WARRANT PRICING - A REVIEW, WITH COMMENTS, OF SAMUELSON'S AND OTHER THEORIES

J. L. Lebowitz*
David Kreps**
P. B. Linhart†

* Belfer Graduate School of Science, Yeshiva University
** Dartmouth College (A.T.& T. Summer Employee)
† Management Sciences Division, A.T.& T.

ABSTRACT

There exist a large number of papers in the literature dealing with warrant prices. These vary from the purely empirical (Kassouf, 1968) to the highly theoretical (Samuelson - McKean, 1965). We present here a review and interpretation of this literature with principal emphasis on Samuelson's paper, as well as some empirical data on the current A.T.& T. warrants. The data shows a very large dispersion of warrant prices for a given stock price. This raises some questions about the validity of Samuelson's and similar models which predict a unique warrant price for a given stock price. We plan to explore this in a future memorandum.

Warrant Pricing - A Review with Comments, of Samuelson's and Other Theories

1. Introduction

A simple warrant is a right to buy a share of a particular stock at a specified price, called the exercise price, E. There are two types of simple warrants, differin the provisions as to when the warrant holder may exercise his right to buy the stock.

1) The European warrant, where the exercise time is some fixed future date.
2) The American warrant, where the right may be exerup to some fixed future date. A perpetual warrant is an American warrant where the fixed future date is infinity; the right may be exercised at any future time.

We study the "proper" price of the warrant, i.e., the time discounted expected value of the excess of the stock price over the exercise price that will accrue to the warrant holder when he exercises his warrant. We assume here that he will exercise his warrant with optimal strategy. No consideration will be given to the existence of a speculative market for warrants, since we assume for the purposes of this paper that all warrant holders and all prospective warrant holders evaluate the warrant similarly. With this restriction, the value of the warrant is dependent only on investors' anticipations as to the future price of the stock. These anticipations will be given probabilistically; a random process will be described that represents the anticipated series of stock prices. From this process the value of the European warrant will be explicitly calculated. Inequalities relating the values of European and American warrants, and of different American warrants, will be determined by arbitrage arguments. A functional form for the calculation of the value of the American warrant will be given, and closed form solutions of this will be given for special cases. This development is taken principally from Samuelson [6] and Chen [3]. Appendices will mention other formulas for warrant prices, empirical and analytic. Mention will also be made in an appendix of rational pricing for 'zero-exercise price warrants'.

2. The Stock Price Series

The anticipated price of the stock will be defined for the model as a random process X_t. The units for this process will be the exercise price of the warrant, E (or if the warrant allows the holder to obtain K units of stock for exercise price E, units will be E/K). In these units, the exercise price for one share of stock will be unity. Several properties of the process X_t are postulated:

A) $\text{Prob}\ \{X_{t+T} \leq Y | X_t = x,\ X_{t-\tau_1} = x_1,\ X_{t-\tau_2} = x_2,\ \ldots\}$

$$= P(Y,x;T) \equiv \int_0^Y p(y,x;T)\,dy\ . \qquad (2.1)$$

That is, the process is stationary and Markov. Consequently, it satisfies the Chapman-Kolmogorov equation:

$$P(X,x;T) = \int_0^\infty P(X,z;T-t) \quad dP(z,x;t) \qquad (2.2)$$

B) The price scales, i.e.:

$$P(X,x;t) = G(X/x;t) \qquad (2.3)$$

Equation (2.3) implies that the stock price has the same probability of doubling (or tripling, or halving), in a given time interval, no matter what its initial value for the period. Thus

$$E(X|x;t) = \int XdP(X,x;t) = \int XdG(\tfrac{X}{x};t)$$

$$= x\int \frac{X}{x}\, dG(\tfrac{X}{x};t) = xK(t)\ . \qquad (2.4)$$

Taking expectation values of X in (2.2) yields

$$xK(t) = xK(t-t_1)\ K(t_1)$$

which has the solution $K(t) = e^{\alpha t}$ or

$$E(X|x;t) = xe^{\alpha t} \quad . \tag{2.5}$$

It will be assumed that $0 < \alpha < \infty$. α may be considered a measure of investors' utility aversion to risk; α is the amount the investment must appreciate so as to compensate for the risk involved in investment and hence determines the present price of the stock if its future expected value is known.

C) A third assumption regarding $P(X,x;t)$ is that it is of log stable form; that is, it can be realized as a limiting form of a product of independent random variables. It then follows that its characteristic equation must be of Levy-Khinchin form. A particular example is the log-normal form, with density function

$$p(y,x;t) = \frac{1}{y}\frac{1}{\sigma\sqrt{2\pi t}} e^{-[\ln y - \ln x - \mu t]^2/2\sigma^2 t} \quad . \tag{2.6}$$

In this case $\alpha = 1/2\sigma^2 + \mu$.

