$

hen for β small enough

$$\sum_{L' \not\sim L} w(L') e^{2n(L')\beta} \leqslant 2n(L)\beta, \tag{5.2}$$

here the sum is taken over all laces L' incompatible with L (see (A.3)).

roof. According to the compatibility rule for laces

$$\begin{aligned}\sum_{L' \not\sim L} w(L') e^{2n(L')\beta} &\leqslant \sum_{i=1}^{n(L)} \sum_{\substack{L':\ I(L') \ni l(t_i(L)) \\ \text{or } I(L') \ni r(t_i(L))}} w(L') e^{2n(L')\beta} \\ &\leqslant 2n(L) \sum_{L':\ I(L') \ni 0} w(L') e^{2n(L')\beta}.\end{aligned} \tag{5.3}$$

Laces L' with fixed $n(L')$ may have $(2n(L')-1)2^{n(L')-1}$ different partitions of e set $\{I_m(L')\}$ into subsets $\{I'_{m'_j}(L')\}$ and $\{I''_{m''_j}(L')\}$. If $I(L') \ni 0$, then the origin n appear at any of $|I(L')| \leqslant (2n(L')-1)|I_{m^*}(L')|$ points of this interval. Hence, r β small enough

$$\begin{aligned}\sum_{L':\ I(L') \ni 0} w(L') e^{2n(L')\beta} &\leqslant \sum_{n=1}^{T-1} 2(2n-1)^4 2^{n-1} \left(\sum_{I^*=1}^{T-1} I^*(I^*)^{u-1} q(I^*) \varepsilon e^{2\beta} \right) \\ &\quad \times \prod_{j=1}^{n-1} \left(\sum_{I'_j=1}^{T-1} \sum_{I''_j=0}^{I'_j} (I'_j)^{u-1} q(I'_j) \varepsilon e^{2\beta} \right) \\ &\leqslant \sum_{n=1}^{T-1} 2(2n-1)^4 2^{n-1} \left(h_1 d^{-1} \varepsilon e^{2\beta}\right) \left(2h_1 d^{-1} \varepsilon e^{2\beta}\right)^{n-1} \\ &\leqslant 3\varepsilon h_1 d^{-1} = \beta.\end{aligned} \tag{5.4}$$

In what follows we need a number of direct corollaries or modifications of our asic calculation (5.4). They are all listed below and hold true for β small enough.

$$\sum_{L:\ l(I(L))=0} 2(2n(L)-1)|I_{m^*}(L)|\, M(L) \prod_{j=1}^{n(L)} \left|I'_{m'_j}(L)\right|^{\frac{3}{h_2}} e^{2n(L)\beta} \leqslant \beta; \tag{5.5}$$

$$\sum_{L:\ l(I(L))=0} 2(2n(L)-1)^u |I_{m^*}(L)|^u M(L) e^{2n(L)\beta} \leqslant \beta; \tag{5.6}$$

$$\sum_{\substack{L:\ l(I(L))=0, \\ n(l) \geqslant 2}} 2(2n(L)-1)^u |I_{m^*}(L)|^u M(L) e^{2n(L)\beta} \leqslant h_4 \varepsilon d^{-1} \beta; \tag{5.7}$$

$$\begin{aligned}\sum_{\substack{L:\ l(I(L))=0, \\ |I_{m^*}(L)| \geqslant I_0^*}} 2(2n(L)-1) M(L) e^{2n(L)\beta} &\leqslant \sum_{n=1}^{T-1} 2(2n-1)^2 2^{n-1} \left(\sum_{I^*=I_0^*}^{T-1} q(I^*) \varepsilon e^{2\beta} \right) \prod_{j=1}^{n-1} \left(\sum_{I'_j=1}^{T-1} \sum_{I''_j=0}^{I'_j} q(I'_j) \varepsilon e^{2\beta} \right) \\ &\leqslant 3 \sum_{I^*=I_0^*}^{T-1} q(I^*) \varepsilon \\ &\leqslant 3(I_0^*)^{-u} \sum_{I^*=I_0^*}^{T-1} (I^*)^u q(I^*) \varepsilon \\ &\leqslant (I_0^*)^{-u} \beta;\end{aligned} \tag{5.8}$$

$$\sum_{\substack{L:\ l(I(L))=0, \\ |I_{m^*}(L)| \geqslant I_0^*}} 2(2n(L)-1)|I_{m^*}(L)|\, M(L) e^{2n(L)\beta} \leqslant (I_0^*)^{1-u} \beta. \tag{5.9}$$

According to Lemma 4.1, $|W(L)| \leqslant w(L)$ for any lace L with $|I(L)| \leqslant T$. Hence, Lemma 5.1 verifies condition (A.3) of Theorem A.1 (see Appendix) and Theorem A.1 holds true for all $\Xi(t), t \leqslant T$. This means that for $\log \Xi(T)$ one has an absolutely convergent polymer expansion of the form (A.4).

For k such that $C(k) \geqslant 2^{-\frac{1}{2h_2}}$ we define $T(k)$ as the solution of the equation

$$\log C(k) = -\frac{\log T}{h_2 T}. \tag{5.10}$$

For k such that $0 \leqslant C(k) < 2^{-\frac{1}{2h_2}}$ we put $T(k) = 1$.

Lemma 5.2. *For any k-small lace L its statistical weight satisfies the bound*

$$|W(k,L)| \leqslant M(L) \prod_{j=1}^{n(L)} \left|I'_{m'_j}(L)\right|^{\frac{3}{h_2}}. \tag{5.11}$$

Proof. Since the lace L is k-small, we have

$$\Xi(k, I_m(L)) \equiv Z(k, I_m(L)) \tag{5.12}$$

for any $m = 1, 2, \dots, 2n(L)-1$. Therefore it follows from (3.3) that

$$C(k)^{(1+h_3\beta)|I_m(L)|} \leqslant \frac{Z(k, I_m(L))}{\Xi(I_m(L))} = G(k, I_m(L)). \tag{5.13}$$

learly $|G(k,L)| \leqslant 1$ and

$$\begin{aligned} |W(k,L)| &= \left| W(L)G(k,L)\left(\prod_{m=1}^{2n(L)-1} G(k,I_m(L))\right)^{-1}\right| \\ &\leqslant |W(L)| \prod_{m=1}^{2n(L)-1} C(k)^{-(1+h_3\beta)|I_m(L)|} \\ &\leqslant M(L) \prod_{m=1}^{2n(L)-1} |I_m(L)|^{\frac{(1+h_3\beta)|I_m(L)|}{\kappa_2 |I_m(L)|}} \\ &\leqslant M(L) \prod_{j=1}^{n(L)} \left|I'_{m'_j}(L)\right|^{\frac{2}{\kappa_2}} \end{aligned} \tag{5.14}$$

r $2h_3\beta \leqslant 1$.

