of the quantized version of the classical sound field.[7] Then

$$\mathbf{k}\cdot\nabla_1\psi_0=(\mathbf{k}\cdot\nabla_1\varphi_0)\exp[\tfrac{1}{2}\sum_{i<j}\delta U(r_{ij})]$$
$$+\tfrac{1}{2}(\sum_j \mathbf{k}\cdot\nabla_1\delta U(r_{1j}))\psi_0. \quad (27)$$

The analysis and conclusions represented by Eqs. (16)–(25) are still valid if (a) δU is identified with U in Eq. (20) with $b=mc/\pi^2\hbar\rho$ and (b) the summands in $C(r_{12}, \cos(\mathbf{r}_{12},\mathbf{k}))$ generated by $\mathbf{k}\cdot\nabla_1\varphi_0$ and $\mathbf{k}\cdot\nabla_2\varphi_0$ contribute nothing to the leading term in the asymptotic formula of Eq. (12). Statements (a) and (b) are highly plausible sufficient conditions for the variance of H to be small in the sense defined by Eq. (24). To avoid possible misunderstanding, we state explicitly that φ_0 is certainly not a product function of the type defined by Eq. (15).

PHYSICAL REVIEW VOLUME 171, NUMBER 1 5 JULY 1968

Time Evolution of the Total Distribution Function of a One-Dimensional System of Hard Rods*

J. L. Lebowitz
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

AND

J. K. Percus
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

AND

J. Sykes
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 21 December 1967)

We continue our investigation of the time evolution of a one-dimensional system of hard rods. At $t=0$ there is one particle with a specified position r' and velocity v', and the remainder are in "equilibrium." Since in this system collisions merely interchange velocities, the "equilibrium" velocity distribution $h_0(v)$ need not be Maxwellian. Exact solutions are obtained for the time-dependent one-particle position-velocity distribution function $f(r-r', v, t/v')$. We investigate in particular the averaged positional part of f, viz., $G(r-r', t)$, which is the time-dependent pair correlation function whose space-time Fourier transform $S(k,\omega)$ describes coherent neutron scattering in realistic systems. It is shown that $S(k,\omega)$ does not generally contain modes corresponding to sound propagation. The exceptions are systems with discrete velocity distributions. In the latter case the space Fourier transform $\chi(k,t)$ of $G(r,t)$ is rigorously a sum of simple damped oscillations. An exact kinetic equation for the time evolution of f is derived and investigated. Also found is an approximate kinetic equation which, however, gives exact values of $S(k,\omega)$.

I. INTRODUCTION

In a previous paper[1] we investigated some of the nonequilibrium properties of a one-dimensional system of hard rods. We were particularly interested in the time evolution of the self-distribution $f_s(r,v,t/v')$. This is the distribution function of a labeled (test) particle of the system starting, at $t=0$, from the origin with a velocity v'. The system is assumed to be in equilibrium at $t=0$, subject to this restriction on the test particle. Since in this system collisions merely interchange velocities, "equilibrium" corresponds to a random distribution of particle positions (with the restriction that the distance between the centers be larger than a, the hard rod diameter), and a velocity distribution which is a product of individual particle velocity distributions $h_0(v)$, with $h_0(v)$ an arbitrary, positive, even function of v normalized to unity. When the labeled (test) particle has itself a velocity distribution $h_0(v)$, then its position distribution at time t,

$$G_s(r,t)=\int dv\int dv' f_s(r,v,t/v')h_0(v'), \quad (1.1)$$

has the significance of the time-dependent self-distribution function introduced by Van Hove for neutron scattering.[2]

In this paper, we shall be interested in the total one-particle distribution function of the system under the same initial conditions as before. This function $f(r,v,t/v')$ is the density of particles at position r with

* Supported by the U. S. Air Force Office of Scientific Research under Grant Nos. AF-AFOSR-508-66 and 945-65 and by the U. S. Atomic Energy Commission, Contract No. AT(30-1)-1480.

[1] J. K. Percus, Bull. Am. Phys. Soc. 10, 722 (1965); J. L. Lebowitz and J. K. Percus, Phys. Rev. 155, 122 (1967).

[2] L. Van Hove, Phys. Rev. 95, 249 (1954).

velocity v when one particle was initially at the origin with velocity v'. In terms of the symmetric N-particle Gibbs ensemble distribution function $\mu(x_1,\cdots,x_N;t/y_0)$, x_i standing for the position and velocity coordinates of the ith particle (we think here of a system of N particles on a line of length L and later take the thermodynamic limit $N\to\infty$, $L\to\infty$, $\rho=N/L$ fixed),

$$f(y_1,t/y_0)=N\int dx_1\cdots\int dx_N \times\mu(x_1,\cdots,x_N;t/y_0)\delta(x_1-y_1), \quad (1.2)$$

where $y_1=(r_1,v_1)$, $y_0=(0,v_0)$. The distribution μ satisfies the Liouville equation with the initial condition

$$\mu(x_1,\cdots,x_N;0/y_0)=\sum_{i=1}^{N}\mu_0(x_1,\cdots,x_N)\delta(x_i-y_0)/f_1(y_0), \quad (1.3)$$

where

$$f_1(y_0)=N\int dx_1\cdots\int dx_N\mu_0(x_1,\cdots,x_N)\delta(x_i-y_0) \quad (1.4)$$

and $\mu_0(x_1,\cdots,x_N)$ is the equilibrium ensemble density. If $\mu_i(x_1,\cdots,x_N;t/y_0)$ is the distribution at time t with the initial condition

$$\mu_i(x_1,\cdots,x_N;0/y_0)=N\mu_0(x_1,\cdots,x_N)\times\delta(x_i-y_0)/f_1(y_0), \quad (1.5)$$

then

$$f(y_1,t/y_0)=\sum_{i=1}^{N}\int dx_2\cdots\int dx_N\mu_i(y_1,x_2,\cdots,x_N;t/y_0) \quad (1.6)$$

$$=f_s(y_1,t/y_0)+f_d(y_1,t/y_0), \quad (1.7)$$

where

$$\begin{aligned} f_s(y_1,t/y_0)&=\int dx_2\cdots\int dx_N\mu_1(y_1,x_2,\cdots,x_N;t/y_0),\\ f_d(y_1,t/y_0)&=\sum_{i=2}^{N}\int dx_2\cdots\int dx_N\mu_i(y_1,x_2,\cdots,x_N;t/y_0)\\ &=(N-1)\int dx_2\cdots\int dx_N\mu_2(y_1,x_2,\cdots,x_N;t/y_0). \end{aligned} \quad (1.8)$$