3. European Warrants

Given $P(X,x;t)$, the values of the European warrant can be readily calculated. We will assume that warrant value will be discounted at a rate $\beta \geq \alpha$, reflecting the possible added risk involved in holding warrants. Indeed, if the stock pays dividends at a rate d, $\beta \geq \alpha + d$, because a warrant holder does not receive dividends. Suppose the European warrant being evaluated has an exercise date T time-units from now. The distribution of stock price at that date will be $P(X,x;T)$, where x is the present price. If the warrant holder decides, at time T, to exercise his right, he gains a net amount X_T-1. He will exercise his right, then, if $X_T > 1$. His expected net gain, discounted at a rate β, is

$$W(x,T) \equiv e^{-\beta T} \int_1^{\infty} [X-1] \, dP(X,x;T). \tag{3.1}$$

$W(x,T)$ is the 'value' of a European warrant when the stock price is x and there is an interval T to the exercise time.

4. American Warrants: Arbitrage

With $W(x,T)$ as above, let $V(x,T)$ be the value of an American warrant, exercisable until (and including) time T from now, if the present stock price is x. Then

$$V(x,T) \geq W(x,T) \tag{4.1}$$

This is because one possible strategy for exercising an American warrant is to exercise at time T if and only if the price X_T exceeds 1. This strategy nets the warrant holder as much as a European warrant would, so that as long as the warrant holder exercises his right with optimum strategy, the added privilege of exercise before time T will not decrease the value of the warrant.

Similar considerations lead to the result

$$x \geq V(x,\infty) \geq V(x,T_1) \geq V(x,T_2) \geq V(x,0)$$

$$= \quad \mathrm{Max}\{x-1,0\}, \quad \text{for } T_1 \geq T_2 \geq 0. \qquad (4.2)$$

This set of inequalities can be interpreted as follows: The value of a warrant cannot exceed x, since for the price x, one can buy the stock itself, which is at least as valuable as the warrant. A warrant that must be exercised immediately is worth nothing if the present stock price is less than the exercise price 1, and is worth the present stock price less the exercise price otherwise. A warrant exercisable until time T_1 is worth no less than one exercisable until $T_2 < T_1$, because one can always exercise the T_1 warrant with the same strategy as the T_2 warrant. For the same reason, a perpetual warrant is as valuable as any finite time warrant. Fig. 1 illustrates the general pattern of these inequalities.

5. American Warrants: The Functional Form

The precise value of an American warrant is not easily found in closed form. An algorithm for finding the value can however be given if time is taken to be a discrete variable.

Suppose the warrant can be exercised until time T, but only at times until expiration $0 = \tau_0, \tau_1, \ldots, \tau_n = T$, with $\tau_j < \tau_{j+1}$; i.e., the warrant may be exercised at times $T = T-\tau_0, T-\tau_1, \ldots, T-\tau_n = 0$. Then, if we know for the various stock-prices X the value of the warrant with time remaining τ_j to be $V(X, \tau_j)$, we can find $V(x, \tau_{j+1})$. At a given time $T-\tau_{j+1}$, when the stock price is x, the warrant holder may burn the warrant, exercise the warrant, or hold it until the next conversion time $T-\tau_j$. Burning the warrant nets the investor 0. Exercising it nets x-1, while holding it nets a range of present worth values $e^{-\beta(\tau_{j+1}-\tau_j)}V(X,\tau_j)$ for the various possible X, the price of the security at time $T-\tau_j$. Holding thus yields an expected return, in present worth, of

$$e^{-\beta(\tau_{j+1}-\tau_j)}\int_0^\infty V(X,\tau_j)\, dP(X,x;\tau_{j+1}-\tau_j)\ . \tag{5.1}$$

The investor will pursue whichever of the three strategies nets him the largest expected gain, and thus

$$V(x,\tau_{j+1}) = \mathrm{Max}\left\{0, x-1,\ e^{-\beta(\tau_{j+1}-\tau_j)}\int_0^\infty V(X,\tau_j)dP(X,x;\tau_{j+1}-\tau_j)\right\} \tag{5.2}$$

with

$$V(x,0) = \mathrm{Max}\,\{0, x-1\}\ . \tag{5.3}$$

Dynamic programming may now be employed in (5.2) and (5.3) to find $V(x,\tau_j)$ and thus $V(x,T)$. But this is only with the constraint of a finite number of exercise times. This dynamic programming technique will give only an approximation to the correct value for the continuous-time case, although it might be supposed that the approximation gets better as the grid of possible exercise times becomes finer. Strictly, however, we can only say that

$$V(x,t+\tau) \geq \text{Max} \left\{ 0, x-1, \; e^{-\beta\tau} \int_0^\infty V(X,t)\,dP(X,x;\tau) \right\} \;, \qquad (5.4)$$

with equality coming in the limit as τ approaches 0. This leads to a technically difficult problem, which was solved by McKean [5] in some very special cases; we shall later mention one of his results.