Lemmas 5.1 and 5.2 verify condition (A.3) of Theorem A.1 for $W(k,L)$ and prove e absolute convergence of the polymer expansion for $\log Z(k,t), t \leqslant T$.

emma 5.3. *For k such that $C(k) \geqslant 0$ the reduced partition function $Z(k,T)$ is unded by*

$$\Xi(T)C(k)^{T(1+h_3\beta)} \leqslant Z(k,T) \leqslant \Xi(T)C(k)^{T(1-h_3\beta)}. \tag{5.15}$$

roof. We need to estimate

$$\begin{aligned} &|\log Z(k,T) - \log \Xi(T) - T \log C(k)| \\ &\leqslant \sum_{\substack{\pi = [L_i^{\alpha_i}] \subseteq [0,T]: \\ \forall i,\ |I_{m^*}(L_i)| \leqslant T(k)}} |W(k,\pi) - W(\pi)| + \sum_{\substack{\pi = [L_i^{\alpha_i}] \subseteq [0,T]: \\ \exists i,\ |I_{m^*}(L_i)| > T(k)}} |W(\pi)|, \end{aligned} \tag{5.16}$$

here

$$W(k,\pi) = r(\pi)\prod_i W(k,L_i)^{\alpha_i} \tag{5.17}$$

d

$$W(\pi) = r(\pi)\prod_i W(L_i)^{\alpha_i}. \tag{5.18}$$

rst we suppose that $C(k) \geqslant 2^{-\frac{1}{2\kappa_2}}$. According to (A.6) the second sum in (5.16) less than

$$\begin{aligned} \sum_{L \subseteq [0,T]:\ |I_{m^*}(L)| > T(k)} \sum_{\pi \ni L} |W(\pi)| &\leqslant \sum_{L \subseteq [0,T]:\ |I_{m^*}(L)| > T(k)} |W(L)| e^{2n(L)\beta} \\ &\leqslant T \sum_{\substack{L:\ l(I_{m^*}(L))=0, \\ |I_{m^*}(L)| > T(k)}} M(L) e^{2n(L)\beta} \\ &\leqslant T\, T(k)^{-u}\beta \\ &\leqslant -T\, 2h_2\beta \log C(k), \end{aligned} \tag{5.19}$$

here we have used (5.8) together with the definition of $T(k)$.

To estimate the first sum in (5.16) we observe that

$$W(k,\pi) = W(\pi)\frac{\varphi(k,\pi)}{\psi(k,\pi)}, \tag{5.20}$$

where

$$\varphi(k,\pi) = \prod_i G(k,L_i)^{\alpha_i} \tag{5.21}$$

and

$$\psi(k,\pi) = \prod_i \prod_{m=1}^{2n(L_i)-1} G(k,I_m(L_i))^{\alpha_i} \tag{5.22}$$

are the characteristic functions of the random variables with the mean square displacements

$$V_{\varphi,\pi} = \sum_i \alpha_i V_{L_i} \tag{5.23}$$

and

$$V_{\psi,\pi} = \sum_i \alpha_i \sum_{m=1}^{2n(L_i)-1} V_{I_m(L_i)} \tag{5.24}$$

respectively. It follows from (3.3) that

$$1 \leqslant \psi(k,\pi)^{-1} \leqslant C(k)^{-(1+h_3\beta)\sum_i \alpha_i |I(L_i)|} \tag{5.25}$$

and clearly

$$\begin{aligned} |W(k,\pi) - W(\pi)| &\leqslant |W(\pi)|(\psi(k,\pi)^{-1} - 1)|\varphi(k,\pi)| + |W(\pi)|(1 - \varphi(k,\pi)) \\ &\leqslant |W(\pi)| \left(\frac{-\log\psi(k,\pi)}{\psi(k,\pi)} + (1 - \varphi(k,\pi)) \right). \end{aligned} \tag{5.26}$$

The first term in the last line of (5.26) can be estimated by means of (5.25):

$$\begin{aligned} \left|\frac{\log\psi(k,\pi)}{\psi(k,\pi)}\right| &\leqslant C(k)^{-(1+h_3\beta)\sum_i \alpha_i |I(L_i)|} \left(-\log C(k)(1+h_3\beta)\sum_i \alpha_i |I(L_i)| \right) \\ &\leqslant -2\log C(k) \left(\sum_i \alpha_i (2n(L_i)-1)|I_{m^*}(L_i)| \right) \prod_i \left(\prod_{m=1}^{2n(L_i)-1} |I_m(L_i)|^{\frac{1}{2\kappa_2}} \right)^{\alpha_i}. \end{aligned} \tag{5.27}$$

To estimate the second term we apply Lemma 4.2, the bound $C(k) \leqslant \exp\left(-\frac{k^2}{2d}\right)$, and the Taylor formula:

$$|W(\pi)|(1-\varphi(k,\pi)) \leqslant |W(\pi)|\frac{1}{2}\left|\sum_{s=1}^{d}\sum_{r=1}^{d}\varphi''_{sr}(vk,\pi)k^{(s)}k^{(r)}\right| \tag{5.28}$$
$$\leqslant |W(\pi)|\frac{1}{2}k^2V_{\varphi,\pi}$$
$$\leqslant -d\,\log C(k)M(\pi)\sum_i\alpha_i(1+\beta)\sum_{j=1}^{n(L_i)-1}|I''_{m''_j}(L_i)|$$
$$\leqslant -2d\,\log C(k)M(\pi)\sum_i\alpha_i(n(L_i)-1)|I_{m^*}(L_i)|.$$

Here $M(\pi)=r(\pi)\prod_i M(L_i)^{\alpha_i}$, $\varphi''_{sr}(vk,\pi)$ denotes the second partial derivative of $\varphi(k,\pi)$ at the intermediate point vk, $0\leqslant v\leqslant 1$, and we have used the fact that $\varphi(k,\pi)$ is the characteristic function of a symmetric random variable (in particular, $\varphi'_s(0,\pi)=0$ for $s=1,\dots,d$). Now

$$-2\log C(k)\sum_{\substack{\pi=[L_i^{a_i}]\subseteq[0,T]:\\ \forall i,\,|I_{m^*}(L_i)|\leqslant T(k)}}|W(\pi)|\left(\sum_i\alpha_i(2n(L_i)-1)|I_{m^*}(L_i)|\right) \tag{5.29}$$
$$\times\prod_i\left(\prod_{m=1}^{2n(L_i)-1}|I_m(L_i)|^{\frac{2}{3h_2}}\right)^{\alpha_i}$$
$$\leqslant -2\log C(k)\sum_{\substack{L\subseteq[0,T]:\\ |I_{m^*}(L)|\leqslant T(k)}}(2n(L)-1)|I_{m^*}(L)|$$
$$\times\sum_{\pi=[L_i^{a_i}]:\ \pi\ni L}\alpha(L,\pi)|W(\pi)|\prod_i\left(\prod_{j=1}^{n(L)}\left|I'_{m'_j}(L_i)\right|^{\frac{2}{h_2}}\right)^{\alpha_i}$$
$$\leqslant -2\log C(k)\,T\sum_{L:\ l(I(L))=0}M(L)(2n(L)-1)|I_{m^*}(L)|\prod_{j=1}^{n(L)}\left|I'_{m'_j}(L)\right|^{\frac{2}{h_2}}e^{2n(L)\beta}$$
$$\leqslant -\log C(k)\,T\beta,$$

where we have used (5.5). We observe that for any polymer π the function $\varphi(k,\pi)$ is the product of the quantities $G(k,L)$ only over those $L\in\pi$ with $n(L)\geqslant 2$. Hence

$$-2d\log C(k)\sum_{\substack{\pi=[L_i^{a_i}]\subseteq[0,T]:\\ \forall i,\,|I_{m^*}(L_i)|\leqslant T(k)}}M(\pi)\sum_{L_i:\ n(L_i)\geqslant 2}\alpha_i(n(L_i)-1)|I_{m^*}(L_i)| \tag{5.30}$$
$$\leqslant -2d\,\log C(k)\sum_{\substack{L\subseteq[0,T]:\ n(L)\geqslant 2,\\ |I_{m^*}(L)|\leqslant T(k)}}(n(L)-1)|I_{m^*}(L)|\sum_{\pi=[L_i^{a_i}]:\ \pi\ni L}\alpha(L,\pi)M(\pi)$$
$$\leqslant -2d\,\log C(k)\,T\sum_{\substack{L:\ l(I(L))=0,\\ n(L)\geqslant 2}}(n(L)-1)|I_{m^*}(L)|\,M(L)e^{2n(L)\beta}$$
$$\leqslant -d\,\log C(k)\,T\,h_4\varepsilon d^{-1}\beta$$
$$= -\log C(k)\,T\,h_4\varepsilon\beta,$$

where we have used (5.7).

