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GRIFFITHS INEQUALITIES FOR ANTI-FERROMAGNETS*

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We prove the existence of new Griffiths type inequalities for anti-ferromagnetic Ising spin systems and the corresponding lattice gases. It is shown in particular that for such a system the two spin (or particle) Ursell function $[\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle]$ is positive when i and j are on the same sublattice and negative when they are on different sublattices. This is true for all values of the external magnetic field (or chemical potential) at the different lattice sites.

In a very interesting recent paper Fortuin et al. [1] made an important extension of the Griffith et al. [2] inequalities for ferromagnetic systems. In lattice gas language they proved that whenever the two particle and higher order interactions are attractive then the inequalities of [2] hold independent of the sign or magnitude of the one body interaction (which includes the chemical potential). It is the purpose of this note to point out that the result of [1] implies new Griffiths type inequalities for anti-ferromagnetic spin systems and corresponding lattice gases.

Let Λ be a set, e.g. a finite lattice, whose sites will be designated by i, $i = 1, \dots, |\Lambda|$. In the usual notation $\rho_i \in (0,1)$ designates the occupation number of the ith site and $\sigma_i = (2\rho_i - 1) \in (-1,1)$ is the corresponding Ising spin variable. We shall denote by $P, Q, R, \dots$ the subsets of Λ and let $\rho_R = \prod_{i\in R}\rho_i$, $\sigma_R = \prod_{i\in R}\sigma_i$, etc. The lattice gas Hamiltonian of this system has the general form,

$$H_L = \sum_{R\subset\Lambda} \varphi(R)\rho_R\,. \tag{1}$$

The result of [1] which we shall need is that

$$\frac{\partial\langle\rho_P\rangle}{\partial\varphi(Q)} = \langle\rho_P\rho_Q\rangle - \langle\rho_P\rangle\langle\rho_Q\rangle \geqslant 0 \tag{2}$$

whenever $\varphi(R) \geqslant 0$ for R containing two or more sites. Here

$$\langle\rho_P\rangle = \sum_{\{\rho_i\}} \rho_P \exp[\textstyle\sum_R \varphi(R)\,\rho_R] / \{\sum_{\{\rho_i\}} \exp[\sum_R \varphi(R)\rho_R]\}\,, \tag{3}$$

and we have included the reciprocal temperature β in the Hamiltonian (or simply set it equal to one). Inequalities for the corresponding Ising spin system can be deduced from (2). In particular if there are only pair interaction between the particles,

$$H_L = -\tfrac{1}{2}\sum_{i\neq j}\varphi(i,j)\,\rho_i\rho_j - \sum\varphi(i)\,\rho_i, \qquad i, j\in\Lambda \tag{4}$$

then the corresponding spin Hamiltonian is

$$H_S = -\tfrac{1}{2}\sum_{i\neq j} J(i,j)\sigma_i\sigma_j - \sum h(i)\sigma_i + \text{const.} \tag{5}$$

with

$$J(i,j) = \tfrac{1}{4}\,\varphi(i,j), \qquad h(i) = \tfrac{1}{2}\,\varphi(i) + \tfrac{1}{4}\sum_{\substack{j\in\Lambda\\ j\neq i}}\varphi(i,j)\,. \tag{6}$$

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In (5) $h(i)$ is the external magnetic field acting on the ith site. It is then a consequence of (2) that independent of the sign of the external field at the different sites,

$$\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle=4[\langle\rho_i\rho_j\rangle-\langle\rho_i\rangle\langle\rho_j\rangle]\geq 0 \tag{7}$$

whenever $J(i,j)\geq 0$.

Consider now a set Λ which can be decomposed into two disjoint subsets A' and B', $A'\cup B'=\Lambda$, e.g. Λ is a subset of an infinite cubic lattice and A' and B' are the intersections of Λ with the two sublattices A and B. Let

$$\lambda_i=\begin{cases}1 \text{ if } i\in A'\\ -1 \text{ if } i\in B'\end{cases},\qquad \lambda_R=\prod_{i\in R}\lambda_i \tag{8}$$

and set

$$\mu_i=\lambda_i\sigma_i,\qquad n_i=\tfrac{1}{2}(\mu_i+1)=\lambda_i\rho_i+\tfrac{1}{2}(1-\lambda_i). \tag{9}$$

It is readily verified now by using the transformation (9) that whenever $\lambda_R\,\varphi(R)\geq 0$ for R containing two or more sites (which for a system having only pair interactions means that the pair potential in (4) (and (5)) has the property that $\lambda_i\lambda_j\,\varphi(i,j)\geq 0$, i.e. $\varphi(i,j)\geq 0$ for i and j on the same sublattice and $\varphi(i,j)\leq 0$ for i and j on different sublattices) then the inequalities of [1] apply to the $\langle n_R\rangle$. We thus have in particular that

$$\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle=\lambda_i\lambda_j\left|\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle\right|. \tag{10}$$

Eq. (10) and its generalizations which can be obtained by using (9) on the inequality (2) of [1] is the main result of this note. It shows that for an anti-ferromagnetic Ising spin system in an arbitrary external magnetic field two spins are positively correlated when they are on the same sublattice and negatively correlated when on different sublattices. In lattice gas language (10) is a statement about the two particle Ursell function $F(i,j)=\langle\rho_i\rho_j\rangle-\langle\rho_i\rangle\langle\rho_j\rangle=\lambda_i\lambda_j\,|F(i,j)|$ which alternates in sign as j is translated along a row or column in a cubical lattice while i is held fixed. This result remains true in the limiting case of the interparticle potential $-\varphi(i,j)$ becoming infinte when i and j are nearest neighbors (hard cube lattice gas).

The results we have obtained may be used to prove the existence of the thermodynamic limit of the correlation functions $\langle\rho_R\rangle$ in these systems under suitable boundary conditions. When Λ is a subset of an infinite lattice X and $\bar{A}$, $\bar{B}$ are parts of the A and B sublattices outside Λ we may consider the two boundary conditions in which all the sites in $\bar{A}\,(\bar{B})$ are occupied by particles and the sites in $\bar{B}\,(\bar{A})$ are empty. We then find that for any fixed R, $\langle n_R\rangle=\langle\prod_{i\in R}[\lambda_i\rho_i+\frac{1}{2}(1-\lambda_i)]\rangle$ decreases (increases) monotonically as $\Lambda\to\infty$ for any preassigned values of the chemical potential or external magnetic fields at all sites of X. This proves the existence of the thermodynamic limit of the $\langle n_R\rangle$ and hence also of the $\langle\rho_R\rangle$ (at least when the interparticle potentials are bounded). The limiting correlation functions need not coincide for the two different boundary conditions. They do however give bounds on the $\langle n_R\rangle$ obtained with any other boundary conditions.

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References

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