The choice between capitalizing and expensing under rate regulation

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In this paper we give a mathematical analysis of some of the consequences, over time, of the decision to capitalize or to expense. Both regulated and unregulated firms are considered. It is shown that for a regulated firm, in contrast to an unregulated one, this decision does have an impact on the customers, who should rationally prefer either capitalization or expensing, depending on their discount rate. It is shown that the attitudes of rational owners will also depend on their discount rate. The point of view of the tax collector is also considered.

1. Introduction

■ This paper examines the relative effects on revenue, taxes, earnings, outside capital requirements, etc., of "capitalizing" certain items of expenditure, that is, raising capital to pay for them, as opposed to "expensing" them, paying for them outright.

It is assumed that these expenditures will be made in any case, so that the choice between capitalization and expensing is an *accounting* choice,[1] which does, however, have economic conse-

[1] Thus we are *not* discussing the economic choice between labor and capital as factors of production, discussed by Averch and Johnson [1], and in many subsequent papers. See, in particular, Bailey and Malone [2].

quences, as we shall see. It should also be emphasized that the items under consideration are those for which such an option is allowed by the tax collectors, the accounting profession, and the regulatory body, if any—for example, items of furniture costing less than $100.

The difference between capitalizing and expensing is as follows. If an item is capitalized, it is paid for out of future revenues, but at the cost of providing a return to the suppliers of capital; capitalization spreads out expenditures into the future, and smooths them. If an item is expensed, it is paid for out of current revenues. One of the essential determinants of such a choice therefore involves a comparison between the cost of capital and the time value of money, as experienced by the various interested parties. Another determinant of the choice will be its effect on outside capital requirements. The interested parties include the customers, the owners, the managers, the regulators, and the tax collectors; the interests of these groups are in part conflicting.

To determine an "optimal" amount of capitalization, it would be necessary to balance off its effects on one group against those on another. We shall not attempt to do that in this paper. Instead, we shall calculate the effect over time of capitalizing in place of expensing on each of a number of variables of interest to the various groups, and leave to the reader any balancing of interests among them.

At this point we should make clear the distinction between an unregulated and a regulated firm. By an *unregulated* firm we mean one whose revenues are determined entirely by the market place, and are therefore independent of the capitalization-expensing decision. The customers of such a firm are clearly indifferent to this decision. For a *regulated* firm (to which this paper is primarily devoted), a public commission sets prices for the firm's products in such a way that an allowed rate of return—the ratio of net earnings to total capital—will be realized.[2] The customers of a regulated firm clearly are affected by the capitalization-expensing decision.

In reality, of course, regulation is intermittent[3] and not absolute. Thus a real regulated firm might be expected to lie between the two extreme cases described above. However, since our main point in this note is to present in simple mathematical form the direct, first-order effects which are caused by a switch from capitalization to expensing, we shall assume here that regulation is perfect. For the same reason, we shall neglect here the price-elasticity of demand, and assume that revenues and expenses (hence earnings) are predictable with complete certainty. In such a world of perfect certainty, it would follow that the cost of capital to the firm, expressed in its discount rate, as well as the rate of return on capital which the regulatory commission should permit, should be equal to the pure interest rate, i. Such certainty is, of course, not present in the real world and the difference between the pure

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[2] For a discussion of price-setting in regulated firms, see Davis [4] and references therein.

[3] See Klevorick [6].

interest rate, on the one hand, and, for example, the cost of equity, on the other, is one reflection of this uncertainty. The appropriate formulation of the exact form of the relationship between risk and expected return on investment is a question which has been much discussed.[4]

In simple cases the results may be seen easily by intuition. Consider, for example, a strictly regulated firm with no taxes. In this case the choice between expensing or capitalizing an item is, from the customers' point of view, equivalent to the choice between buying something outright and buying it on time. The "interest" the customers pay if the item is capitalized is equal to the company's rate of return on capital, and the payback schedule is the company's depreciation schedule for that item.

Complications such as taxes, however, can have a considerable effect on the results. In the presence of the corporate income tax, the customers' "interest" mentioned above becomes the *pre-tax* rate of return, which may be considerably higher than the commonly quoted after-tax rate of return set by the regulator.