If $0\leqslant C(k)<2^{-\frac{1}{3h_2}}$, then $Z(k,T)=C(k)^T$ and (5.15) is reduced to the estimate

$$|\log\Xi(T)|=\left|\sum_{\pi\subseteq[0,T]}W(\pi)\right|\leqslant\sum_{L\subseteq[0,T]}|W(L)|e^{2n(L)\beta}\leqslant -T\log C(k)\frac{2h_2}{\log 2}\beta, \tag{5.31}$$

which follows from (A.6) and (5.5). Clearly (5.19), (5.29)–(5.31) imply (5.15) with $h_3=2h_2+1+h_4\varepsilon+3h_2$.

Lemma 5.4. *The mean square displacement V_T of a weakly self-avoiding walk of length T satisfies the estimate $|V_T-T|\leqslant\beta T$.*

Proof. For a fixed configuration of mutually external laces the corresponding mean square displacement is the sum of mean square displacements of laces and mean square displacements of simple random walks on time intervals between laces: $\sum_j V_{L_j}+\left(T-\sum_j|I(L_j)|\right)$. Hence

$$V_T=\sum_{\{L_j\}^{\text{ext}}\subseteq[0,T]}\mu_T\left(\{L_j\}^{\text{ext}}\right)\left(T-\sum_j\left(|I(L_j)|-V_{L_j}\right)\right) \tag{5.32}$$
$$=T-\sum_{L\subseteq[0,T]}\mu_T(L^{\text{ext}})\left(|I(L)|-V_L\right)$$
$$=T-\sum_{L\subseteq[0,T]}\mu_T(L)\left(\sum_{m=1}^{2n(L)-1}V_{I_m(L)}-V_L\right),$$

where the last equality follows from Lemma A.5. In view of Lemma A.4 and the bound (A.6)

$$\left|\sum_{L\subseteq[0,T]}\mu_T(L)\left(\sum_{m=1}^{2n(L)-1}V_{I_m(L)}-V_L\right)\right|$$
$$\leqslant\sum_{L\subseteq[0,T]}|W(L)|e^{2n(L)\beta}\left(\sum_{m=1}^{2n(L)-1}V_{I_m(L)}+V_L\right). \tag{5.33}$$

Using Lemmas 4.1 and 4.2, the right-hand side of (5.33) can be estimated from above by

$$\begin{aligned}
&\sum_{L\subseteq[0,T]} M(L)e^{2n(L)\beta}\left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} + \sum_{j=1}^{n(L)-1} V_{I''_{m_j''}}\right) \\
&\leqslant T \sum_{L:\, l(I(L))=0} M(L)e^{2n(L)\beta}\left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} + \sum_{j=1}^{n(L)-1} V_{I''_{m_j''}}\right) \\
&\leqslant T \sum_{L:\, l(I(L))=0} M(L)e^{2n(L)\beta}(1+\beta)(3n(L)-2)|I_{m^*}(L)| \\
&\leqslant \beta T,
\end{aligned} \tag{5.34}$$

where we used the induction hypothesis (3.2) and the relation (5.5).

Lemmas 5.3 and 5.4 lead to the induction assumptions (3.2) and (3.3) when $t = T$.

§6. Induction step: estimate of $\sum t^u q(t)$

In this section we show that the induction assumption (3.1) is true when $t = T$.

Lemma 6.1. *For* $u = \min\left(2, \frac{d}{2} - \frac{5}{4}\right)$ *we have*

$$\sum_{t=1}^{T} t^u q(t) \leqslant h_1 d^{-1}. \tag{6.1}$$

Proof. Our calculation is based on the representation

$$\Xi(k,T) = \sum_{\{L_j\}^{l,\mathrm{ext}}\subseteq[0,T]} \prod_r Z\big(k, I_r(\{L_j\}^{l,\mathrm{ext}})\big) \prod_j \Xi(k, L_j), \tag{6.2}$$

where the sum is taken over all collections of mutually external k-large laces. The intervals $I_r(\{L_j\}^{l,\mathrm{ext}})$ form the complement $[0,T]\setminus(\cup_j I(L_j))$. The cluster expansions for $\Xi(t), t \leqslant T$ (statements (A.4) and (A.6) of Theorem A.1) and (5.5) imply that

$$\Xi(T)^{-1}\prod_r \Xi\big(I_r(\{L_j\}^{l,\mathrm{ext}})\big)\prod_j \prod_{m=1}^{2n(L_j)-1} \Xi(I_m(L_j)) \leqslant \exp\Big(2\beta\sum_j n(L_j)\Big). \tag{6.3}$$

It follows from (6.3) and the induction assumption (3.3) that

$$\begin{aligned}
|G(k,T)| &= \left|\frac{\Xi(k,T)}{\Xi(T)}\right| \\
&\leqslant \sum_{\{L_j\}^{l,\mathrm{ext}}\subseteq[0,T]} \prod_r \frac{Z\big(k, I_r(\{L_j\}^{l,\mathrm{ext}})\big)}{\Xi\big(I_r(\{L_j\}^{l,\mathrm{ext}})\big)} \prod_j |G(k,L_j)W(L_j)|e^{2\beta n(L_j)} \\
&\leqslant \sum_{\{L_j\}^{l,\mathrm{ext}}\subseteq[0,T]} C(k)^{(1-h_3\beta)(T-\sum_j |I(L_j)|)} \prod_j M(L_j)e^{2\beta n(L_j)}.
\end{aligned} \tag{6.4}$$

In view of (6.4)

$$\begin{aligned}
\sum_{t=1}^{T} t^u q(t) &\leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d} \sum_{t=1}^{T} t^u |G(k,t)|\, dk \\
&\leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d} \sum_{t=1}^{T} t^u \sum_{r=0}^{t} \sum_{\substack{s_1+t_1+\cdots+s_r+t_r+s_{r+1}=t;\\ s_i\geqslant 0,\ t_i > T(k)}} \prod_{i=1}^{r+1} |C(k)|^{(1-h_3\beta)s_i} \\
&\quad\times \prod_{i=1}^{r}\left(\sum_{L:\, l(I(L)=0, |I(L)|=t_i} M(L)e^{2n(L)\beta}\right) dk.
\end{aligned} \tag{6.5}$$

As we know $q(1)$ and $q(2)$ exactly, it is sufficient to estimate $\sum_{t=3}^{T} t^u q(t)$ only. The contribution to the right-hand side of (6.5) coming from the terms corresponding to $r = 0$ does not exceed

$$\begin{aligned}
&(2\pi)^{-d}\int_{[-\pi,\pi]^d} \sum_{t=3}^{T} t^u |C(k)|^{(1-h_3\beta)t}\, dk \\
&\quad\leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d} \frac{9|C(k)|^{(1-h_3\beta)3}}{(1-|C(k)|^{(1-h_3\beta)})^{u+1}}\, dk \leqslant h_5 d^{-\frac{5}{4}},
\end{aligned} \tag{6.6}$$

where β is assumed to be small enough and h_5 is an absolute constant. Here the exponent $-\frac{5}{4}$ comes from the rough estimate $(2\pi)^{-d}\int_{[-\pi,\pi]^d} |C(k)|^{(1-h_3\beta)3} dk \leqslant \mathrm{const}\, d^{-\frac{3}{2}}$.

