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Analyticity Properties of Regular and Random Systems*

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The thermodynamic and correlation functions of a classical system are analytic in the two variables β and z (the reciprocal temperature and fugacity respectively), if they are analytic in z at fixed β over a suitable range of values of β and z.[1] Similar results hold also when β is replaced by other variables such as those describing the action of external potentials. Using this and other known results it is possible to show for an Ising spin system with purely ferromagnetic interactions (or the corresponding lattice gas model) that (i) the free energy per site and the distribution functions $n_s(x_1, \cdots, x_s)$ are analytic in the two variables β and H if $\beta>0$ and the complex magnetic field H is not a limit point of zeros of the grand partition function Ξ and (ii) the Ursell functions $u_s(x_1, \cdots, x_s)$ tend to 0 as $\Delta_s = \max_{ij} |x_i - x_j| \to \infty$ if $\beta>0$ and $\mathscr{R}H \neq 0$; in particular, if the interaction potential vanishes for separations exceeding some fixed cutoff value λ, then $|u_s| < C \exp[(-\beta m|\mathscr{R}H| + \epsilon)\Delta_s/\lambda]$ where m is the magnetic moment of each spin, ϵ is any small positive number and C is a constant independent of Δ_s. One consequence of the result (i) is that a phase transition can occur as β is varied at constant H only if H is a limit point of zeros of Ξ (which can happen only if $\mathscr{R}H=0$); this supplements Lee and Yang's result that the same condition is necessary for a phase transition when H is varied at constant $\beta>0$. For a cutoff potential we also have that the thermodynamic susceptibility χ is given by $\sum_x u_2(x)$ of the infinite volume Ursell function for $\mathscr{R}H \neq 0$. Similar results hold for other thermodynamic quantities which can be expressed as fluctuations.

For a lattice gas with non-negative interaction potential, similar results are shown to hold provided $\beta>0$ and the complex fugacity z is less than the radius of convergence of the Mayer z expansion. Similar results are also proved for the continuum gas, but here n_s and u_s must be replaced by their values integrated over small volumes surrounding each of the points $x_2, \cdots, x_s$.

It is shown further that, except for continuum systems without hard cores, the pressure p, n_s and u_s have convergent Maclaurin expansions in β for small enough z.

Griffith and Lebowitz[2] have also investigated systems of spins (Heisenberg or Ising) on a lattice in which the various lattice sites are occupied at random. It is shown that the free energy per site exists in the limit of an infinite system and is a continuous function of concentration. For Ising systems some of the previous results about analyticity in β and H can now be extended also to the concentration variable p, the probability that a site is occupied by a spin. It is also proven that random Ising ferromagnets exhibit a spontaneous magnetization (in two and three dimensions) at sufficiently high concentrations and low temperatures. When the potential between spins at sites i and j $J_{ij}>0$ for all i and j spontaneous magnetization occurs far all values of p at sufficiently low temperatures.

We have also shown,[3] that the Laplace transform of the density of energy levels of a particle moving among randomly located scattering centers is an entire function of the density of scatterers, *i.e.* $Z(\beta, \rho) = \int_{-\infty}^{\infty} e^{-\beta E} n(E, \rho) dE$ is analytic in ρ in the whole complex ρ-plane. Here $n(E, \rho)$ is the average density of energy levels per unit volume of a particle whose Hamiltonian is

$$H = p^2/2m + \sum_j V(r - R_j)$$

where R_j is the position of a fixed scattering center. These centers are distributed at random with an average density ρ.

References

1) J. L. Lebowitz and O. Penrose: to be published in Comm. Math. Phys.
2) R. B. Griffith and J. L. Lebowitz: J. math. Phys. 9 (1968) 1284.
3) T. Burke and J. L. Lebowitz: to be published in J. math. Phys.

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DISCUSSION

R. Kubo: You said everything is analytic if the magnetic field is finite. Now, could you make a comment or a conjecture how singularities can appear when you make the magnetic field vanish?

J. L. Lebowitz: I cannot say very much about this. Our proof of analyticity in β covers all regions of the complex $\rho = e^{2\beta H}$ plane which is free of zeros of the partition function in the thermodynamic limit. This includes arcs of the circle $|\rho| = 1$. It follows from this that at the critical temperature β_c found by Onsager for $H=0$ the zeros in ρ-plane must converge on the real axis. For small values of β and Miracle-Sole have proven analyticity in β for all Gallavotti $\rho \geq 0$.

E. J. Verboven: In the algebraic approach of the two dimensional Ising model, the limiting property

$$\lim_{|i-j|\to\infty} |\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle| \to 0$$

is easily demonstrated for $H \neq 0$. This is the so called strong cluster property. It is equivalent with the analyticity of the state. For $H=0$ this property is not valid any more on and below the critical temperature, which corresponds to a phase transition. The state may be written as a superposition of two states, each corresponding to a pure phase, for which the property is valid.

Reference G. G. Emch, H. J. F. Knops and E. J. Verboven: Comm. math. Phys. 8 (1968) 300, and forthcoming papers.