The position distribution function $G(r,t)$, that is, the density at r at time t when the particle at the origin at $t=0$ has the same velocity distribution $h_0(v)$ as the other particles,

$$G(r,t)=\int dv\int dv' f(r,v,t/v')h_0(v'), \quad (1.9)$$

is the time-dependent spatial correlation whose space-time Fourier transform $S(k,\omega)$ gives the coherent neutron scattering,[2]

$$S(k,\omega)=\frac{1}{2\pi}\int dr\int dt\, e^{ikr-i\omega t}\{G(r,t)-\rho\}. \quad (1.10)$$

In Eq. (1.10), $G(r,t)=G(r,-t)$ and ρ is the average number density which $G(r,t)$ must approach as $r\to\infty$ or $t\to\infty$. For $t=0$

$$\begin{aligned} f(r,v,0/v')&=\delta(v-v')\delta(r)+\rho g(r)h_0(v)\\ &=\rho h_0(v)+\delta(v-v')\delta(r)+\rho\{g(r)-1\}h_0(v) \end{aligned} \quad (1.11)$$

and

$$G(r,0)=\delta(r)+\rho g(r), \quad (1.12)$$

where $g(r)$ is the equilibrium radial distribution function. $g(r)\to 1$ for distances beyond the molecular correlation length which, away from phase transitions, is of the same order as the range of the interparticle potential—this is equal to a for hard rods. The time evolution of $G(r,t)$ thus represents the dissipation of a microscopic density fluctuation.

A most striking result that we find in the study of the one-dimensional hard-rod system relates to the case where $h_0(v)$ is the sum of two δ functions,

$$h_0(v)=\tfrac{1}{2}\{\delta(v-c)+\delta(v+c)\}, \quad (1.13)$$

when, in the thermodynamic limit,

$$\chi(k,t)=\int dr\, e^{ikr}\{G(r,t)-\rho\} \quad (1.14)$$

satisfies rigorously a simple damped harmonic oscillator equation

$$\partial^2\chi(k,t)/\partial t^2+\Omega^2(k)\chi(k,t)+R(k)\partial\chi(k,t)/\partial t=0. \quad (1.15)$$

Here we have

$$\Omega^2(k)=c^2k^2\chi^{-1}(k,0), \quad (1.16)$$

with

$$(\chi k,0)=1+\rho\int dr\, e^{ikr}\{g(r)-1\},$$

and

$$R(k)=2nc(1-\cos ka)=c^2(1-\cos ka)/D, \quad (1.17)$$

where

$$n=\rho/(1-\rho a)=\rho g(a), \quad D=c/2n. \quad (1.18)$$

D is the diffusion constant.

When the velocity distribution function consists of a sum of δ functions at velocities $\pm c_j$, $j=1,\cdots,n$, then $\chi(k,t)$ is the sum of n damped oscillatory terms satisfying a partial differential equation of order $2n$. In the particular case where $n=2$, $c_1=0$, and $c_2=c\neq 0$, we obtain one diffusional mode (zero frequency) and one damped sound mode. For a continuous $h_0(v)$, e.g., a Maxwellian distribution of velocities, there are no simple damped oscillations in $\chi(k,t)$.

Equation (1.15), which is a rigorous irreversible equation arising from reversible dynamical equations in the thermodynamic limit, is of striking simplicity. We shall now discuss briefly the structure of $\Omega^2(k)$ and $R(k)$ to see if an equation like (1.15) can be used as an approximation for $\chi(k,t)$ in higher dimensions. First, for small values of k,

$$\chi^{-1}(k,0)\sim d\beta p/d\rho=(1+na)^2=(1-\rho a)^{-2},$$

where p is the pressure and $\beta^{-1}=\langle v^2\rangle=c^2$ for this system, so that

$$\Omega(k)\rightarrow uk \text{ as } k\rightarrow 0,$$

with u the isothermal speed of sound (the mass of the particles has been set to unity here). The form of $\Omega^2(k)$ has a simple intuitive interpretation even for large k. As shown elsewhere,[3]

$$\chi^{-1}(k,0)=1-\rho C(k)=\delta^2\beta A/\delta\rho_k\delta\rho_{-k}, \tag{1.19}$$

where $C(k)$ is the direct correlation function, A is the Helmholtz free energy in an equilibrium system considered as a function of the nonuniform density $\rho(r)$ (produced by an external potential) whose Fourier transform is

$$\rho_k=\langle\hat{\rho}_k\rangle'=\langle\sum_j e^{ikr_j}\rangle'.$$

$\hat{\rho}_k$ is the Fourier transform of the microscopic density operator and the prime indicates that the average is taken over a nonuniform ensemble. $\chi(k,t)$ itself may of course also be written as

$$\chi(k,t)=(1/N)\langle\hat{\rho}_k(t)\hat{\rho}_{-k}(0)\rangle$$
$$=\frac{1}{N}\sum_{i,j}\langle e^{ik[r_i(t)-r_j(0)]}\rangle,\quad (k\neq 0), \tag{1.20}$$

the average now being taken over a uniform equilibrium ensemble. We may thus think of the increase in the free energy, due to the nonuniform density,

$$\delta\beta A=\tfrac{1}{2}\sum_k\{1-\rho C(k)\}\delta\rho_k\delta\rho_{-k}, \tag{1.21}$$

as providing the restoring force for bringing the system back to uniformity. [The term $c^2k^2=\langle v^2\rangle k^2$ in $\Omega^2(k)$ is essentially the reciprocal of the mass associated with the kth normal mode.]