The contribution of the case $r = 1$ and $\sum_{i=1}^{r+1} s_i = 0$ to the right-hand side of (6.5) is estimated by

$$\begin{aligned}
&(2\pi)^{-d}\int_{[-\pi,\pi]^d} \sum_{\substack{L:\, l(I(L)=0, n(L)\geqslant 2,\\ |I_{m^*}(L)|\geqslant T(k)}} (2n(L)-1)^u |I_{m^*}(L)|^u M(L)e^{2n(L)\beta}\, dk \\
&\quad\leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d} h_4\varepsilon d^{-1}\beta\, dk = 3h_4 6^2 d^{-1} h_1 d^{-1},
\end{aligned} \tag{6.7}$$

where we have used (5.7) and the fact that the lace L with $|I(L)| = t$ and $n(L) = 1$ does not contribute to $C(k,t)$.

In the last three cases we exploit the simple inequalities

$$\frac{T(k)^{-u}}{1-|C(k)|^{(1-h_3\beta)}} \leqslant 2h_2 \tag{6.8}$$

and

$$t^u \leqslant (2r+1)^u \max\Big(\max_{1\leqslant i\leqslant r+1}(s_i^u), \max_{1\leqslant i\leqslant r}(t_i^u)\Big). \tag{6.9}$$

The contribution to the right-hand side of (6.5) coming from the terms with $r \geqslant 2$ and $\sum_{i=1}^{r+1} s_i = 0$ is bounded by

$$\begin{aligned}(2\pi)^{-d}\int_{[-\pi,\pi]^d}\sum_{r=2}^{T} r^u r\Bigg(\sum_{\substack{L:\,l(I(L))=0,\\ |I_{m^*}(L)|\geqslant T(k)}}(2n(L)-1)^u|I_{m^*}(L)|^u M(L)e^{2n(L)\beta}\Bigg)\\ \times\Bigg(\sum_{\substack{L:\,l(I(L))=0,\\ |I_{m^*}(L)|\geqslant T(k)}} M(L)e^{2n(L)\beta}\Bigg)^{r-1} dk\\ \leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d}\sum_{r=2}^{T} r^u r\beta^r\,dk\\ \leqslant 16\beta^2 = 144h_1\varepsilon^2 d^{-1}h_1 d^{-1},\end{aligned} \tag{6.10}$$

where the additional factor r takes into account the number of possible values of i at which $\max_{1\leqslant i\leqslant r} t_i$ is attained, β is assumed to be small enough, and (5.6) has been used in the first inequality.

If $r \geqslant 1$, $\sum_{i=1}^{r+1} s_i \geqslant 1$, and $\max_{1\leqslant i\leqslant r+1}(s_i^u) \geqslant \max_{1\leqslant i\leqslant r}(t_i^u)$, then the right-hand side of (6.5) does not exceed

$$\begin{aligned}(2\pi)^{-d}\int_{[-\pi,\pi]^d}\sum_{r=1}^{T}(2r+1)^u(r+1)\Bigg(\sum_{s=1}^{\infty}s^u|C(k)|^{(1-h_3\beta)s}\Bigg)\\ \times\Bigg(\sum_{s=0}^{\infty}|C(k)|^{(1-h_3\beta)s}\Bigg)^r\\ \times\Bigg(\sum_{\substack{L:\,l(I(L))=0,\\ |I_{m^*}(L)|\geqslant T(k)}} M(L)e^{2n(L)\beta}\Bigg)^r dk\\ \leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d}\frac{2|C(k)|^{(1-h_3\beta)}}{(1-|C(k)|^{(1-h_3\beta)})^{u+1}}\\ \times\sum_{r=1}^{T}(2r+1)^u(r+1)\left(\frac{\beta T(k)^{-u}}{1-|C(k)|^{(1-h_3\beta)}}\right)^r dk\\ \leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d}\frac{|C(k)|^{(1-h_3\beta)}}{(1-|C(k)|^{(1-h_3\beta)})^{u+1}}72(2h_2\beta)\,dk\\ \leqslant h_6 d^{-\frac{1}{4}}72(2h_2\beta) \leqslant h_7\varepsilon d^{-\frac{1}{4}}h_1 d^{-1},\end{aligned} \tag{6.11}$$

where $(r+1)$ is the number of possible values of i at which $\max_{1\leqslant i\leqslant r+1} s_i$ is attained, β is assumed to be small enough, h_6, h_7 are absolute constants, (5.8) has been used in the first inequality, and (6.8) has been used in the second inequality.

If $r \geqslant 1$, $\sum_{i=1}^{r+1} s_i \geqslant 1$, and $\max_{1\leqslant i\leqslant r+1}(s_i^u) < \max_{1\leqslant i\leqslant r}(t_i^u)$, then the right-hand side of (6.5) does not exceed

$$\begin{aligned}(2\pi)^{-d}\int_{[-\pi,\pi]^d}\sum_{r=1}^{T}(2r+1)^u r(r+1)\Bigg(\sum_{s=1}^{\infty}|C(k)|^{(1-h_3\beta)s}\Bigg)\\ \times\Bigg(\sum_{s=0}^{\infty}|C(k)|^{(1-h_3\beta)s}\Bigg)^r\\ \times\Bigg(\sum_{\substack{L:\,l(I(L))=0,\\ |I_{m^*}(L)|\geqslant T(k)}}(2n(L)-1)^u|I_{m^*}(L)|^u M(L)e^{2n(L)\beta}\Bigg)\\ \times\Bigg(\sum_{\substack{L:\,l(I(L))=0,\\ |I_{m^*}(L)|\geqslant T(k)}} M(L)e^{2n(L)\beta}\Bigg)^{r-1} dk\\ \leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d}\frac{|C(k)|^{(1-h_3\beta)}\beta}{(1-|C(k)|^{(1-h_3\beta)})^2}\\ \times\sum_{r=1}^{T}(2r+1)^u r(r+1)\left(\frac{\beta T(k)^{-u}}{1-|C(k)|^{(1-h_3\beta)}}\right)^{r-1} dk\\ \leqslant (2\pi)^{-d}\int_{[-\pi,\pi]^d}\frac{|C(k)|^{(1-h_3\beta)}\beta}{(1-|C(k)|^{(1-h_3\beta)})^2}\sum_{r=1}^{T}(2r+1)^u r(r+1)(2h_2\beta)^{r-1}\,dk\\ \leqslant h_8 d^{-\frac{1}{4}}\beta 36 \leqslant h_9\varepsilon d^{-\frac{1}{4}}h_1 d^{-1},\end{aligned} \tag{6.12}$$

where $(r+1)$ takes into account the number of ways of choosing i with $s_i \geqslant 1$, the factor r takes into account the number of ways of choosing a value of i giving $\max_{1\leqslant i\leqslant r} t_i$, β is assumed to be small enough, h_8, h_9 are absolute constants, (5.6) and (5.8) have been used in the first inequality, and (6.8) has been used in the second inequality.

Clearly $1^2q(1) + 2^2q(2) = \frac{5}{2}d^{-1}$. Hence for any $h_1 > \frac{5}{2} + h_5$ the inequality (6.1) follows from the estimates (6.6), (6.7) and (6.10)–(6.12) for ε/d small enough.

The induction step is completed.