One can also consider situations in which the third term in the brackets of (5.4) is always the largest, for any τ. In that case, the warrant is held until expiration and its value is the same as if it were a European warrant. This is always the situation when $\beta = \alpha$, a case which we now analyze directly. We will then return to the more general case.

6. The Case $\beta = \alpha$

Assume that $\beta = \alpha$. This implies that there are no dividends being paid on the stock, and that investors require only the same mean gain from warrants as they do from shares of stock. Then, it will be shown, a warrant will not be exercised before its exercise date. This will be done by proving that for any τ in Eqn. (5.4)

$$e^{-\beta\tau} \int_0^\infty V(X,t)dP(X,x;\tau) \geq x-1. \qquad (6.1)$$

(It is clear that the integrand is non-negative, as $V(X,t) \geq 0$.) We have already observed that $V(X,t) \geq \mathrm{Max}\,\{x-1,0\}$. Thus, the integral majorizes

$$e^{-\beta\tau} \int_1^\infty (X-1)dP(X,x;t). \qquad (6.2)$$

We show this to be greater than x-1:

$$e^{-\beta\tau}\int_1^\infty (X-1)dP(X,x;\tau) = e^{-\beta\tau}\int_0^\infty (X-1)dP(X,x;\tau) + e^{-\beta\tau}\int_0^1 (1-X)dP(X,x;\tau)$$

$$= e^{-\beta\tau}(xe^{\alpha\tau}-1) + e^{-\beta\tau}\Phi(x,t) = xe^{(\alpha-\beta)\tau} + e^{-\beta\tau}(\Phi(x,\tau)-1)$$

$$= x + e^{-\beta\tau}(\Phi(x,\tau) -1),$$

where we have used (2.5), set $\alpha = \beta$, and have defined

$$\Phi(x,t) = \int_0^1 (1-X)dP(X,x;t) .$$

It is clear that $0 \leq \Phi(x,t) \leq 1$, and hence Eqn. (6.1) follows.

Note especially what happens as $t \to \infty$.

$$V(x,t+\tau) \geq e^{-\beta\tau} \int_0^\infty V(X,t)\, dP(X,x;\tau) \geq e^{-\beta\tau}\int_1^\infty (X-1)\, dP(X,x;\tau)$$

$$= x + e^{-\beta\tau}(\Phi(x,t) - 1) \quad . \qquad (6.4)$$

Thus as $t \to \infty$, $V(x,t+\tau)$ exceeds $X-\varepsilon$ for any positive ε. But arbitrage guarantees that $V(x,t) \leq x$. Hence,

$$\lim_{t\to\infty} V(X,t) = x. \qquad (6.5)$$

Then $V(x,\infty) = x$, again by arbitrage. And indeed, x is a solution for $V(x,\infty)$ in the analogue of (5.3) for infinite t:

$$V(x,\infty) = \text{Max}\left\{0, x-1,\ e^{-\beta\tau} \int_0^\infty V(X,\infty)\, dP(X,x;t)\right\} \qquad (6.6)$$

The result $V(x,\infty) = x$ seems to be contrary to experience as there are certainly perpetual warrants that trade at a price below their associated common. But what has been modeled here cannot be the real world. We are assuming a stock whose value will grow forever at a constant mean rate, and that $\beta = \alpha$, i.e., that investors require the same rate of return from warrants as from stocks, (which implies at the least that there will never be dividends). Moreover $V(x,\infty) = x$ only if all warrant holders adopt the optimal strategy, which is, never to exercise the warrant. If in any finite time it is exercised the mean return may not be x. Under these assumptions, the value of the warrant is indeed the price of the stock, but this is an asset that is frozen forever.

7. The Case $\beta > \alpha$

If $\beta = \alpha$, then, the warrant has the same value as a European warrant, given in (3.1), and depicted in Fig. 2. Suppose instead that $\beta > \alpha$. In this case there is a stock price at which the warrant should no longer be held to expiration but should be exercised. Intuitively, as the price of the stock becomes very large ($x >> 1$), then because $x \geq V(x,t) \geq x-1$, the expected gain on the warrant will approach α. A proof runs as follows: For any warrant with remaining life t, if the warrant is not to be exercised until expiration, it must be true that

$$e^{-\beta\tau} \int_0^\infty V(X,0)\, dP(X,x;t) \geq x-1. \tag{7.1}$$

But $V(X,0) \leq X$, or

$$e^{-\beta\tau} \int_0^\infty V(X,0)\, dP(X,x;t) \leq e^{-\beta\tau} \int_0^\infty X dP(X,x;t) = xe^{(\alpha-\beta)t}. \tag{7.2}$$

For large enough x, this last term is less than x-1, which contradicts (7.1).