If the distinction between book and tax depreciation (allowed by present law) is recognized, the mathematics becomes more complicated and additional terms appear in the results.

In order to simplify the formulas and circumvent tedious calculations, we make use of the *convolution* notation. This is particularly helpful in connection with present worths, which we use to compare different expenditure streams and hence to express the preferences of the various parties. If the meaning of the convolution is understood, the compact notation should also aid in the interpretation of results.

2. The model

■ The model on which our calculations are based is illustrated by the diagram of Figure 1, in which the symbols are also defined.[5] This diagram shows the cash flows per unit of (continuous) time, into, out of, and within the firm as here modeled. Revenues and outside capital flow in, and expenses, interest, etc. flow out.

Looking at the top node, we see that the total revenues, R, which the firm collects from its customers go after covering expenses, E, and book depreciation, D_B, into the pre-tax earnings, A, or

$$R = E + A + D_B. \tag{1}$$

The bottom node shows that the construction budget, C, is provided by the sum of the book depreciation, D_B, and the increase, per unit time, in the total capital of the firm, K. This capital consists of two parts: first that raised from the outside through the sale of stocks and bonds, denoted by K_O, and second that accumulated internally through the retention of earnings, denoted by K_I. Thus

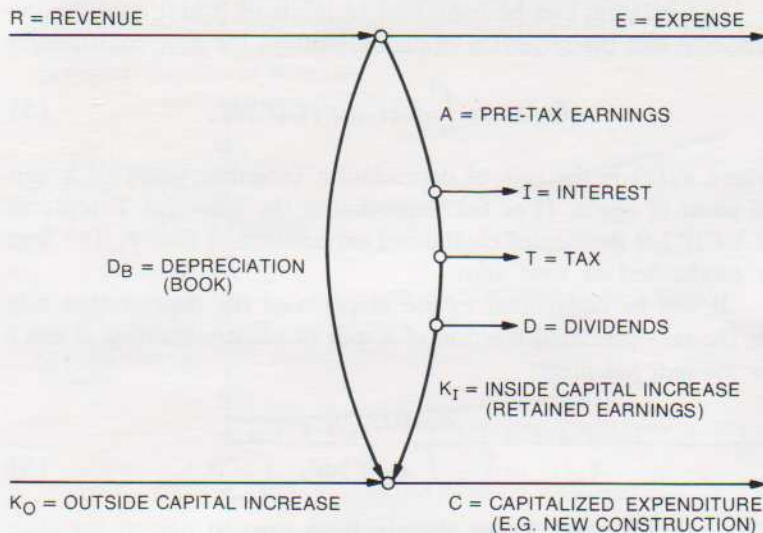$$C = D_B + \dot{K}_O + \dot{K}_I, \tag{2}$$

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[4] See, for example, Cootner and Holland [3] and Sharpe [8].
[5] Some of the methods used here are similar to those of Linhart [7].

FIGURE 1

CASH FLOWS OF THE MODEL FIRM

R = REVENUE

E = EXPENSE

A = PRE-TAX EARNINGS

I = INTEREST

T = TAX

D_B = DEPRECIATION
(BOOK)

D = DIVIDENDS

K_I = INSIDE CAPITAL INCREASE
(RETAINED EARNINGS)

K_O = OUTSIDE CAPITAL INCREASE

C = CAPITALIZED EXPENDITURE
(E.G. NEW CONSTRUCTION)

> EACH ARROW REPRESENTS A (TIME-VARIABLE) FLOW WITH THE UNITS:
> DOLLAR PER UNIT TIME. VARIABLES NOT SHOWN ARE DEFINED BELOW:
>
> $K = K_O + K_I$ = TOTAL CAPITAL
>
> δ = DEBT RATIO, i.e., δK = debt, $(1-\delta)K$ = EQUITY
>
> $a = A/K$ = PRE-TAX RATE OF RETURN
>
> $\rho = (A-T)/K$ = POST-TAX RATE OF RETURN
>
> τ = TAX RATE
>
> $i = I/\delta K$ = INTEREST RATE
>
> $d = D/(1-\delta)K$ = DIVIDEND RATE
>
> X = UNIT NET PLANT (SEE TEXT)
>
> SUBSCRIPTS B AND T REFER TO BOOK AND TAX DEPRECIATION.

where the dot denotes the time derivative, e.g.,

$$\dot{K}_I = \frac{dK_I}{dt}.$$

$\dot{K}_I$ represents retained earnings, which are obtained from pre-tax earnings A, as shown in Figure 1, by the subtraction of taxes T, interest I, and dividends D,

$$\dot{K}_I = A - I - T - D. \qquad (3)$$

In addition, we have the tax equation

$$T = \tau(R - E - D_T - I), \qquad (4)$$

where D_T is depreciation as calculated for tax purposes. (Only income taxes are considered here.)