§7. Proof of Theorem 1.1

Theorem 1.1 easily follows from the variety of facts established in the previous sections. The calculations below are quite standard for the polymer expansion technique. Let

$$D = 1 - \sum_{L:\, l(I(L))=0} \mu_\infty(L^{\rm ext})(|I(L)| - V_L)$$
$$= 1 - \sum_{L:\, l(I(L))=0} \mu_\infty(L)\left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right). \qquad (7.1)$$

It follows from (5.34) that $|D-1| \leqslant \beta$. More precisely,

$$|TD - V_T| = \Bigg| \sum_{L\subseteq[0,T]} \left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right)(\mu_T(L) - \mu_\infty(L)) \qquad (7.2)$$
$$- \sum_{\substack{L:\,|I(L)|\leqslant T,\\ I(L)\cap[0,T]\neq\varnothing,\\ I(L)\cap[T+1,\infty)\neq\varnothing}} \left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right)\mu_\infty(L)$$
$$- T \sum_{\substack{L:\,l(I(L))=0,\\ |I(L)|>T}} \left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right)\mu_\infty(L)\Bigg|.$$

According to Lemma A.5 and the bound (A.6),

$$|\mu_\infty(L)| \leqslant |W(L)| e^{2n(L)\beta}. \qquad (7.3)$$

In view of Lemmas 4.1 and 4.2, and similarly to (5.34), the absolute value of the second sum in (7.2) does not exceed

$$\sum_{L:\, l(I(L))=0,\ |I(L)|\leqslant T} |I(L)||W(L)|e^{2n(L)\beta}\left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} + V_L\right) \qquad (7.4)$$
$$\leqslant \sum_{L:\, l(I(L))=0,\ |I(L)|\leqslant T} \frac{3}{2}(1+\beta)|I(L)|^2 M(L) e^{2n(L)\beta}$$
$$\leqslant T^{2-u} \sum_{L:\, l(I(L))=0} 2(2n(L)-1)^u |I_{m^*}(L)|^u M(L) e^{2n(L)\beta}$$
$$\leqslant T^{2-u}\beta,$$

where we have used (5.6).

Similarly, the absolute value of the third sum in (7.2) does not exceed

$$T \sum_{L:\, l(I(L))=0,\ |I(L)|>T} |W(L)|\, e^{2n(L)\beta}\left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} + V_L\right) \qquad (7$$
$$\leqslant T \sum_{L:\, l(I(L))=0,\ |I(L)|>T} 2(2n(L)-1)|I_{m^*}(L)|M(L)e^{2n(L)\beta}$$
$$\leqslant T\, T^{1-u}\beta,$$

where we have used (5.9).

Finally, to estimate the absolute value of the first sum in (7.2) we use Lemma A

$$\sum_{L\subseteq[0,T]} \left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right)(\mu_T(L) - \mu_\infty(L)) \qquad (7$$
$$= - \sum_{L\subseteq[0,T]} \left(\sum_{m=1}^{2n(L)-1} V_{I_m(L)} - V_L\right) \sum_{\pi:\, \pi\ni L,\ I(\pi)\cap[0,T]^c\neq\varnothing} \alpha(L,\pi)W(\pi)$$
$$= - \sum_{\substack{\pi=\{L_i^{\alpha_i}\}:\, I(\pi)\cap[0,T]\neq\varnothing,\\ I(\pi)\cap[0,T]^c\neq\varnothing}} W(\pi)\left(\sum_{L_i\subseteq[0,T]} \alpha_i\left(\sum_{m=1}^{2n(L_i)-1} V_{I_m(L_i)} - V_{L_i}\right)\right),$$

where $[0,T]^c = (-\infty,\infty)\setminus[0,T]$ and $I(\pi) = \bigcup_{L_i\in\pi} I(L_i)$.

Given the polymer $\pi = \{L_i^{\alpha_i}\}$ entering the right-hand side of (7.6), we denote L_{i^*} the lace giving $\max_{L_i\in\pi, L_i\subseteq[0,T]} |I(L_i)|$ and observe that

$$\min\Big(l(I(L_{i^*})),\ T - r(I(L_{i^*}))\Big) \leqslant |I(L_{i^*})| \sum_{L_i\in\pi,\ L_i\subseteq[0,T]} \alpha_i. \qquad (7$$

Let $w_0(L_i) = 2M(L_i)$ and $w_0(\pi) = r(\pi)\prod_i w_0(L_i)^{\alpha_i}$. Lemma 5.1 establishes $w_0(L)$ the truth of condition (A.3) of Theorem A.1. Thus one can estimate fr above the absolute value of the right-hand side of (7.6) by

$$\sum_{\substack{\pi=[L_i^{\alpha_i}]:\ I(\pi)\cap[0,T]\neq\varnothing,\\ I(\pi)\cap[0,T]^c\neq\varnothing}} 2^{-\sum_{L_i\subseteq[0,T]}\alpha_i}\, w_0(\pi)\left(\frac{3}{2}(1+\beta)\sum_{L_i\subseteq[0,T]}\alpha_i\,|I(L_i)|\right) \tag{7.8}$$

$$\leqslant \sum_{\substack{\pi=[L_i^{\alpha_i}]:\ I(\pi)\cap[0,T]\neq\varnothing,\\ I(\pi)\cap[0,T]^c\neq\varnothing}} 2^{-\sum_{L_i\subseteq[0,T]}\alpha_i}\, w_0(\pi)\frac{3}{2}(1+\beta)\,|I(L_{i^*})|\sum_{L_i\subseteq[0,T]}\alpha_i$$

$$\leqslant \sum_{s=1}^{\infty}\ \sum_{\substack{L:\ |I(L)|\leqslant T,\\ 0\leqslant l(I(L))\leqslant s|I(L)|\ \text{or}\\ 0\leqslant T-r(I(L))\leqslant s|I(L)|}}\ \sum_{\substack{\pi=[L_i^{\alpha_i}]:\ L_{i^*}(\pi)=L,\\ \sum_{L_i\subseteq[0,T]}\alpha_i=s}} 2^{-s}w_0(\pi)\frac{3}{2}(1+\beta)\,|I(L_{i^*})|\,s$$

$$\leqslant \sum_{s=1}^{\infty} 2\sum_{\substack{L:\ l(I(L))=0\\ |I(L)|\leqslant T}} s|I(L)| \sum_{\substack{\pi=[L_i^{\alpha_i}]:\ L_{i^*}(\pi)=L,\\ \sum_{L_i\subseteq[0,T]}\alpha_i=s}} 2^{-s}w_0(\pi)\frac{3}{2}(1+\beta)\,|I(L_{i^*})|\,s$$

$$= 3(1+\beta)\sum_{\substack{L:\ l(I(L))=0\\ |I(L)|\leqslant T}} |I(L)|^2\sum_{s=1}^{\infty}s^2 2^{-s}\sum_{\substack{\pi=[L_i^{\alpha_i}]:\ L_{i^*}(\pi)=L,\\ \sum_{L_i\subseteq[0,T]}\alpha_i=s}} w_0(\pi)$$

$$\leqslant 3(1+\beta)\sum_{\substack{L:\ l(I(L))=0\\ |I(L)|\leqslant T}} 6|I(L)|^2 w_0(L)e^{2n(L)\beta}$$

$$\leqslant 18(1+\beta)T^{2-u}\sum_{L:\ l(I(L))=0} 2(2n(L)-1)^u|I_{m^*}(L)|^u M(L)e^{2n(L)\beta}$$

$$\leqslant 20T^{2-u}\beta,$$

where in the last inequality we have used (5.6). Now (1.11)–(1.12) is a consequence of (7.4), (7.5) and (7.8).