It is now much more difficult to find closed form solutions for V(x,t). McKean [5] has shown, however, that for log normal $P(X,x;t)$,

$$V(x,\infty) = \begin{cases} [c-1](x/c)^{\gamma} & \text{for } x \leq c \\ x-1 & \text{for } x \geq c, \end{cases} \tag{7.3}$$

where $c = \gamma/(\gamma-1) \geq 1$, and, in terms of the log normal distribution parameters given in (2.6), and α, β,

$$\gamma = \left[\frac{1}{2} - \frac{\alpha}{\sigma^2}\right] + \sqrt{\left[\frac{1}{2} + \frac{\alpha}{\sigma^2}\right]^2 + 2\left[\frac{\beta}{\sigma^2} - \frac{\alpha}{\sigma^2}\right]} \tag{7.4}$$

A diagram of warrant prices for $\beta > \alpha$, adapted from Samuelson [6], and Chen [3], is found in Figure 3.

8. Deficiencies of the Model

The model of warrant pricing described above is open to a number of criticisms. The pricing of warrants in the real world is quite unlike what is theorized: The model implies for a certain stock price a warrant value that will diminish slowly as the time of expiration approaches. But as seen in Fig. 4, for a given stock price there is a range of warrant prices that may be as large as twenty-five percent (with some of the higher prices occurring at later times). This variation might be explained within the context of the model: changes in the parameters α, β, and the investors' expectations as to the stock price series, represented by P, may all be changing.

But it is also possible that the price of warrants fluctuates because of the stochastic nature of a speculative market in them. Future models should include provisions for speculative markets in warrants, and should consider what effects such markets would have on warrant pricing. (Attempts have been made to model a securities market in order to gauge these effects. A memo on this subject will appear shortly.)

References

1. W. M. Boyce, "Stopping Rules for Selling Bonds," The Bell Journal of Economics and Management Science, Vol. 1, No. 1, (Spring, 1970).

2. W. M. Boyce, "On the Value of A.T.& T. Warrants," Unpublished memorandum.

3. A. Chen, "A Model of Warrant Pricing in a Dynamic Market," The Journal of Finance, pp. 1041-1059.

4. S. T. Kassouf, "Warrant Price Behavior - 1945 to 1964," The Financial Analyst's Journal (January-February, 1968).

5. H. P. McKean, "A Free Boundary Problem for the Heat Equation Arising from a Problem of Mathematical Economics," Industrial Management Review, Vol. 6, (Spring, 1965).

6. P. Samuelson, "Rational Theory of Warrant Pricing," Industrial Management Review, Vol. 6, (Spring, 1965)

7. J. P. Shelton, "The Relation of the Price of a Warrant to the Price of Its Associated Stock," The Financial Analyst's Journal, (May-June and July-August, 1967).