The damping term $R(k)$ is much more difficult to interpret in a way in which it can be generalized to other systems. First we note that when k is equal to an integral multiple of $2\pi/a$, $R(k)$ vanishes and $\chi(k,t)$ has a nondecaying oscillatory behavior. This is certainly an artifact of the model [indeed, it is not true when $h_0(v)$ is different from a δ function]. For small k, $R(k)\sim c^2k^2a^2/2D$, which is to be compared with the sound damping $(\frac{4}{3}\eta+\zeta)k^2$ obtained from hydrodynamics in three dimensions, where η is the coefficient of shear viscosity and ζ of bulk viscosity.[4]

A similar situation arises in the analysis of the self-part of $\chi(k,t)$, $\chi_s(k,t)$, related to incoherent neutron scattering.[2] For this model, with the δ-function velocity distribution, $\chi_s(k,t)$ also satisfies a damped oscillator equation with

$$\Omega_s^2(k)=c^2k^2,\quad R_s(k)=c^2/D. \tag{1.22}$$

One way of generalizing the frequency and friction coefficients is to consider the behavior of $\chi(k,t)$ [and $\chi_s(k,t)$] for short times. When the interaction potential between the particles is continuous, all odd derivatives of $\chi(k,t)$ vanish as $t\rightarrow 0$. For hard-core potentials, on the other hand, $\partial^3\chi(k,0+)/\partial t^3\neq 0$, which together with the general result that $\partial^2\chi(k,0+)/\partial t^2=-k^2\langle v^2\rangle$ can be used to determine $\Omega(k)$ [leading again to Eq. (1.16)] and $R(k)$. Such determination of $R(k)$ does not, however, lead in general to the correct asymptotic behavior of $\chi(k,t)$ for large t. Indeed, the assumption that $\chi_s(k,t)$ satisfies the damped harmonic-oscillator equation (1.15) with R_s and Ω_s defined in (1.22) implies that the velocity autocorrelation function $\Psi(t)$ has a simple exponential decay.

The general form of the time-dependent velocity and spatial distributions for the one-dimensional system of hard rods is derived in Sec. II. The spatial part is then computed explicitly for some forms of $h_0(v)$ in Sec. III. In Sec. IV we discuss the asymptotic form of $S(k,\omega)$ for small k and ω and show that it will have hydrodynamical type of behavior only for the case where $h_0(v)$ contains some δ functions. There is thus no hydrodynamical regime for a one-dimensional system of hard rods with a Maxwellian distribution of velocities. In Sec. V we derive a kinetic equation for the time evolution of $f(r,v,t)$, the total distribution at time t when there was initially one particle with an arbitrary distribution of position and velocity and the rest had their equilibrium distribution. Due to the initial conditions, this equation is linear in f and has the property, since collisions merely interchange velocities, that $h(v,t)=\int dr\, f(r,v,t)$ is independent of time. We also consider low density and "molecular chaos" type of approximations to $\partial f/\partial t$ and show that the latter yields the exact $S(k,\omega)$.

II. FORM OF $f(r,v,t/v')$ AND $S(k,\omega)$

In the derivation[1] of $f_s(r,v,t/v')$ use was made of the fact that f_s depends on the density ρ and the diameter a of the rods only through the combination $n=\rho/(1-\rho a)$,

[3] J. K. Percus, in *The Equilibrium Theory of Classical Fluids*, edited by H. L. Frisch and J. L. Lebowitz (W. A. Benjamin, Inc., New York, 1964), p. II-61.

[4] L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) **24**, 419 (1963).

i.e., the only length appearing in the problem is the effective interparticle distance given by $\rho^{-1}-a=n^{-1}$. Thus f_s has the same form for finite a as it would have for $a=0$, impenetrable point particles. When $a=0$, the dynamics of the whole system is identical, except for the relabelling of particles during a collision, to the dynamics of an ideal gas where the particles pass each other without interaction. We thus had a simple alogarithm, first developed by Jepsen,[5] for finding f_s. The situation is a bit more complicated in computing f. Here we have to take into account explicitly the fact that in a collision of particles with diameter a having velocities v and v' there is a jump, by displacement $\pm a$, in the locations of the particles with these velocities. Indeed, this is the only effect which makes the time evolution of f different from that of an ideal gas. For $a=0$, there is no such difference and

$$f(r,v,t/v')=\rho h_0(v)+\delta(r-v't)\delta(v-v').$$

The hard-core system is in fact most readily analyzed by setting up a correspondence between it and the trivial point core system. Suppose that we consider a rigid box from O to $L-Na$, containing N, free particles $X_1,\cdots,X_N$. If $X(j)$ denotes the jth ordered particle, then the $X(j)$ move as a set of zero-diameter hard cores. Now define

$$x_j=X(j)+(j-\tfrac{1}{2})a \tag{2.1}$$

in a box from O to L. The only effect has been to place an additional space between successive particles

$$x_{j+1}-x_j=X(j+1)-X(j)+a \tag{2.2}$$

(and $\frac{1}{2}a$ for each wall). The irreducible length a thus represents a hard core, and corresponding to each dynamical state of the cores, x_j, is a dynamical state of the coreless $X(j)$ and hence a state of the free particles X_j. Our only problem is that of establishing the order or relative position of a labeled free particle X_j. But this is just one more than the number of particles to the left of X_j; hence setting

$$\begin{aligned}\epsilon(x)&=1 \text{ for } x>0\\ &=\tfrac{1}{2} \text{ for } x=0\\ &=0 \text{ for } x<0,\end{aligned} \tag{2.3}$$

we have

$$\text{order of } X_j=\sum_{s=1}^{N}\epsilon(X_j-X_s)+\tfrac{1}{2}. \tag{2.4}$$

We conclude that the relation

$$x_j=X_j+a\sum_{s=1}^{N}\epsilon(X_j-X_s) \tag{2.5}$$

transforms the set of free particles to the set of core center positions.

The evaluation of $f(r,v,t/v')$ is most easily performed in Fourier k space. In taking the thermodynamic limit it is more convenient to define

$$f(r,v,t/v')=\frac{1}{Nh_0(v')}\int dR\, f_2(R+\tfrac{1}{2}r,\, v,\, t;\, R-\tfrac{1}{2}r,\, v',\, 0), \tag{2.6}$$

i.e., to average over the center of mass. Since the two-particle distribution function is given by

$$f_2(r,v,t;\,r',v',t')=\sum_{i,j}\langle\delta(r-x_i(t))\delta(r'-x_j(t'))\times\delta(v-v_i(t))\delta(v'-v_j(t'))\rangle, \tag{2.7}$$

we have

$$f(k,v,t/v')=\int dr\, e^{ikr}f(r,v,t/v')=\frac{1}{Nh_0(v')}\sum_{i,j}\times\langle e^{ik[x_i(t)-x_j(0)]}\delta(v-v_i(t))\delta(v'-v_j(0))\rangle \tag{2.8}$$

or by virtue of (2.5)

$$f(k,v,t/v')=\frac{1}{Nh_0(v')}\sum_{i,j}\langle e^{ik[X_i(t)-X_j(0)]}\times\exp\{ika\sum_{s=1}^{N}[\epsilon(X_i(t)-X_s(t))-\epsilon(X_j(0)-X_s(0))]\}\times\delta(v-V_i(t))\delta(v'-V_j(0))\rangle. \tag{2.9}$$