In the case of a regulated firm, there is a further accounting relation,

$$R = \rho K + T + D_B + E,$$

which expresses the fact that post-tax earnings, $A - T$, are a fixed fraction of capital K. This fraction, ρ, is the allowed rate of re-

turn set by the regulator. As mentioned earlier, we shall assume here, when dealing with a regulated firm, that this allowed rate of return is in fact realized.

Depreciation can be expressed in terms of a unit depreciation schedule and the sequence of capital outlays for new construction

$$D_B(t) = \int_0^t d_B(t - t')C(t')dt', \qquad (5)$$

where $d_B(s)$ is the rate of depreciation (fraction/year) of a unit of plant of age s. (For tax depreciation the subscript T replaces B.) $C(t')$ is the rate of capitalized expenditure at time t'. The firm is established at time zero.

It will be convenient to use in place of the depreciation rate d_B the undepreciated fraction of a unit of plant remaining at age t or the *unit net plant*:

$$X_B(0) = 0$$
$$X_B(t) = 1 - \int_0^t d_B(t')dt', \quad t > 0. \qquad (6)$$

The unit net plant jumps sharply from zero to one at the time ($t = 0$) of acquisition and then falls gradually back to zero as the plant is depreciated. A typical time course for $X_B(t)$ is plotted in Figure 2.

FIGURE 2

SCHEMATIC FOR DEPRECIATION OF UNIT NET PLANT
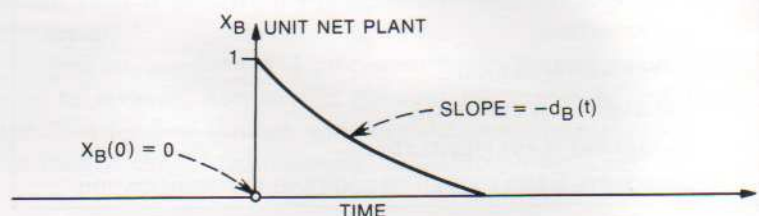
Setting $X_B(0) = 0$ has the effect of including the acquisition jump within the nonnegative time domain, which is mathematically convenient.

3. Convolution notation

■ The integral in equation (5) is a *convolution* of the functions $d_B(t)$ and $C(t)$. Using a standard notation, equation (5) can be written

$$D_B = d_B * C. \qquad (5a)$$

The convolution has many simple properties useful in calculation. For example, a change in the variable of integration shows that

$$d_B * C = C * d_B.$$

Other properties are given in the Appendix. The convolution may be thought of as superimposing weighted translates of a fixed function. The integral in equation (5), for example, superimposes the depreciation flows generated by various vintages of plant. That generated by vintage t' plant is the per unit depreciation schedule

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$d_B(t)$, translated to start at t' and weighted by the amount $C(t')dt'$ of vintage t' plant that was installed.

A useful result states that total capital equals net plant, which is the sum of undepreciated plant from all past vintages:

$$K = X_B * C.$$

■ The total rate of expenditure required to run the business is $[E(t) + C(t)]$ (see Figure 1). $E(t)$ is the portion treated on the books as current expense; $C(t)$ is the portion treated as capital. Taking the physical operations and the sum $E(t) + C(t)$ as given, we wish to examine the effects of transferring, on the books, a stream of money $F(t)$ from the expense stream $E(t)$ to the capital outlay stream $C(t)$. $F(t)$ is an arbitrary function, zero until the time of transfer, and positive thereafter. $F(t)$ could be taken to be negative; we would then be considering a shift from capitalization to expensing. The reader can easily supply the corresponding sign changes in the results.