To get (1.13) we observe that for given k and T and s large enough

$$-\log C\left(\frac{k}{\sqrt{s}}\right)\leqslant\frac{\log(sT)}{h_2 sT}. \tag{7.9}$$

Since $G(k,T)=\Xi(k,T)/\Xi(T)$, using polymer expansions for $\log(C(k)^T\Xi(k,T))$ and $\log\Xi(T)$ we get

$$G\left(\frac{k}{\sqrt{s}},sT\right)=C\left(\frac{k}{\sqrt{s}}\right)^{sT}\exp\left(\sum_{\pi\subseteq[0,sT]}\left(W\left(\frac{k}{\sqrt{s}},\pi\right)-W(\pi)\right)\right). \tag{7.10}$$

Obviously

$$\lim_{s\to\infty}C\left(\frac{k}{\sqrt{s}}\right)^{sT}=\exp\left(-\frac{T}{2d}k^2\right). \tag{7.11}$$

Differentiating (5.20), it is not hard to check that for given k and $\pi=[L_i^{\alpha_i}]$

$$\lim_{s\to\infty}\left(W\left(\frac{k}{\sqrt{s}},\pi\right)-W(\pi)\right)\frac{s}{k^2}=\frac{1}{2d}\sum_{r=1}^{d}W''_{rr}(0,\pi)$$
$$=\frac{1}{2d}W(\pi)\left(\sum_i\alpha_i\left(\sum_{m=1}^{2n(L_i)-1}V_{I_m(L_i)}-V_{L_i}\right)\right). \tag{7.12}$$

Let

$$W_s(k,\pi)=\left(W\left(\frac{k}{\sqrt{s}},\pi\right)-W(\pi)\right)\frac{s}{k^2}\left(\sum_i\alpha_i\left(3n(L_i)-2)|I_{m^*}(L_i)|\right)\right)^{-1}. \tag{7.13}$$

Then the limit of the series in the exponential term in (7.10) is equal to

$$\lim_{s\to\infty}\Bigg[\frac{k^2}{s}sT\sum_{L:\ l(I(L))=0}((3n(L)-2)|I_{m^*}(L)|)\sum_{\pi\ni L}\alpha(L,\pi)W_s(k,\pi) \tag{7.14}$$
$$-\frac{k^2}{s}\sum_{L\subseteq[0,sT]}((3n(L)-2)|I_{m^*}(L)|)\sum_{\substack{\pi\ni L,\\ \pi\cap[0,sT]^c\neq\varnothing}}\alpha(L,\pi)W_s(k,\pi)$$
$$-\frac{k^2}{s}\sum_{\substack{L:\ I(L)\cap[0,sT]\neq\varnothing\\ I(L)\cap[sT+1,\infty)\neq\varnothing}}((3n(L)-2)|I_{m^*}(L)|)\sum_{\pi\ni L}\alpha(L,\pi)W_s(k,\pi)\Bigg].$$

Using the translation invariance, it is not hard to see that

$$k^2T\sum_{L:\ l(I(L))=0}((3n(L)-2)|I_{m^*}(L)|)\sum_{\pi\ni L}\alpha(L,\pi)W_s(k,\pi)$$
$$=k^2T\sum_{\pi:\ l(I(\pi))=0}\left(W\left(\frac{k}{\sqrt{s}},\pi\right)-W(\pi)\right)\frac{s}{k^2}. \tag{7.15}$$

It follows from the proof of Lemma 5.3 that the sum on the right-hand side of (7.15) converges uniformly with respect to s. Therefore as $s\to\infty$ this sum tends to

$$k^2T\sum_{\pi:\ l(I(\pi))=0}\frac{1}{2d}W(\pi)\left(\sum_i\alpha_i\left(\sum_{m=1}^{2n(L_i)-1}V_{I_m(L_i)}-V_{L_i}\right)\right) \tag{7.16}$$
$$=\frac{1}{2d}k^2T\sum_{L:\ l(I(L))=0}\left(\sum_{m=1}^{2n(L)-1}V_{I_m(L)}-V_L\right)\sum_{\pi\ni L}\alpha(L,\pi)W(\pi)$$
$$=\frac{1}{2d}k^2T\sum_{L:\ l(I(L))=0}\left(\sum_{m=1}^{2n(L)-1}V_{I_m(L)}-V_L\right)\mu_\infty(L)$$
$$=\frac{(1-D)T}{2d}k^2.$$

A calculation similar to (5.26)–(5.28) shows that

$$|W_s(k,\pi)| \leqslant \text{const } dM(\pi)\prod_i\left(\prod_{m=1}^{2n(L_i)-1}|I_m(L_i)|^{\frac{3}{2h_2}}\right)^{\alpha_i}, \tag{7.17}$$

and by arguments similar to (7.2)–(7.8) the absolute value of the expressions in the second and the third lines of (7.14) can be estimated from above by

$$-\text{const}\, d\frac{k^2}{s}(Ts)^{2-u+\frac{3}{h_2}}, \tag{7.18}$$

and for $h_2 > 15$ it tends to zero as $s \to \infty$. This completes the proof of Theorem 1.1.

Appendix.
The polymer expansion theorem

Consider a finite or countable set Θ, whose elements are called (abstract) *contours* and denoted by θ, θ', and so on. We fix some reflexive and symmetric realtion on $\Theta\times\Theta$. A pair $\theta,\theta' \in \Theta\times\Theta$ is called incompatible ($\theta \not\sim \theta'$) if it belongs to the given relation, and compatible ($\theta\sim\theta'$) otherwise. A collection $\{\theta_j\}$ is called a *compatible collection of contours* if any two of its elements are compatible. Every contour θ is assigned a (generally speaking) complex-valued *statistical weight*, denoted by $w(\theta)$, and for any finite $\Lambda\subseteq\Theta$ an (abstract) *partition function* (statistical sum) is defined as

$$Z(\Lambda) = \sum_{\{\theta_j\}\subseteq\Lambda}\prod_j w(\theta_j), \tag{A.1}$$

where the sum is extended to all compatible collections of contours $\theta_i \in \Lambda$. The empty collection is compatible by definition, and it is included in $Z(\Lambda)$ with statistical weight 1.

A *polymer* $\pi = [\theta_i^{\alpha_i}]$ is an (unordered) finite collection of different contours $\theta_i \in \Theta$, taken with positive integer multiplicities α_i, such that for every pair $\theta', \theta'' \in \pi$ there exists a sequence $\theta' = \theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_s} = \theta'' \in \pi$ with $\theta_{i_j} \not\sim \theta_{i_{j+1}}$, $j = 1, 2, \ldots, s-1$. The notation $\pi\subseteq\Lambda$ means that $\theta_i\in\Lambda$ for every $\theta_i\in\pi$.

With every polymer π we associate an (abstract) graph $\Gamma(\pi)$ which consists of $\sum_i\alpha_i$ vertices labelled by the contours from π and edges joining every two vertices labelled by incompatible contours. It follows from the definition of $\Gamma(\pi)$ that it is connected, and we denote by $r(\pi)$ the quantity

$$r(\pi) = \prod_i(\alpha_i!)^{-1}\sum_{\Gamma'\subset\Gamma(\pi)}(-1)^{|\Gamma'|}, \tag{A.2}$$

where the sum is taken over all connected subgraphs Γ' of $\Gamma(\pi)$ containing all the $\sum_i\alpha_i$ vertices, and $|\Gamma'|$ denotes the number of edges in Γ'. For any $\theta\in\pi$ we denote by $\alpha(\theta,\pi)$ the multiplicity of θ in the polymer π.