If we insert the free-particle equations of motion

$$X_i(t)=X_i+V_it,\quad V_i(t)=V_i$$

[where $X_i(0)=X_i$, $V_i(0)=V_i$], and separate the $i=j$ and $i\neq j$ terms, and then choose $i=1$, $j=2$, Eq. (2.9) becomes

$$f(k,v,t/v')=\frac{\delta(v-v')}{h_0(v')}\langle e^{ikvt}\exp\{ika\sum_{s=2}^{N}[\epsilon(X_1-X_s+t(v-V_s))-\epsilon(X_1-X_s)]\}\delta(v-V_1)\rangle+\frac{(N-1)}{h_0(v')}\langle e^{ik(X_1-X_2+vt)}\times\exp\{ika\sum_{s=3}^{N}[\epsilon(X_1-X_s+t(v-V_s))-\epsilon(X_2-X_s)]\}\exp\{ika[\epsilon(X_1-X_2+t(v-v'))-\epsilon(X_2-X_1)]\}\times\delta(v-V_1)\delta(v'-V_2)\rangle. \tag{2.10}$$

[5] D. W. Jepsen, J. Math. Phys. 6, 405 (1965).

The expectation with respect to particle s in (2.10) is of the form

$$\begin{aligned}\langle\exp\{ika[\epsilon(y-X_s-tV_s)-\epsilon(x-X_s)]\}\rangle_{X_s,V_s}\\ =\langle 1+i\sin ka[\epsilon(y-X_s-tV_s)-\epsilon(x-X_s)]\\ -(1-\cos ka)\epsilon(-(y-X_s-tV_s)(x-X_s))\rangle_{X_s,V_s}\\ =\langle 1\rangle+\frac{i\sin ka}{L-Na}\langle y-x-tV_s\rangle_{V_s}\\ -\frac{(1-\cos ka)}{L-Na}\langle|y-x-tV_s|\rangle_{V_s}. \quad (2.11)\end{aligned}$$

If we assume that $\langle V_s\rangle_{V_s}=0$ and define

$$\mu(w)=\langle|w-V|\rangle_V=\int dV\, h_0(V)|w-V|, \quad (2.12)$$

(2.11) becomes finally

$$1+\frac{i\sin ka}{L-Na}(y-x)-\frac{(1-\cos ka)}{L-Na}|t|\mu\left(\frac{y-x}{t}\right). \quad (2.13)$$

It follows that as $N\to\infty$ with $N/L=\rho$, (2.10) reduces to

$$\begin{aligned}f(k,v,t/v')=&\frac{\delta(v-v')}{h_0(v')}e^{ikvt}\langle\delta(v-V_1)\ \exp\{n[itv\sin ka-|t|\mu(v)(1-\cos ka)]\}\rangle_{X_1,V_1}+\frac{N}{h_0(v')}\langle\delta(v-V_1)\delta(v'-V_2)e^{ik(X_1-X_2+vt)}\\ &\times\exp\{n[i(X_1-X_2+vt)\sin ka-|t|\mu(v+(X_1-X_2)/t)(1-\cos ka)]\}\\ &\times\exp\{ika[\epsilon(X_1-X_2+t(v-v'))-\epsilon(X_2-X_1)]\}\rangle_{X_1,V_1;X_2,V_2} \quad (2.14)\end{aligned}$$

or, performing the final integrations, for $t>0$

$$\begin{aligned}f(k,v,t/v')=&\delta(v-v')e^{iktv}F(nt,ka,v)\\ &+nh_0(v)\int_{-\infty}^{\infty}dX\, e^{ikX}F(nt,ka,X/t)\\ &\times\exp\{ika[\epsilon(X/t-v')-\epsilon(v-X/t)]\}, \quad (2.15)\end{aligned}$$

where

$$F(\tau,ka,v)=e^{-\tau[\mu(v)(1-\cos ka)-iv\sin ka]}. \quad (2.16)$$

The first term in (2.15) has a very simple interpretation in (r,v) space where the term multiplying $\delta(v-v')$ can be written in the form

$$\sum_{j=-\infty}^{\infty}P_j(v,t)\delta(r-vt-ja).$$

$P_j(v,t)$ is the probability that a particle starting from the origin with velocity v will have transferred at time t its velocity to its jth neighbor. Similarly, the second term can be decomposed into jth neighbor pairs.

The Laplace transform of (2.15) yields

$$\begin{aligned}\mathcal{T}(k,s;v,v')=&\int_0^{\infty}dt\, e^{-st}f(k,v,t/v')=\delta(v-v')A(v,s)\\ &+nh_0(v)\int_{-\infty}^{\infty}dw\, A^2(w,s)e^{ika[\epsilon(w-v')-\epsilon(v-w)]}, \quad (2.17)\end{aligned}$$

where

$$A(v,s)=[s-i\beta(k)v+\alpha(k)\mu(v)]^{-1} \quad (2.18)$$

and

$$\alpha(k)=n(1-\cos ka), \quad \beta(k)=k+n\sin ka. \quad (2.19)$$

The space-time distribution $G(r,t)$ is defined in (1.9). Its Fourier space and Laplace time transforms are given by[6]

$$\begin{aligned}\chi(k,t)=&\int dr\, e^{ikr}G(r,t)\\ =&\int dv\int dv' f(k,v,t/v')h_0(v') \quad (2.20)\end{aligned}$$

and

$$\begin{aligned}\tilde{\chi}(k,s)=&\int_0^{\infty}dt\, e^{-st}\chi(k,t)\\ =&\int dv\int dv'\mathcal{T}(k,s;v,v')h_0(v'). \quad (2.21)\end{aligned}$$

Using (2.15) and (2.17), we find after some manipulations

$$\chi(k,t)=-k^2\int_{-\infty}^{\infty}\frac{dv\, h_0(v)e^{i\beta(k)vt-\alpha(k)\mu(v)t}}{[\alpha(k)\mu'(v)-i\beta(k)]^2} \quad (2.22)$$

and

$$\begin{aligned}\tilde{\chi}(k,s)=&-k^2\int_{-\infty}^{\infty}dv\\ &\times\frac{h_0(v)}{[\alpha(k)\mu'(v)-i\beta(k)]^2[s-i\beta(k)v+\alpha(k)\mu(v)]}\\ =&\frac{k^2}{2\alpha(k)}\int_{-\infty}^{\infty}\frac{dv}{[s-i\beta(k)v+\alpha(k)\mu(v)]^2}, \quad (2.23)\end{aligned}$$

where use has been made of the relation

$$\mu''(v)=2h_0(v). \quad (2.24)$$

When $h_0(v)$ has a δ-function peak at some velocity $v=c$ so that $\mu'(v)$ is discontinuous at c, the denominator in (2.22) and (2.23) is to be interpreted as the product

$$[\alpha(k)\mu'(c+)-i\beta(k)][\alpha(k)\mu'(c-)-i\beta(k)].$$

[6] See note following Eq. (5.5).