The subscript E (for "expensing") will indicate the value of a variable without the transfer, the subscript C (for "capitalization") its value with the transfer. Thus,

$$C_C = C_E + F \qquad (7)$$
$$E_C = E_E - F.$$

We consider first the case of a strictly regulated firm ($\rho_C = \rho_E$ given) with book and tax depreciation the same ($D_B = D_T$). In this case the pre-tax rate of return is unaffected (see Appendix):

$$a = a_C = a_E = (\rho - \tau i\delta)/(1 - \tau). \qquad (8)$$

We shall now consider the effects of the decision on various groups.

□ **Customers.** The customers are interested only in the effect on revenues, since these are what they pay. It is shown in the Appendix that

$$R_C - R_E = (aX_B - \dot{X}_B) * F. \qquad (9)$$

The convolution on the right may be interpreted as a superposition of "loans" to the body of customers. The amount of the loan made at time t is equal to the amount $F(t)$ transferred to capital at that time. The "interest rate" paid by the customers is equal to the pre-tax of return a, and the unrepaid fraction is equal to the unit net plant X. Thus the term aX represents "interest" on the unrepaid balance, and $-\dot{X}$ represents repayments. Equation (6) shows that these follow the depreciation schedule. Since we included the initial jump to unity in our definition of X, the $-\dot{X}$ term includes the initial payment of the "loan" to the customers as well as the repayment back to the firm.

If F rises faster than e^{at}, then the current "loan" is always more than enough to cover current "interest," so that $R_C - R_E$ always remains negative. In this case capitalization yields a perpetual saving in revenue. This situation is analogous to a stock

market bubble or a chain letter and is wonderful in that it generates something for nothing.

If we rule out this unlikely case, then $R_C - R_E$ must eventually become positive, so that early revenue savings are bought at the cost of a later revenue penalty. How are the customers to judge the value to them of such an ambiguous revenue differential? A simple answer is to suppose that they look at the present worth of $R_C - R_E$ with respect to some discount rate k related to the way they trade time against money.

If we let $PW_k\{Z\}$ stand for the present worth,[6] i.e.,

$$PW_k\{Z\} \equiv \int_0^\infty e^{-kt} Z(t)\,dt,$$

then it follows from equation (9) (see the Appendix) that

$$PW_k\{R_C - R_E\} = (a - k)PW_k\{X\}PW_k\{F\}. \qquad (10)$$

Under the present worth criterion, then, customers prefer capitalization if and only if the expression on the right is negative. Since both $PW_k\{X\}$ and $PW_k\{F\}$ are positive, this occurs if and only if the customers' discount rate k exceeds the pre-tax rate of return a.

Although this result is simple, it must be applied with some caution. It assumes, tacitly, that the body of customers is static, infinite-lived and possessed of a definable interest as a whole. In fact, the customers come and go in an endless stream and have only individual interests. Thus a revenue differential $(R_C - R_E)$ that varies in time necessarily trades off the interests of some customers against those of others. Specifically, capitalization favors early customers at the expense of later ones. The present worth, therefore, should be regarded at best as only a crude indicator of the customers' interest and should not be used without regard to the shape of $R_C - R_E$ over time.

More refined analysis would also recognize that the quantity of interest to the individual customer is not total revenue, but revenue per unit of service or perhaps revenue per customer. These quantities can, of course, be calculated, given the appropriate data.

Nevertheless, the time shape of $R_C - R_E$ and even its present worth give a simple and useful first cut into the very knotty problem of determining the customers' interest in capitalization versus expensing for a regulated firm.

☐ **Owners.** The customers do not constitute the only interested party. One concern of the owners (shareholders) and their representatives the managers—e.g., the treasurer—may well be the amount of new capital they must raise. It is shown in the Appendix that the effect on outside capital requirements of the transfer $F(t)$ from expense to capital is

$$\dot{K}_{OC} - \dot{K}_{OE} = -[(\rho - i\delta - d(1 - \delta))X_B - \dot{X}_B]*F. \quad (11)$$

This equation has the same form as the revenue equation (9). It therefore represents a superposition of "loans," but this time

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[6] The present worth is a Laplace transform. For further discussion of properties of the Laplace transform, see Widder [9].