The polymer expansion theorem below is a modification of results of [10] and [11] proved in [12].

Theorem A.1. *Suppose that there exists a function $\alpha(\theta):\Theta\mapsto\mathbb{R}^+$ such that for any contour θ*

$$\sum_{\theta':\,\theta'\not\sim\theta}|w(\theta')|e^{a(\theta')} \leqslant a(\theta). \tag{A.3}$$

Then, for any finite Λ,

$$\log Z(\Lambda) = \sum_{\pi\subseteq\Lambda} w(\pi), \tag{A.4}$$

where the statistical weight of a polymer $\pi = [\theta_i^{\alpha_i}]$ is equal to

$$w(\pi) = r(\pi)\prod_i w(\theta_i)^{\alpha_i}. \tag{A.5}$$

Moreover, the series (A.4) for $\log Z(\Lambda)$ is absolutely convergent in view of the estimate

$$\sum_{\pi:\,\pi\ni\theta}\alpha(\theta,\pi)|w(\pi)| \leqslant |w(\theta)|e^{a(\theta)}, \tag{A.6}$$

which holds for any contour θ.

Corollary A.2. *For any polymer $\pi = [\theta_i^{\alpha_i}]$*

$$|r(\pi)| \leqslant \min_{\theta_i\in\pi}\left(\alpha(\theta_i,\pi)^{-1}|w(\theta_i)|e^{a(\theta_i)}\right)\prod_i|w(\theta_i)|^{-\alpha_i}. \tag{A.7}$$

Proof. We denote by θ^* the contour from π such that

$$\alpha(\theta^*,\pi)^{-1}|w(\theta^*)|e^{a(\theta^*)} = \min_{\theta_i\in\pi}\left(\alpha(\theta_i,\pi)^{-1}|w(\theta_i)|e^{a(\theta_i)}\right). \tag{A.8}$$

According to (A.6)

$$\alpha(\theta^*,\pi)|w(\pi)| \leqslant \sum_{\pi':\,\pi'\ni\theta^*}\alpha(\theta^*,\pi')|w(\pi')| \leqslant |w(\theta^*)|e^{a(\theta^*)}, \tag{A.9}$$

and (A.7) now follows from definition (A.5).

Corollary A.3. *For any function $b(\theta):\Theta\mapsto\mathbb{R}^+$ we consider the modified statistical weights of contours $\tilde{w}(\theta)$ such that*

$$|\tilde{w}(\theta)| = |w(\theta)|e^{-b(\theta)}. \tag{A.10}$$

Then for the corresponding statistical weights of polymers $\tilde{w}(\pi)$ we have the bound

$$|\tilde{w}(\pi)| \leqslant \min_{\theta_i\in\pi}\left(\alpha(\theta_i,\pi)^{-1}|w(\theta_i)|e^{a(\theta_i)}\right)\exp\left(-\sum_i\alpha_i b(\theta_i)\right). \tag{A.11}$$

Proof. Substituting (A.7) into the definition of $\tilde{w}(\pi)$, we immediately get (A.11).

Suppose that for a given compatiblility relation for contours one can find the maximal statistical weight $w(\theta)$ satisfying (A.3) with some $a(\theta)$. Then Corollary A.3 says that for smaller (in absolute value) statistical weights $\tilde{w}(\theta)$ of contours the corresponding statistical weights of polymers decay exponentially.

For any finite or infinite $\Lambda \subseteq \Theta$ we define the Λ-*correlation function* of the compatible collection $\{\theta_j\} \subseteq \Lambda$ as

$$\mu_\Lambda(\{\theta_j\}) = Z(\Lambda)^{-1} \sum_{\{\theta_i\}\subseteq\Lambda:\ \{\theta_i\}\supseteq\{\theta_j\}} \prod_i w(\theta_i), \tag{A.12}$$

and put $\mu_\Lambda(\theta) = \mu_\Lambda(\{\theta\})$. We denote by $F(\theta)$ the set of all contours $\theta' \in \Theta$ incompatible with θ (the forbidden set for θ). Then it follows from the polymer expansion (A.4) that

$$\begin{aligned}\mu_\Lambda(\theta) &= Z(\Lambda)^{-1} w(\theta) Z(\Lambda \setminus F(\theta)) \\ &= w(\theta)\exp\Bigg(-\sum_{\pi\subseteq\Lambda:\ \exists\theta'\in\pi,\theta'\not\sim\theta} w(\pi)\Bigg).\end{aligned} \tag{A.13}$$

A simple expression for $\mu_\Lambda(\theta)$ is given by the following lemma.

Lemma A.4. *For any* $\Lambda \in \Theta$ *and any* $\theta \in \Lambda$

$$\mu_\Lambda(\theta) = \sum_{\pi\subseteq\Lambda:\ \pi\ni\theta} \alpha(\theta,\pi) w(\pi). \tag{A.14}$$

Proof. We fix any $\theta \in \Lambda$ and change the statistical weight of this θ from $w(\theta)$ to $w(v,\theta) = v w(\theta)$. We denote by $Z(v,\Lambda)$ the partition function calculated via (A.1) with $w(v,\theta)$ taken instead of $w(\theta)$. Then

$$\mu_\Lambda(\theta) = \frac{d\log Z(v,\Lambda)}{dv}\bigg|_{v=1}. \tag{A.15}$$

On the other hand, in view of (A.4)

$$\frac{d\log Z(v,\Lambda)}{dv}\bigg|_{v=1} = \sum_{\pi\subseteq\Lambda} \frac{dw(v,\pi)}{dv}\bigg|_{v=1} = \sum_{\pi\subseteq\Lambda:\ \pi\ni\theta} \alpha(\theta,\pi) w(\pi), \tag{A.16}$$

where $w(v,\pi)$ is defined by (A.5) with $w(v,\theta)$ instead of $w(\theta)$.

Suppose that the partial order $<$ is defined on Θ in such a way that $\theta' < \theta$ implies that $\theta' \sim \theta$ and conversely $\theta' \sim \theta$ implies that either $(\theta \cup I(\theta)) \cap (\theta' \cup I(\theta')) = \varnothing$ or $\theta' \cup I(\theta') \subseteq I(\theta)$. Here $I(\theta)$ denotes the set of contours

$$I(\theta) = \{\theta' \in \Theta \mid \theta' < \theta\}, \tag{A.17}$$

called the *interior* of θ. Two contours θ and θ' are called *mutually external* if $(\theta \cup I(\theta)) \cap (\theta' \cup I(\theta')) = \varnothing$. Clearly the partition function $Z(\Lambda)$ can be represented in the form

$$Z(\Lambda) = \sum_{\{\theta_j\}^{ext}\subseteq\Lambda} \prod_j w(\theta_j) Z(I(\theta_j)), \tag{A.18}$$

where the sum extends over all compatible collections of mutually external contours only. We introduce the notation $F^{ext}(\theta) = F(\theta) \cup \{\theta' \in \Theta \mid \theta < \theta'\}$. Then

$$\mu_\Lambda(\theta^{ext}) = Z(\Lambda)^{-1} w(\theta) Z(\Lambda \setminus F^{ext}(\theta)) \tag{A.19}$$

is the correlation function of θ as the external contour (see (A.13)).