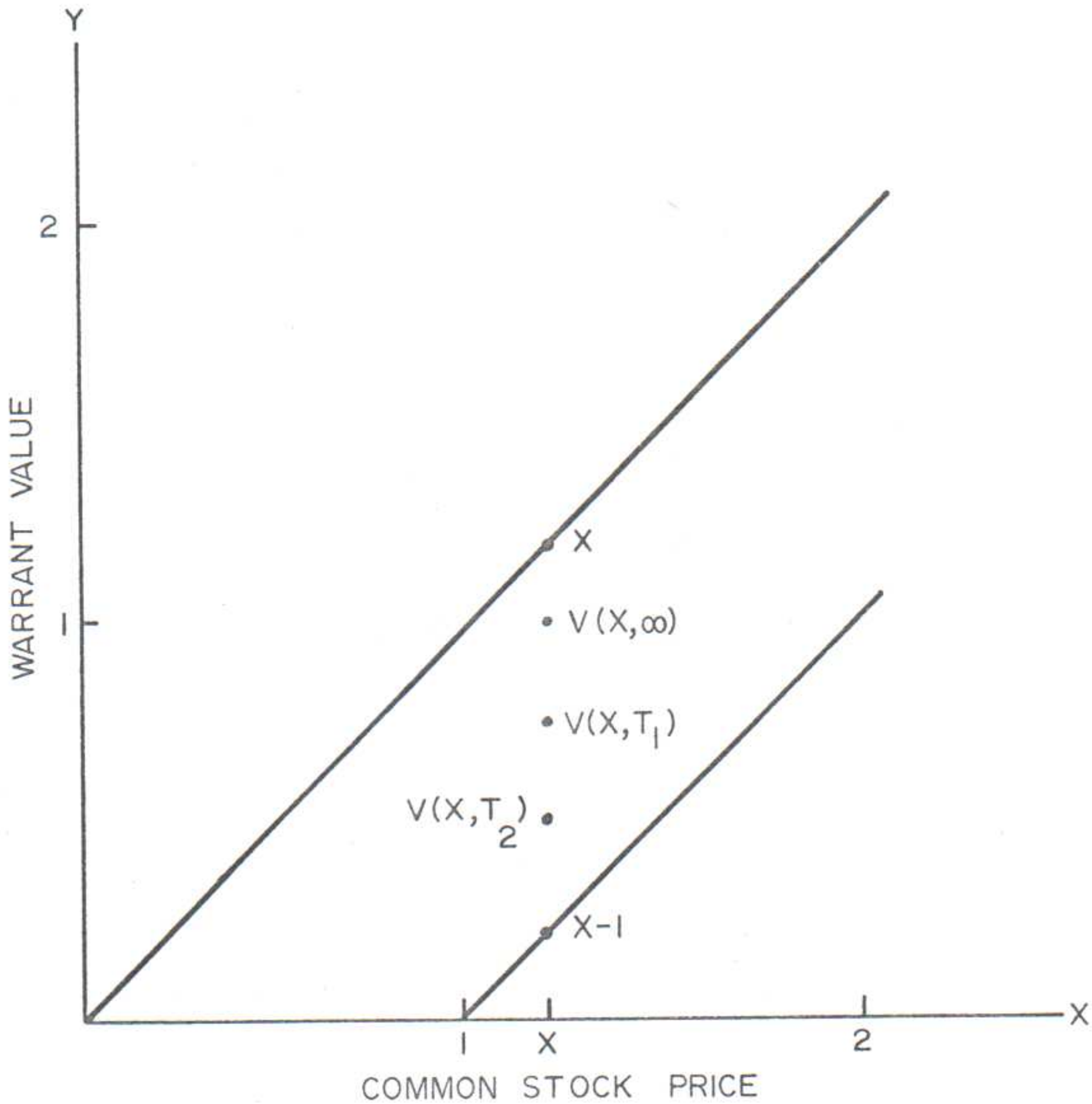

Fig. 1. Arbitrage Conditions of the Value of American Warrants:

For a given common stock price X, we have

$$X \geq V(X,\infty) \geq V(X,T_1) \geq V(X,T_1) \geq V(X,0)$$

$$= \max \{0, X-1\} \text{ where } T_1 \geq T_2 > 0.$$

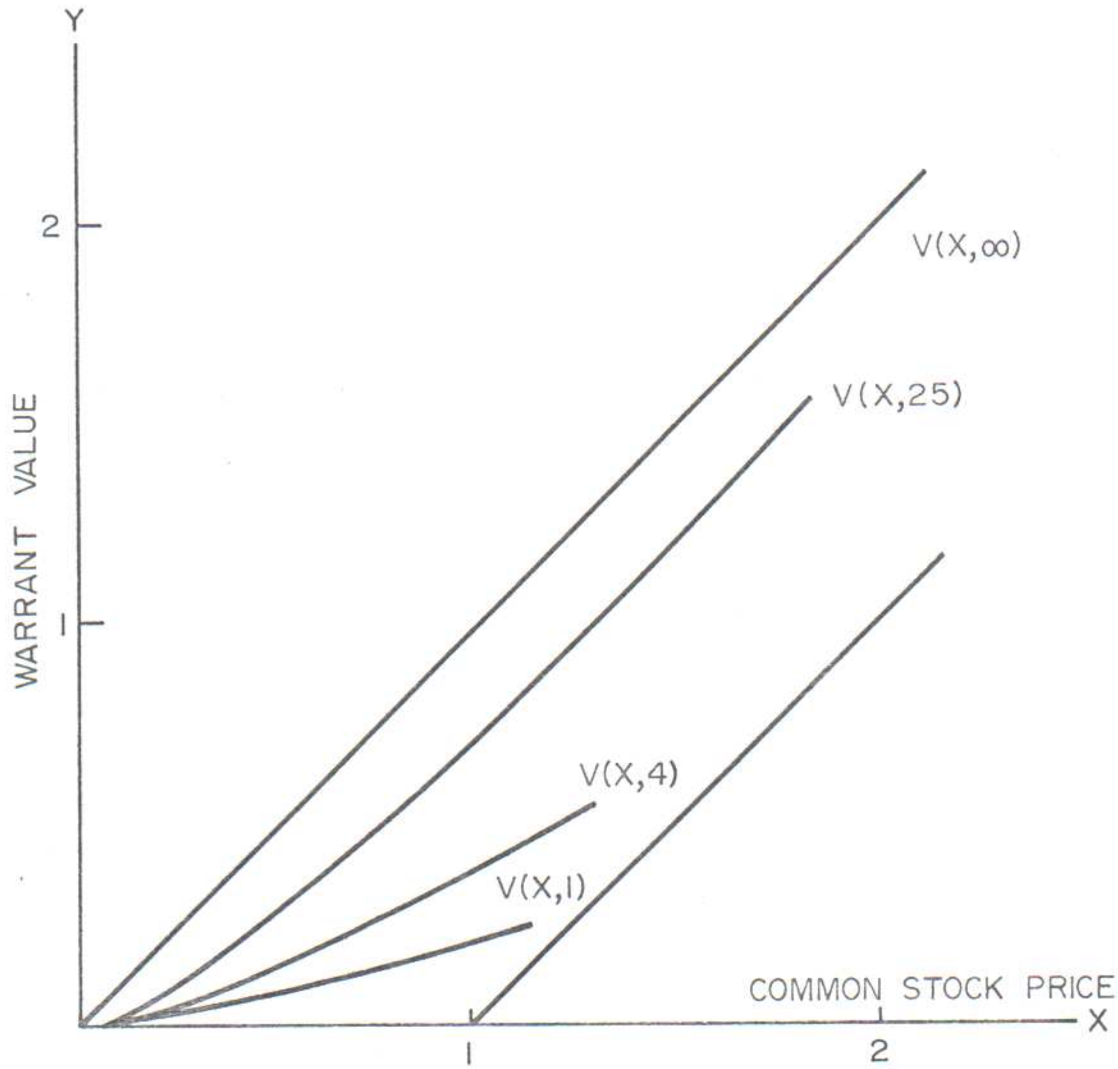

Fig. 2. The $\beta = \alpha$ case for American Warrants, and European Warrants:

In the case $\beta = \alpha$, the value of a perpetual warrant is x, and for finite time American warrants, their value is that of the corresponding European warrant.

(Adapted from [3])