The coherent scattering function $S(k,\omega)$ is given by

$$S(k,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\chi(k,|t|) =(1/\pi)\,\mathrm{Re}\tilde{\chi}(k,i\omega). \quad (2.25)$$

It is seen from (2.22) and (2.23) that χ will have a certain kind of semi-periodicity in k with period $2\pi/a$ since $\alpha(k)$ and $\beta(k)-k$ are periodic in k. Furthermore, when $k=2\pi j/a$ with j an integer (unequal to 0), $\alpha(k)=0$ and $\chi(k,t)$ may have no damping for some $h_0(v)$ (cf. next section). This is almost certainly an artifact of the one-dimensional hard-rod system where the equilibrium

$$\chi(k,0)=k^2/[\alpha^2(k)+\beta^2(k)] \quad (2.26)$$

has the same properties.

The function $\mu(v)$ together with the velocity-independent functions $\alpha(k)$ and $\beta(k)$ determine the time evolution of $\chi(k,t)$. Physically, $n\mu(v)$ is the rate at which a particle with velocity v will have collisions with its neighbors when these are in equilibrium, i.e., have a velocity distribution $h_0(v)$ and a density at contact $n=\rho g(a)$. We have no similar direct interpretation of $\alpha(k)$ and $\beta(k)$. It is true that if $\epsilon(s)$ is the Laplace transform of the spatial correlation $g(r)$, then[7]

$$\epsilon(ik)=(n/\rho)[e^{-ika}/(\alpha+i\beta)],$$

but there is no obvious extension of this result to three dimensions.

III. EXPLICIT FORM OF $S(k,\omega)$ FOR SPECIAL DISTRIBUTIONS

As seen from Eqs. (2.22) and (2.23), the form of χ is determined by $\mu(v)$. Before discussing some general properties of χ, we shall give here explicit expressions and/or graphs of χ for several different velocity distributions $h_0(v)$. The simplest of these is a discrete distribution where the particles can only have velocities $\pm c$. We have

$$h_0(v)=\tfrac{1}{2}[\delta(v+c)+\delta(v-c)], \quad (3.1)$$

$$\mu(v)=\max[c,|v|], \quad (3.2)$$

and

$$\chi(k,t)=\chi(k,0)e^{-\alpha(k)ct}\{\cos\beta(k)ct +[\alpha(k)/\beta(k)]\sin\beta(k)ct\}, \quad (3.3)$$

where $\chi(k,0)$ is defined in (2.26). The corresponding $\tilde{\chi}(k,s)$ and coherent scattering function are given by

$$\tilde{\chi}(k,s)=\chi(k,0)\frac{s+2\alpha(k)c}{[s+\alpha(k)c]^2+\beta^2(k)c^2} \quad (3.4)$$

and

$$S(k,\omega)=2k^2c^3\alpha(k)\pi^{-1}[\{\omega^2+c^2[\alpha^2(k)-\beta^2(k)]\}^2 +4c^4\alpha^2(k)\beta^2(k)]^{-1}. \quad (3.5)$$

The more general discrete velocity distribution

$$h_0(v)=\tfrac{1}{2}\sum_{j=1}^{n}A_j[\delta(v+c_j)+\delta(v-c_j)] \quad (3.6)$$

is dealt with in Appendix A and we only write down here the result for the case $n=2$ with $c_1=0$ and $c_2=c$. For this case there is just one diffusion mode in addition to the oscillatory mode:

$$\chi(k,t)=\mathrm{Re}[D_1(k)e^{-\lambda_1(k)t}+D_2(k)e^{i\omega_2(k)t-\lambda_2(k)t}] \quad (3.7)$$

and

$$S(k,\omega)=\frac{1}{4\pi}\Bigg[\frac{2\lambda_1(D_1+D_1^*)}{\lambda_1^2+\omega^2} +\frac{(D_2+D_2^*)\lambda_2+i(D_2-D_2^*)(\omega+\omega_2)}{\lambda_2^2+(\omega+\omega_2)^2} +\frac{(D_2+D_2^*)\lambda_2-i(D_2-D_2^*)(\omega-\omega_2)}{\lambda_2^2+(\omega-\omega_2)^2}\Bigg]. \quad (3.8)$$

λ_1, λ_2, ω_2, D_1, and D_2 are defined in Appendix A. $S(k,\omega)$, calculated from (3.8), is shown in Figs. 1 and 2 with the parameters

$$\mu(0)=A_2c=1,\quad n=a=1,\quad A_2=\tfrac{2}{3},$$

so that

$$c=\tfrac{3}{2},\quad A_1=\tfrac{1}{3},\quad \rho=\tfrac{1}{2}.$$

A very different kind of distribution, which was also investigated in Ref. 1, is a very long-ranged $\frac{3}{2}$ power distribution

$$h_0(v)=\tfrac{1}{2}c^2(c^2+v^2)^{-3/2} \quad (3.9)$$

for which

$$\mu(v)=(c^2+v^2)^{1/2}. \quad (3.10)$$

For this distribution the mean kinetic energy is infinite but the diffusion constant which depends on $\langle|v|\rangle$ exists. We find

$$\chi(k,t)=\chi(k,0)ct[\alpha^2(k)+\beta^2(k)]^{1/2} \times K_1\{ct[\alpha^2(k)+\beta^2(k)]^{1/2}\}, \quad (3.11)$$

where K_1 is the modified Bessel function which has an exponential decay and no oscillatory character. Similarly,

$$S(k,\omega)=\frac{k^2}{[\alpha^2(k)+\beta^2(k)]^{3/2}}h_0\left(\frac{\omega}{[\alpha^2(k)+\beta^2(k)]^{1/2}}\right). \quad (3.12)$$

For the Maxwellian distribution χ cannot be calculated analytically. The results of numerical computation are given in Fig. 3. In these graphs the variables have been chosen in the following way:

$$\mu(0)=1,\quad n=a=1.$$

IV. GENERAL FORM OF $S(k,\omega)$

The general form of $S(k,\omega)$ for real three-dimensional systems has been the subject of many experimental and

[7] E. H. Lieb and D. C. Mattis, *Mathematical Physics in One Dimension* (Academic Press Inc., New York, 1966), p. 7.