with a negative sign and with "interest" equal to $\rho - i\delta - d(1 - \delta)$. The latter quantity is readily recognized to be the *retained earnings relative to capital*. We will denote it by r:

$$r = \rho - i\delta - d(1 - \delta) = \frac{\dot{K}_I}{K}. \tag{12}$$

Then equation (11) can be rewritten:

$$\dot{K}_{OC} - \dot{K}_{OE} = - (rX_B - \dot{X}_B)*F. \tag{13}$$

If F grows less rapidly than e^{rt}, *then* $\dot{K}_{OC} - \dot{K}_{OE}$ eventually becomes negative [see argument following equation (9)]. But this means that capitalization eventually leads to a *savings* in outside capital requirements! This seemingly anomalous result is explained by noting that capitalization generates a stream of retained earnings which helps to satisfy capital requirements. If F grows slowly enough, the cumulative stream of retained earnings from past capitalizations may more than offset new capital requirements.

However, since r may be numerically small, the case in which F grows faster than e^{rt} is not unreasonable. In this case outside capital requirements are always greater under capitalization than under expensing, but (if $r > 0$) not so great as $F(t)$.

We now suppose that the quantity which the owner (or collectivity of owners) wishes to maximize is the net outflow to him. In terms of our model this is dividends minus the equity inflow. Denoting it by S, we have

$$S = D - (\dot{K}_O - \delta\dot{K}).$$

The effect on this outflow of the transfer of $F(t)$ from expense to capitalization is

$$S_C - S_E = (1 - \delta) \left(\frac{\rho - i\delta}{1 - \delta} X_B - \dot{X}_B \right) *F.$$

As before, this represents a superposition of "loans." The transfer to capitalization, in effect, causes the owner to lend money at the "interest rate"

$$\epsilon = \frac{\rho - i\delta}{1 - \delta}.$$

As always, the payback follows the depreciation schedule $\dot{X}$. The quantity ϵ is readily recognized to be the rate of return on equity. (Multiply it by equity capital $(1 - \delta)K$.) Taking the present worth relative to the owner's discount rate k' gives

$$PW_{k'}\{S_C - S_E\} = (1 - \delta)(\epsilon - k')PW_{k'}\{X\}PW_{k'}\{F\}. \tag{14}$$

From this we can read the intuitively plausible result that the owner prefers capitalization over expensing if and only if his discount rate k' is less than the rate of return on equity ϵ.

It is interesting to contrast the owner of a regulated firm, considered above, with the owner of an unregulated firm. For the unregulated firm we take the revenue R as given, rather than the rate of return ρ. Then equation (14) is replaced by

$$PW_{k'}\{S_C - S_E\} =$$
$$(\delta - \tau)\left(k' - \frac{(1-\tau)\delta i}{\delta - \tau}\right)PW_{k'}\{X\}PW_{k'}\{F\}. \quad (14a)$$

The present worth can be positive, i.e., capitalization can be preferable to the owner, only if

$$\delta > \tau,$$

i.e., only if the debt ratio is higher than the tax rate. Under present circumstances this would rarely be the case for an unregulated firm. Thus we would expect unregulated owners generally to favor expensing and to capitalize only as much as necessary to remain solvent.

A parenthetical comment: to understand equation (14a) intuitively, it is useful to decompose the "loan" into the sum

$$\delta(\dot{X} - iX) - \tau(\dot{X} - \delta iX).$$

The first term is a loan *to* the owner at interest rate i; the second is a loan *from* the owner at the lower interest rate δi. The first term is a debt effect; the second is a tax effect.

☐ **Tax collector.** The tax collector is interested in the effect on taxes, which (see the Appendix) is

$$T_C - T_E = (a - \rho)X_B*F = \frac{\tau}{1 - \tau}(\rho - i\delta)X_B*F. \quad (15)$$

Taxes are always greater under capitalization. Incidentally, the dependence of tax on the depreciation rate is clearly shown by this formula. The faster the depreciation, the smaller the net plant X at any time, hence the lower the tax. This assumes, of course, that the same depreciation schedule is used for book and tax purposes.