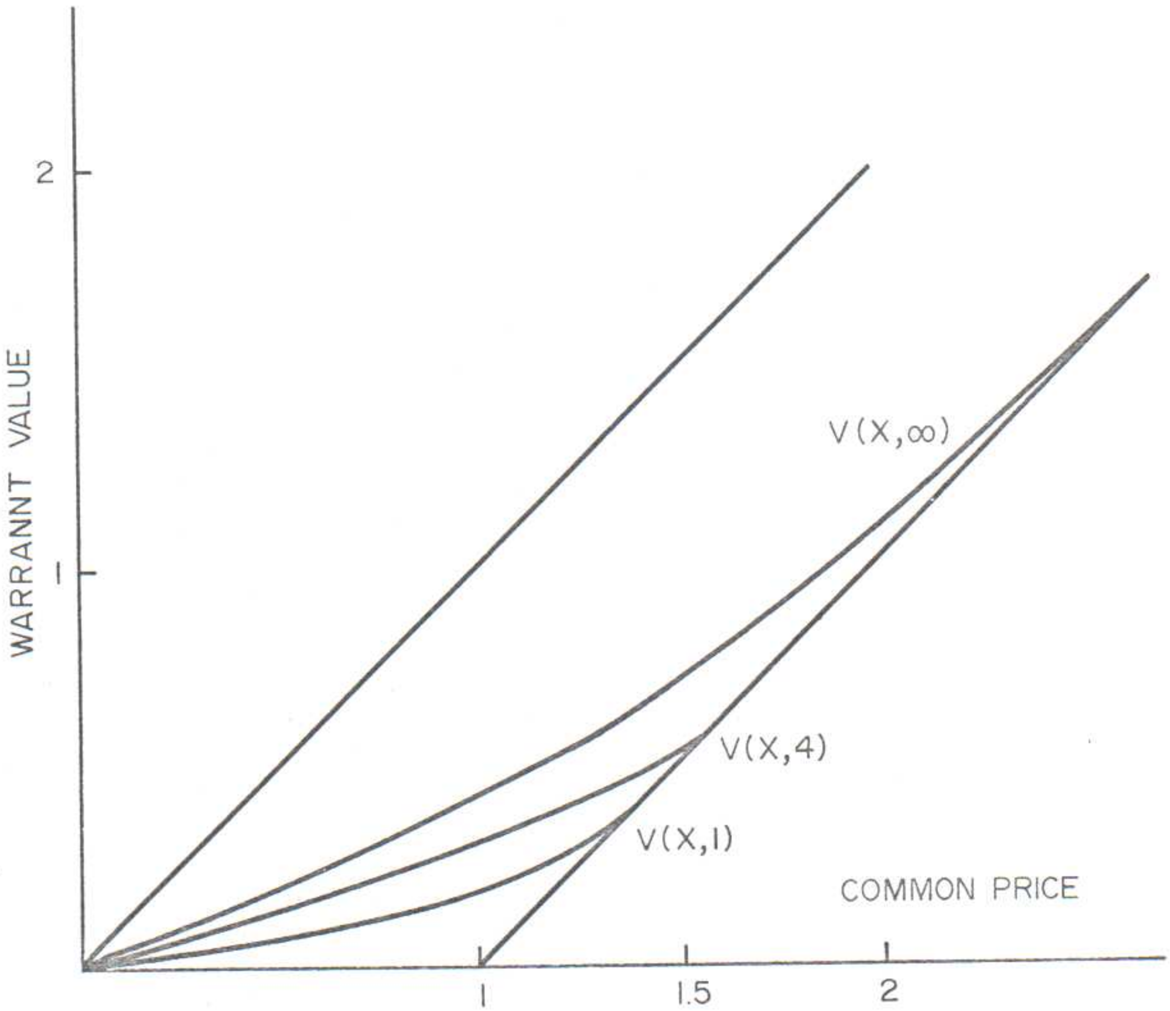

Fig. 3. Warrant Pricing, the $\beta \geq \alpha$ Case.

In the case $\beta > \alpha$ there are prices sufficiently large to require immediate exercise of the right.

(Adapted from [3])

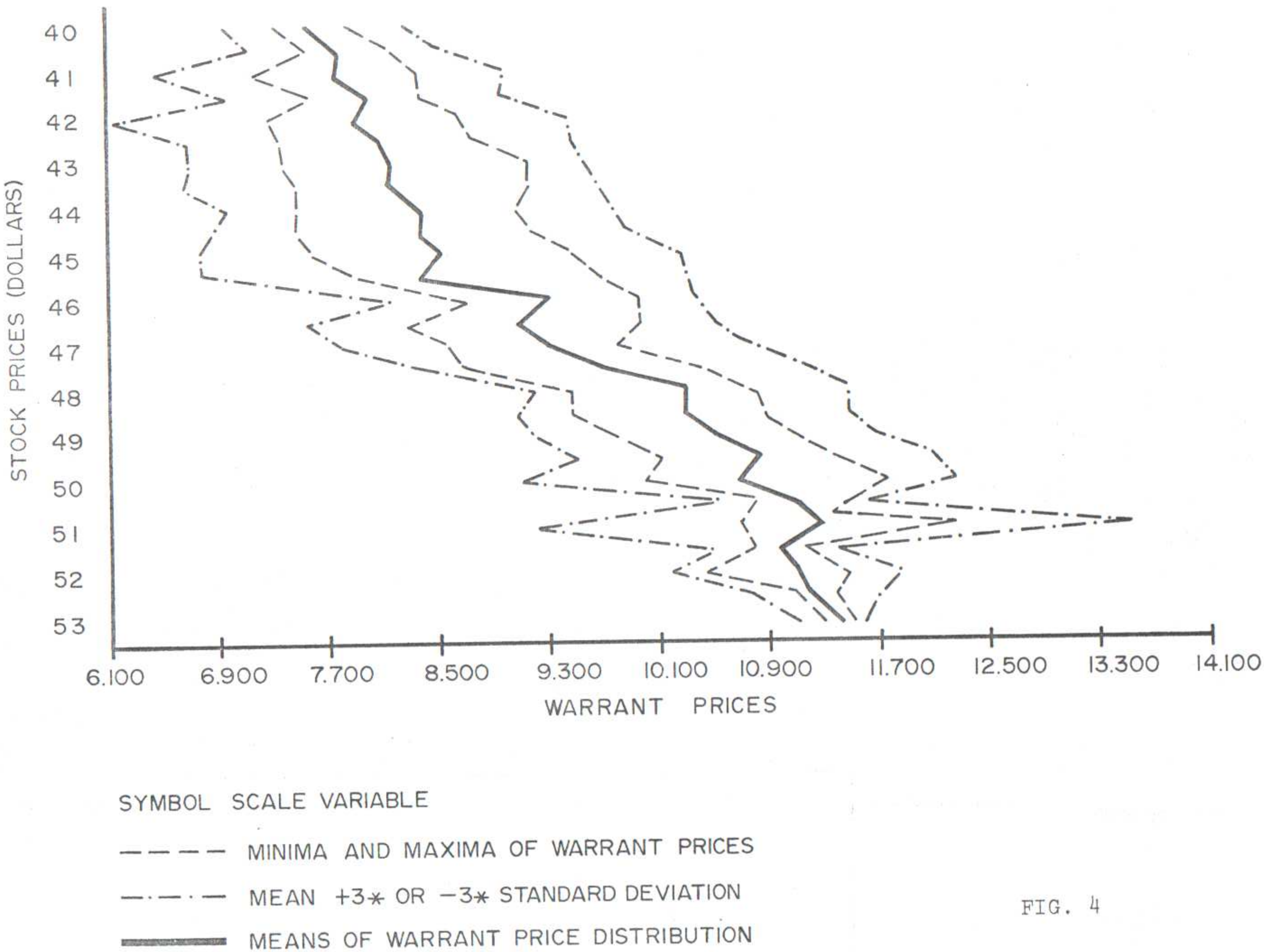

FIG. 4

DISPERSION OF WARRANT PRICES FOR GIVEN STOCK PRICES

Appendix 1: Empirical Formulae

Several empirical formulae for warrant pricing have been advanced. The simplest is that of Giguére (1958) for a perpetual American warrant:

$$W^* \equiv \frac{W}{E} = \frac{1}{4}X^2 \equiv \frac{1}{4}\,(S/E)^2 \tag{A1.1}$$

Here W is the price of the warrant, S the price of the stock, E the exercise price of the warrant. W^* and X are then the warrant and common price, rescaled properly, in the sense of this paper. Note that this is precisely the McKean formula for $\gamma = c = 2$.

Kassouf (1962) as quoted by Shelton [8] has a formula

$$W^* = \sqrt{1+X^2} - 1, \tag{A1.2}$$

while Kassouf [4] gives the more general equation

$$W^* = (1+X^m)^{1/m} - 1 \quad , \tag{A1.3}$$

where m is determined by

$$m = 1.2 + 5.3\frac{1}{T} + 14.8d + .3D + .4X + .4\ln\,(X/X_1), \tag{A1.4}$$

where T = number of months to expiration, d = annual dividend,
D = number of warrants/number of common,
X_1 = price of common eleven months earlier.

Shelton also gives a 'zone of plausible prices' for the warrant

$$(3/4)X \geq W^* \geq \text{Max}\,[0, X-1] \qquad \text{for } X \leq 4$$

$$W^* = X-1 \qquad \text{for } X > 4 \qquad (A1.5)$$

The formula for where in the 'zone' the price should be (fraction measured from the top) is

$$W = 4\sqrt{M/72}\,[.47 - 4.25(\text{yield}) + .17 \text{ if listed on the AMEX}] \qquad (A1.6)$$

where M = months to expiration, yield = dividend/stock price. (For perpetual warrants, take M = 120.)

<u>Barron's</u>, Dec. 7, 1970, writes for warrant pricing

$$W^* \equiv X(1+\frac{R}{100}) - 1 = X-1 + (\frac{R}{100})\,X \qquad (A1.7)$$

so that R is the percentage rise in the stock price necessary to make the warrant worth its price if exercised now.

Appendix 2: Boyce's Model

William Boyce (BTL) presents in a memo [2] a model of warrant pricing substantially different from Samuelson's. He assumes a speculative market for warrants, and that the warrant price in that market is precisely the value of a European warrant given in equation (3.1). The function $P(X,x;t)$ appearing in that equation, corresponds in Boyce's model to the expectations of the aggregate of investors. These investors expect that the stock price will behave as a stationary process, which Boyce assumes to be Brownian motion with a drift. A single investor is then considered, who 'knows' that the stock price series is not a simple Brownian motion with a drift. This investor expects the price series to behave as a Wiener Process constrained to a certain normal end distribution (as in [1]). The single investor is then able to speculate profitably in warrants.

For example, Boyce postulates that A.T.& T. common will behave as a constrained Wiener process in which day-to-day variances are, in sum, greater than the total expected variance for the period. (This is the $\sigma < 1$ case as considered in [1]). Then a rise in stock price will cause the aggregate of all investors, save the 'smart' one, to expect an end distribution of stock prices that has a higher mean than what 'will' happen. Warrants will be overvalued, and the 'smart' investor will sell them. A fall in prices will result in undervalued warrants; they should be held.

Appendix 3: Zero Exercise Price Warrants

Recently, some attention has been given to the question of zero exercise price warrants, warrants that can be exercised for no (or nominal) fee. It is clear that a zero exercise price American warrant must have value the same as the stock; the value of the warrant cannot exceed the value of the stock, and by exercising the warrant immediately, a gain equal to the price of the stock can be obtained.

For a zero exercise price European warrant, exercisable at time T from now, the optimal strategy is obvious; at time T, claim your free stock. The value of the warrant is then

$$W^{*}(x,T) = e^{-\beta T}\int_{0}^{\infty} X \, dP(X,x;T)$$

$$= e^{-\beta T} e^{\alpha T} x = x e^{-(\beta-\alpha)T} \quad . \qquad (A3.1)$$

This is less than x so long as $\beta > \alpha$, i.e., at least whenever the stock pays dividends. (We do not consider the possible tax advantages to the investor that might reside in such a scheme.)