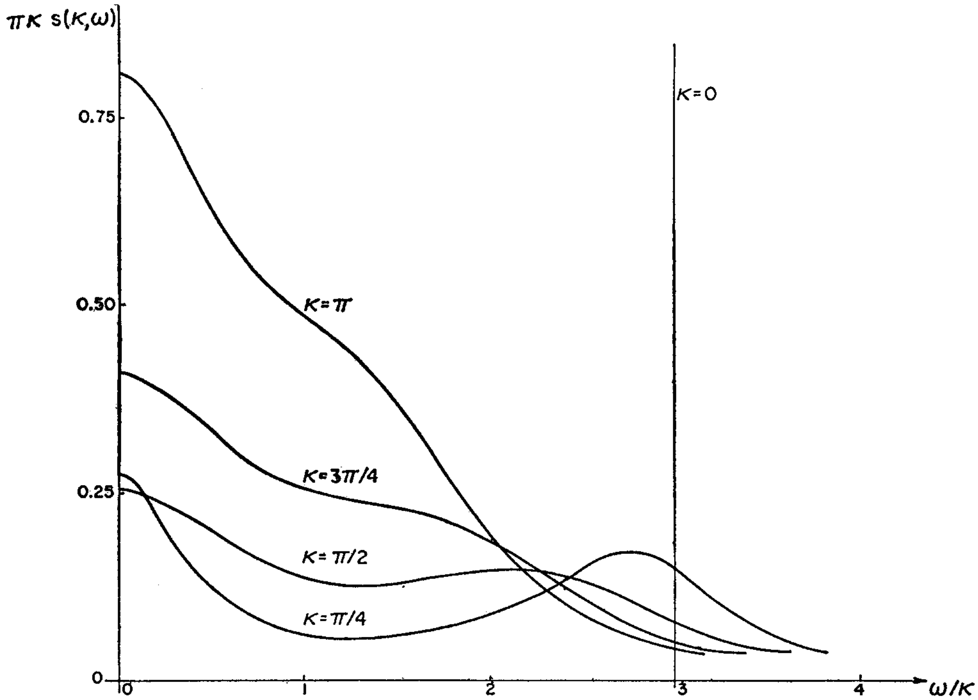

FIG. 1. $\pi kS(k,\omega)$ as a function of ω/k at different values of k for the velocity distribution function $h_0(v)=\frac{1}{3}\delta(v+\frac{3}{2})+\frac{1}{3}\delta(v)+\frac{1}{3}\delta(v-\frac{3}{2})$ and $n=a=1$.

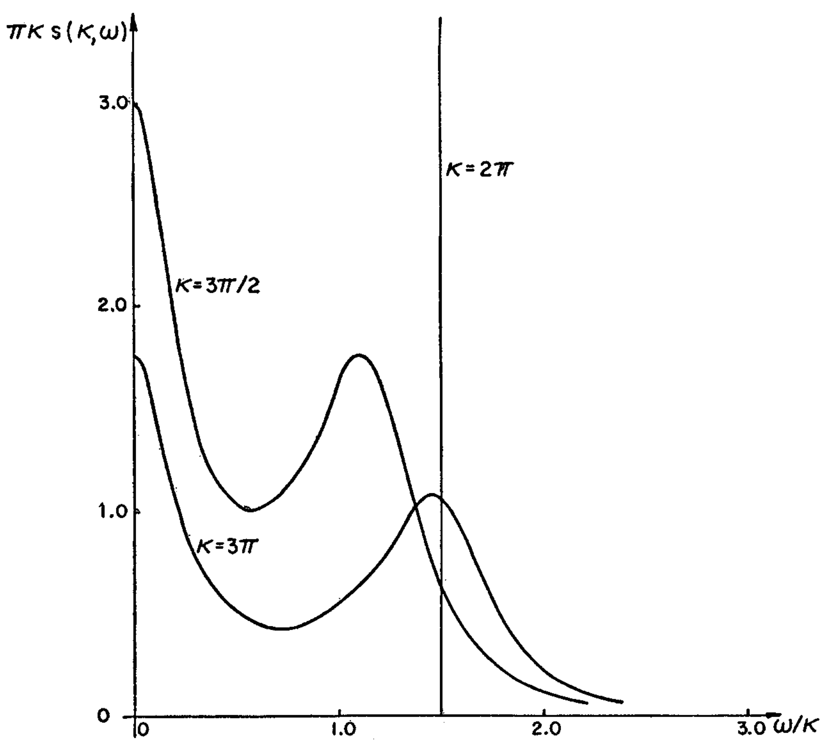

FIG. 2. $\pi kS(k,\omega)$ as a function of ω/k at different values of k for the velocity distribution function $h_0(v)=\frac{1}{3}\delta(v+\frac{3}{2})+\frac{1}{3}\delta(v)+\frac{1}{3}\delta(v-\frac{3}{2})$ and $n=a=1$.