A numerical example is displayed in Figure 3. The differentials $R_C - R_E$, $\dot{K}_{OC} - \dot{K}_{OE}$, $T_C - T_E$, and $S_C - S_E$ are graphed as functions of time for an exponentially increasing $F(t)$. This case would occur, for example, if an exponentially growing regulated firm decided at time $t = 0$ to transfer a fixed fraction of its expenditures from expense accounts to capital accounts.

5. Book and tax depreciation difference

■ Present law permits public utilities to use an accelerated depreciation schedule for tax purposes, while retaining an unaccelerated schedule for book and regulatory purposes. This results in lower taxes early in the life of a capitalized item and higher taxes later. The tax differential is treated in either of two ways: it is allowed to "flow through" to revenues or it is set aside in a "normalization" reserve, which is used for capital but is not included in the rate base.[7] The legal and other merits of the two treatments are currently a matter of considerable controversy.
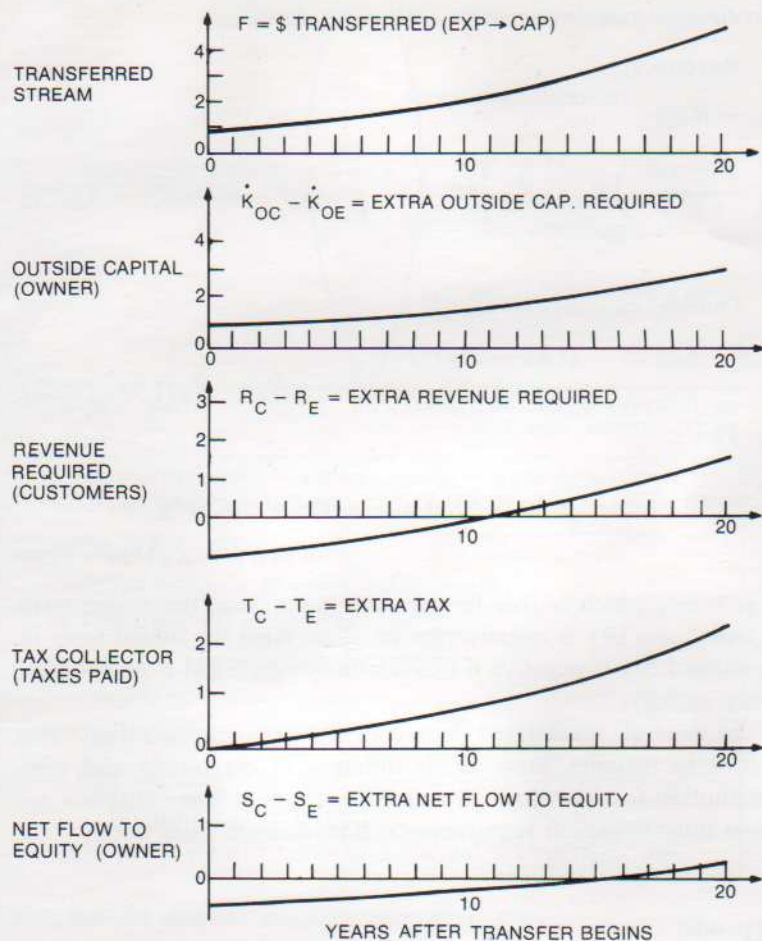
To take account of these possibilities our results must be modified as follows. (The subscripts B and T refer to book and tax depreciation.)

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[7] See [5] for the tax law. See Linhart [7] for a discussion of alternative methods of accounting for accelerated tax depreciation.

FIGURE 3

EFFECTS OF TRANSFERRING STREAM F(t) FROM EXPENSE TO CAPITALIZATION.
IN THIS EXAMPLE, F(t) IS EXPONENTIAL. THIS WOULD BE THE CASE, FOR
EXAMPLE, IF AN EXPONENTIALLY GROWING FIRM TRANSFERRED A FIXED
FRACTION OF ITS EXPENDITURES. ASSUMPTIONS: REGULATED FIRM WITH
$F = e^{0.08t}$, $\rho = 0.09$, $\delta = 0.45$, $i = 0.06$, $\tau = 0.50$, $d = 0.06$, USING STRAIGHT LINE
DEPRECIATION WITH 20 YEAR LIFE.