Lemma A.5. *Let us fix a function* $V_\theta : \Theta \mapsto \mathbf{R}^+$ *such that for the parameter* v *varying in some interval containing 1 the modified statistical weight*

$$w(v,\theta) = v^{V_\theta} w(\theta) \frac{Z(I(\theta))}{Z(v,I(\theta))}, \tag{A.20}$$

with

$$Z(v,\Lambda) = \sum_{\{\theta_j\}^{ext}\subseteq\Lambda} \prod_j v^{V_{\theta_j}} w(\theta_j) Z(I(\theta_j)), \tag{A.21}$$

satisfies (A.6) *for some* $a(\theta)$. *For any* $\Lambda \subseteq \Theta$ *we put*

$$V_\Lambda = \sum_{\theta\subseteq\Lambda} V_\theta \mu_\Lambda(\theta^{ext}). \tag{A.22}$$

Then

$$V_\Lambda = \sum_{\pi=\{\theta_i^{\alpha_i}\}\subseteq\Lambda} w(\pi)\Big(\sum_i \alpha_i \left(V_{\theta_i} - V_{I(\theta_i)}\right)\Big) = \sum_{\theta\subseteq\Lambda} \left(V_\theta - V_{I(\theta)}\right) \mu_\Lambda(\theta). \tag{A.23}$$

Proof. According to definition (A.21),

$$V_\Lambda = \frac{d\log Z(v,\Lambda)}{dv}\bigg|_{v=1}. \tag{A.24}$$

On the other hand,

$$Z(v,\Lambda) = \sum_{\{\theta_j\}\subseteq\Lambda} \prod_j w(v,\theta_j), \tag{A.25}$$

and in view of (A.4)

$$\frac{d\log Z(v,\Lambda)}{dv}\bigg|_{v=1} = \sum_{\pi\subseteq\Lambda} \frac{dw(v,\pi)}{dv}\bigg|_{v=1} = \sum_{\pi=\{\theta_i^{\alpha_i}\}\subseteq\Lambda} w(\pi)\Big(\sum_i \alpha_i \left(V_{\theta_i} - V_{I(\theta_i)}\right)\Big), \tag{A.26}$$

where $w(v,\pi)$ is defined by (A.5) with $w(v,\theta)$ instead of $w(\theta)$. Using (A.14) we easily get

$$\begin{aligned}&\sum_{\pi=\{\theta_i^{\alpha_i}\}\subseteq\Lambda} w(\pi)\Big(\sum_i \alpha_i \left(V_{\theta_i} - V_{I(\theta_i)}\right)\Big) \\ &= \sum_{\theta\subseteq\Lambda} \left(V_\theta - V_{I(\theta)}\right) \sum_{\pi\ni\theta:\ \pi\subseteq\Lambda} \alpha(\theta,\pi) w(\pi) = \sum_{\theta\subseteq\Lambda} \left(V_\theta - V_{I(\theta)}\right) \mu_\Lambda(\theta),\end{aligned} \tag{A.27}$$

which proves the lemma.

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On the centre of the Sklyanin Poisson algebra

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The ground ring K is assumed to be a field of characteristic 0.

The goal of this paper is to compute the centre of the Sklyanin Poisson algebra ([1], [2], English p. 13) by starting out from the fact that this algebra is a homotopy deformation of the canonical Poisson algebra on the symmetric algebra of the one-dimensional trivial extension of the Lie algebra $\mathfrak{so}(3, K)$. The main concepts and facts are given in detail in [3].

1. The homotopy algebra. Let $\mathfrak{H}$ be a Poisson algebra with associative multiplication $(u, v) \mapsto uv$ and commutator $(u, v) \mapsto [u, v]$, $u, v \in \mathfrak{H}$. We denote by $\operatorname{Hmt}\mathfrak{H}$ the set of solutions of the family of linear equations $\operatorname{cycl}_{(u,v,w)}([u,v][w,t]) = 0$, $u, v, w \in \mathfrak{H}$, with respect to $t \in \mathfrak{H}$. If $t \in \operatorname{Hmt}\mathfrak{H}$, then the K-space $\mathfrak{H}$ with the associative multiplication of the Poisson algebra $\mathfrak{H}$ and the commutator $(u, v) \mapsto t[u, v]$ is a Poisson algebra, which is called a homotopy of $\mathfrak{H}$ and denoted by $\mathfrak{H}(t)$.

$\operatorname{Hmt}\mathfrak{H}$ contains the centre $Z(\mathfrak{H})$ of the Lie algebra $\mathfrak{H}$ and is a subalgebra of the associative algebra $\mathfrak{H}$. Moreover, we have the following result.

Proposition. $\operatorname{Hmt}\mathfrak{H}$ *is an ideal of the Lie algebra* $\mathfrak{H}$.

Proof. The following identity holds in a Poisson algebra:

$$\operatorname*{cycl}_{(u,v,w)}([u,v][w,[x,t]]) = [x, \operatorname*{cycl}_{(u,v,w)}([u,v][w,t])] - \operatorname*{cycl}_{([x,u],v,w)}([[x,u],v][w,t])$$
$$- \operatorname*{cycl}_{(u,[v,x],w)}([u,[x,v]][w,t]) - \operatorname*{cycl}_{(u,v,[x,w])}([u,v][[x,w],t]), \qquad t, u, v, w, x \in \mathfrak{H}.$$

Corollary. $\operatorname{Hmt}\mathfrak{H}$ *is a subalgebra of the Poisson algebra* $\mathfrak{H}$.

It is curious that $\operatorname{Hmt}\mathfrak{H}$ appears under certain deformations.

Let $\mathfrak{a}$ be a Lie algebra. Suppose that $\mathfrak{H} = \mathcal{E}(\mathfrak{a}) :=$ the Poisson algebra on the symmetric algebra $S(\mathfrak{a})$ with canonical commutator, $d \in \mathbb{N}$, $q \in S^d(\mathfrak{a})$, and $\mathfrak{b} := K \cdot e_0 \oplus \mathfrak{a}$ is the one-dimensional trivial extension of the algebra $\mathfrak{a}$, and suppose that in $S(\mathfrak{b})$ we are given the skew-symmetric multiplication

$$[u, v]_\bullet := de_0^{d-1}[u, v], \quad [e_0, u]_\bullet := [u, q], \qquad u, v \in \mathfrak{H},$$

subject to the Leibnitz identity with respect to multiplication in $S(\mathfrak{b})$. If $q \in \operatorname{Hmt}\mathfrak{H}$, then $\mathfrak{g} := (S(\mathfrak{b}), [\,]_\bullet)$ is a homogeneous Poisson algebra of degree d. For $q \in Z(\mathfrak{H})$ we have $\mathfrak{g} = \mathcal{E}(\mathfrak{b})(de_0^{d-1})$.

The centre $Z(\mathfrak{g})$ obvously contains $Z(\mathfrak{H})$.

2. Strictly regular elements. The centre $Z(\mathfrak{H})$ is a subalgebra of the associative algebra $\mathfrak{H}$. Let $x \in \mathfrak{H}$, let $Z_{\mathfrak{H}}(x)$ be the centralizer of x in the Lie algebra $\mathfrak{H}$, and let $Z(\mathfrak{H})[x]$ be the subalgebra generated by x in the associative algebra $\mathfrak{H}$ over $Z(\mathfrak{H})$. The centralizer $Z_{\mathfrak{H}}(x)$ certainly contains the algebra $Z(\mathfrak{H})[x]$. An element x is said to be *strictly regular* in $\mathfrak{H}$ if $Z_{\mathfrak{H}}(x) = Z(\mathfrak{H})[x]$ (cf. [4], p. 71).

It is not hard to see that the centre $Z(\mathfrak{g})$ contains the subalgebra $Z(\mathfrak{H})[e_0^d + q]$ generated by the element $e_0^d + q$ in the algebra $S(\mathfrak{b})$ over $Z(\mathfrak{H})$.