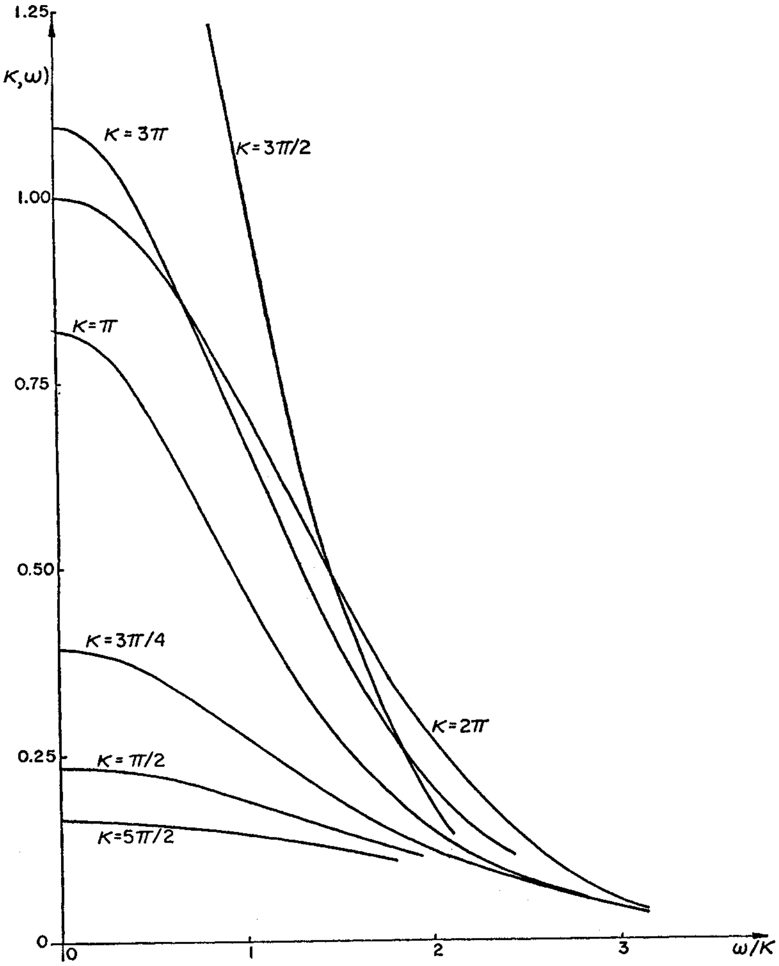

FIG. 3. $\pi k S(k,\omega)$ as a function of ω/k at different values of k for the velocity distribution function $h_0(v)=(1/\pi)e^{-v^2/\pi}$ and $n=a=1$.

theoretical investigations.[8] Since the exact form of $S(k,\omega)$ is unobtainable for real systems or even for any two- or three-dimensional "model" fluids, various approximations of uncertain validity have been used in these investigations. Of particular interest has been the form of $S(k,\omega)$ for small values of k and ω (Ref. 4) (k^{-1} large compared with the "mean free path," ω^{-1} large compared with the velocity relaxation time, and ω/k fixed). The laws of macroscopic hydrodynamics are believed to govern the behavior of these "slowly varying" components. These laws result in a damped oscillatory sound mode and a diffusion mode.

For the one-dimensional system of hard rods there are no hydrodynamical equations [although the self-distribution $G_s(r,t)$ does obey the diffusion equation in the hydrodynamical limit]. The form of $S(k,\omega)$ depends very essentially, as we have seen, on the nature of the velocity distribution $h_0(v)$. The existence of a damped oscillatory behavior of $\chi(k,t)$ appears to require δ functions in $h_0(v)$, as can be seen most readily from (2.23).

From its definition $\mu(v)$ is an even, positive, convex function of v rising monotonically from $\mu(0)=\langle|W|\rangle_W=2nD$ (D is the diffusion constant) to $|v|$ as $|v|\to\infty$. Hence from (2.23) (k,s) is analytic, in the complex s plane, to the right of the curve $l(v)$ defined by

$$\begin{aligned}\mathrm{Im}s&=\beta(k)v\,,\\ \mathrm{Re}s&=-\alpha(k)\mu(v)\leq-\alpha(k)\mu(0)\,.\end{aligned}\tag{4.1}$$

Now if $h_0(v)=0$ for some range of v, $\tilde{\chi}(k,s)$ can be extended to the left of l and will be analytic in the whole s plane when cuts are made along $l(v)$. [When $h_0(v)$ does not vanish identically for large $|v|$, the error made by cutting it off at some large velocity v_0 which leaves $\mu'(v)$ continuous can presumably be made arbitrarily small for $\mathrm{Re}s>0$ at $\alpha(k)\neq0$.] The existence of damped oscillatory behavior for $\chi(k,t)$ will then depend on

[8] (a) Analysis of liquid data: B. N. Brockhouse, Nuovo Cimento Suppl. 9, 45 (1958); A. Rahman, K. S. Singwi, and A. Sjolander, Phys. Rev. 122, 9 (1961); ibid. 126, 997 (1962); B. A. Dasannacharya and G. Venkataraman, ibid. 156, 196 (1967). (b) Computer experiments: A. Rahman, ibid. 136, A405 (1964); Phys. Rev. Letters 19, 420 (1967). (c) Kinetic equation theory: M. Nelkin and A. Ghatak, Phys. Rev. 135, A4 (1964); S. Yip and M. Nelkin, ibid. 135, A1241 (1964); J. M. J. Van Leeuwen and S. Yip, ibid. 139, A1138 (1965); R. C. Desai and M. Nelkin, Nucl. Sci. Engr. 24, 142 (1966).

whether $\tilde{\chi}(k,s)$ has poles on l. This in turn depends on whether $\mu'(v)$ has any discontinuities corresponding to a δ-function peak in $h_0(v)$.

When $h_0(v)$ consists of a sum of δ functions, $\chi(k,t)$ will be a sum of damped oscillators. Hence, when there is a δ function at $v=0$ and at $v=\pm c$, there will be just one diffusive and one "sound" mode and we obtain in the hydrodynamical limit

$$S(x,y)=\frac{1}{2\pi}\frac{n^2a^2}{(1+na)^2}\frac{A_2}{nc}\left\{\frac{A_1}{x^2+n^4a^4A_2{}^2/4y^2}+\frac{[(2+A_1)(1+na)^2-A_1x^2]}{[x^2-(1+na)^2]^2+n^3a^3[x^2(4+7na)-(1+na)^2(4+na)]/6y^2}\right\}, \quad (4.2)$$

with A_1 and A_2 the fraction of particles with $v=0$ and $|v|=c$,

$$A_1+A_2=1\,,$$

and

$$x=\omega/kc\,,\quad y=n/k\,.$$

The form of (4.2) is very similar to that obtained from hydrodynamics in three dimensions.[9] The reasons for the agreement of this distribution with hydrodynamical calculations and the lack of such behavior for "smooth" distributions are not entirely clear to us at present. The general form (2.23) represents a weighted distribution of frequencies and relaxation times along the curve l. The density of zeros is determined by the shape of l, particularly by its curvature. Now while the analytic character of $S(k,\omega)$ depends very much on the precise form of $h_0(v)$, it seems reasonable to expect that the general "shape" of $S(k,\omega)$ for real k and ω will only depend on the general shape of $h_0(v)$, i.e., any $h_0(v)$ with a maximum at $v=0$ and $v=\pm c$ will produce a similar shaped curve. It is perhaps possible to conjecture that the situation might be somewhat similar in higher dimensions where $\mu(|v|)$ is always singular at $v=0$ and $h_0(|v|)$, which is Maxwellian, has a peak at some characteristic speed dependent on the temperature, thus giving rise to the characteristic hydrodynamical peaks.