TRANSFERRED STREAM — F = $ TRANSFERRED (EXP→CAP)

OUTSIDE CAPITAL (OWNER) — $\dot{K}_{OC} - \dot{K}_{OE}$ = EXTRA OUTSIDE CAP. REQUIRED

REVENUE REQUIRED (CUSTOMERS) — $R_C - R_E$ = EXTRA REVENUE REQUIRED

TAX COLLECTOR (TAXES PAID) — $T_C - T_E$ = EXTRA TAX

NET FLOW TO EQUITY (OWNER) — $S_C - S_E$ = EXTRA NET FLOW TO EQUITY

YEARS AFTER TRANSFER BEGINS

Accelerated tax depreciation with flow-through:

Revenues:

$$R_C - R_E = \tag{16}$$

$$\left[\left(\frac{\rho - \tau i\delta}{1 - \tau} \right) X_B - \dot{X}_B \right] *F + \frac{\tau}{1 - \tau} (\dot{X}_T - \dot{X}_B)*F$$

Outside capital:

$$\dot{K}_{OC} - \dot{K}_{OE} = - (rX_B - \dot{X}_B)*F \tag{17}$$

Taxes:

$$T_C - T_E = \frac{\tau}{1 - \tau} ((\rho - i\delta)X_B*F + (\dot{X}_T - \dot{X}_B)*F). \tag{18}$$

The revenue differential is the same as before except for an added

term. [See equations (8) and (9).[8]] This term has the effect of augmenting both the near-term revenue saving and the far-term revenue penalty resulting from capitalization. Note that the differential in outside capital requirements is unaffected. [Compare equations (17) and (13).]

Accelerated tax depreciation with normalization:

Revenues:

$$R_C - R_E =$$

$$\left[\left(\frac{\rho - \tau i\delta}{1 - \tau}\right)X_B - \dot{X}_B\right]*F + \frac{1-\tau}{\tau}(\rho - \delta i\tau)(X_T - X_B)*F. \tag{19}$$

Outside capital:

$$\dot{K}_{OC} - \dot{K}_{OE} = -(rX_B - \dot{X}_B)*F + \tau[(\dot{X}_T - \dot{X}_B) - r(\dot{X}_T - \dot{X}_B)]*F. \tag{20}$$

Taxes:

$$T_C - T_E = \frac{\tau}{1-\tau}(\rho - i\delta)X_B*F - \tau(X_B - X_T)*F + \tau(\dot{X}_T - \dot{X}_B)*F. \tag{21}$$

If $\rho > \tau i\delta$, which is true in any reasonable case, the added term in equation (19) is negative for all time. Also the added term in equation (20) is negative if r is relatively small and F grows relatively rapidly.

In general, accelerated depreciation favors capitalization. The benefit to revenue, however, is different in magnitude and time distribution for flow-through and normalization. Normalization relieves outside capital requirements; flow-through does not.

Appendix

■ In this Appendix we define some properties of the convolution, and use them to derive various formulae and results employed in the text.

□ **Convolution.**[9]

Definition: $A*B \equiv \int_0^t A(t-t')B(t')dt'.$

The following properties are easily verified:

$A*B = B*A$

$\hat{\delta}*A = A$, $\hat{\delta} = $ Dirac delta function[10]

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[8] In equation (16) we cannot substitute the pre-tax rate of return a for the expression $(\rho - \tau i\delta)/(1 - \tau)$ because a is no longer equal to this, but involves convolutions with X, etc.

[9] See Widder [9].

[10] $\hat{\delta}(t)$ is a unit impulse at $t = 0$. It is the derivative of the unit step function. If $s(t) = 0$ for $t < 0$ and $s(t) = 1$ for $t > 0$, then $\hat{\delta}(t) = \dot{s}(t)$.

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$$1 * A = \int_0^t A \, dt'$$

$(A*B)^{\cdot} = \dot{A}*B \text{ if } A(0) = 0$

$PW_k(A*B) = PW_k(A) \cdot PW_k(B)$

$PW_k(\dot{A}) = A(0) + kPW_k(A).$

☐ **Depreciation and capital.** From the definition of $X_B(t)$ [equation (6)]:

$$\dot{X}_B(t) = \hat{\delta}(t) - d_B(t).$$

By the convolution properties and equation (5a):

$$\dot{X}_B * C = \hat{\delta} * C - d_B * C = C - D_B. \qquad (A1)$$

Since $\dot{K} = \dot{K}_0 + \dot{K}_I$ (Figure 1), equation (2) yields

$$C = B_B + \dot{K}. \qquad (A2)$$

Combining (A1) and (A2) gives

$$\dot{K} = \dot{X}_B * C. \qquad (A3)$$

Assuming capital is zero at $t = 0$, the convolution properties also give

$$K = X_B * C. \qquad (A4)$$

This says that capital equals the sum of net plant over all vintages.

☐ **Basic forms.** We shall take E, C, X_B, X_T, ρ, i, d, δ, τ as given functions of time (X_B and X_T stand for unit net plant under book and tax depreciation respectively), and calculate other quantities in terms of them.

The pre-tax rate of return a: From the definitions of ρ and a we get

$$aK = \rho K + T.$$

Substituting for T by (4) and then for $R - E$ by (1) gives

$$aK = \rho K + \tau(aK + D_B - D_T - I).$$

Solving for a, and using (A1), (A4), and the definition of i, we obtain

$$a = \frac{\rho - \tau i \delta}{1 - \tau} + \frac{\tau}{1 - \tau} \left[\frac{(\dot{X}_T - \dot{X}_B)*C}{X_B * C} \right]. \qquad (A5)$$

For book and tax depreciation the same, the second term is zero. This gives equation (8).

The revenue R: Equation (1) can be written

$$R = E + aK + D_B.$$

Substituting for a, K, and D_B by means of (A5), (A4), and (1) gives

$$R = E + C + \left[\frac{\rho - \tau i \delta}{1 - \tau} X_B - \dot{X}_B \right] * C \qquad (A6)$$

$$+ \frac{\tau}{1 - \tau} (\dot{X}_T - \dot{X}_B)*C.$$

For book and tax depreciation the same, the last term is zero and a can be substituted for the expression $(\rho - \tau i\delta)/(1 - \tau)$. Subtracting F from E and adding F to C gives equation (9).

Outside capital requirements. Using $\dot{K} = \dot{K}_o + \dot{K}_I$, we can rewrite equation (3) as

$$\dot{K}_o = \dot{K} - A + T + I + D.$$

Appropriate substitutions (definitions of ρ, i, d) give

$$\dot{K}_o = -[(\rho - i\delta - d(1 - \delta))X_B - \dot{X}_B]{*}C. \qquad \text{(A7)}$$

Tax. The tax formula is conveniently derived by rewriting the definition of ρ as

$$T = A - \rho K = (a - \rho)K,$$

and substituting (A5) and (A4):

$$T = \frac{\tau}{1 - \tau}[(\rho - i\delta)X_B{*}C + (\dot{X}_T - \dot{X}_B){*}C]. \qquad \text{(A8)}$$

Net flow to equity. Using the definitions of $\dot{K}_o$ and ρ,

$$\dot{K}_o = \dot{K} - \rho K + I + D.$$

Substituting this into

$$S = D - (\dot{K}_o - \delta\dot{K}),$$

and then using $K = X{*}C$, etc., yields

$$S = -[(1 - \delta)\dot{X} - (\rho - i\delta)X]{*}C.$$

For the unregulated firm (R given instead of ρ) we use equation (1) in place of the definition of ρ and obtain

$$S = (1 - \tau)(R - E - P) + [(\delta - \tau)\dot{X} - (1 - \tau)\delta i X]{*}C,$$

from which the equation stated in the text follows.

☐ **Normalization.** To handle this case, the model must be modified to take account of the normalization reserve. If $\dot{N}$ is the flow to the normalization reserve, then

$$\dot{N} = T_B - T_T = \tau(D_T - D_B).$$

Equations (2) and (3) and the definition of ρ must be replaced by:

$$C = D_B + \dot{K}_o + \dot{K}_I + \dot{N}$$
$$\dot{K}_I = A - I - T - D - \dot{N}$$
$$\rho = (A - T - \dot{N})/K$$

Substitutions similar to those outlined above yield the results stated in the text.

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