V. KINETIC EQUATION FOR f

As noted in the Introduction, the only effect of collisions is to interchange the velocities of the colliding particles. The usual BBGKY hierarchy equation describing the time evolution of the singlet distribution $f(r,v,t)$ will thus take on the form

$$\begin{aligned}\partial f(r,v,t)/\partial t+v\partial f(r,v,t)/\partial r \\ =\int dv'|v-v'|[f_2(r,v';r+a,v:t)\epsilon(v'-v) \\ -f_2(r,v;r+a,v':t)\epsilon(v-v')+f_2(r,v';r-a,v:t) \\ \times\epsilon(v-v')-f_2(r,v;r-a,v':t)\epsilon(v'-v)]\,,\quad (5.1)\end{aligned}$$

where $f_2(r_1,v_1;r_2,v_2:t)$ is the usual time-dependent two-particle distribution function computed from the same ensemble density as f. It is seen from (5.1) that when f_2 is spatially uniform, i.e., f_2 depends only on $|r_2-r_1|$, then the right-hand side of (5.1), the collision term, vanishes.

Following our procedure for the self-distribution function, we shall now construct a kinetic equation for $f(t)$ with a non-Markoffian collision operator which will give $\partial f(t)/\partial t$ in terms of $f(t')$ for $t'\leq t$. This can be done either by solving explicitly for $f_2(t)$ or by noting that due to our initial condition for μ [Eq. (1.3)], the nonasymptotic part of $f(t)$, $\eta(t)=f(t)-\rho h_0(v)$, depends linearly on its initial value $\eta(0)$. Thus if we define as in (1.2)

$$\begin{aligned}f(y_1,t)&=f(r_1,v_1,t)\\ &=N\int dx_1\cdots\int dx_N\mu(x_1,\cdots,x_N;t)\delta(x_1-y_1)\quad (5.2)\end{aligned}$$

and let the initial value of μ be given by a linear superposition of ensembles, of the type considered in (1.3),

$$\begin{aligned}\mu(x_1,\cdots,x_N;0)=\frac{1}{N}\int dy_0 \\ \times\sum_{i=1}^{N}\mu_i(x_1,\cdots,x_N;0/y_0)f_s(y_0,t)\,,\quad (5.3)\end{aligned}$$

then

$$f(y_1,t)=\int dy_0f(y_1,t/y_0)f_s(y_0,t)$$

or

$$\begin{aligned}f(r,v,t)-\rho h_0(v)&=\eta(r,v,t)\\ &=\int dr'\int dv'T(r-r',v,t/v')f_s(r',v',0)\\ &=\int dr'\int dv'K(r-r',v,t/v')\eta(r',v',0)\,,\quad (5.4)\end{aligned}$$

and

$$T(r,v,t/v')=f(r,v,t/v')-\rho h_0(v)\quad (5.5)$$

is the inverse Fourier transform of (2.15). [By carrying out the passage to the thermodynamic limit in k space for $k\neq 0$, we have automatically subtracted the constant $\rho h_0(v)$ term.] Here $f_s(r,v,0)$ is essentially what corresponds to the self-distribution at $t=0$, i.e., there is one particle in the fluid whose initial distribution is given by $f_s(r,v,0)$ while the rest of the fluid is in equilibrium with respect to this particle. The "Green's func-

[9] S. Yip and S. Ranganathan, Phys. Fluids 8, 1956 (1965); J. H. Ferziger and D. L. Feinstein, Phys. Rev. 158, 97 (1967).

tion" K may be found most simply in the form of an operator or matrix in r and v space by rewriting (5.4) in the form

$$\eta(t)=T(t)f_s(0)=T(t)[T^{-1}(0)\eta(0)]=K(t)\eta(0)\,, \quad (5.6)$$

with $T^{-1}(0)$, the operator inverse to $T(0)$, given in (1.11). In the thermodynamic limit, the case that we are interested in,

$$\rho T(0)=\mathfrak{F}(r-r';v,v')=\rho\delta(r-r')\delta(v-v') +\rho^2h_0(v)[g(r-r')-1] \quad (5.7)$$

and

$$\rho^{-1}T^{-1}(0)=\mathfrak{F}^{-1}=\rho^{-1}\delta(r-r')\delta(v-v') -h_0(v)C(r-r')\,, \quad (5.8)$$

where $\mathfrak{F}$ is a generalization of the usual modified spatial Ursell function and $C(r)$ is the direct correlation function.[10]

We may now write down a linear equation for $\eta(t)$:

$$\partial\eta(r,v,t)/\partial t+v\partial\eta(r,v,t)/\partial r=\int dr'\int dv'\int_0^t dt' \times B(r-r',t-t';v,v')\eta(r',v',t')\,. \quad (5.9)$$

The form of B may be obtained, as in Ref. 1, by taking the Fourier space and Laplace time transforms of Eqs. (5.6) and (5.9). Then

$$\tilde{\eta}(k,v,s)=\int dv'\mathcal{K}(k,s;v,v')\eta(k,v',0)=\mathcal{K}\eta(k,v,0)\,, \quad (5.10)$$

$$(s-ikv)\tilde{\eta}(k,v,s)=\eta(k,v,0)+\int dv'\mathcal{B}(k,s;v,v')\tilde{\eta}(k,v',s) =\eta(k,v,0)+\mathcal{B}\tilde{\eta}(k,v,s)\,, \quad (5.11)$$

or

$$\mathcal{B}=\mathbf{s}-ik\mathbf{v}-\mathcal{K}^{-1}\,, \quad (5.12)$$

where $\mathcal{B}$, $\mathcal{K}$, $\mathbf{s}$, and $\mathbf{v}$ are operators in velocity space, with

$$\mathcal{K}^{-1}=T(0)\,\mathcal{T}^{-1} \quad (5.13)$$

according to (5.6). The script letters are used to represent the Laplace transforms of the functions or operators. We can then write

$$\mathcal{B}=T(0)\mathcal{B}'+(1-T(0))(\mathbf{s}-ik\mathbf{v})\,, \quad (5.14)$$

where

$$\mathcal{B}'=(\mathbf{s}-ik\mathbf{v})-\mathcal{T}^{-1}. \quad (5.15)$$

Putting

$$\mathcal{B}'(k,s;v,v')=[s-ikv-A^{-1}(v,s)] \times\delta(v-v')+h_0(v)\mathcal{H}(k,s;v,v')A^{-1}(v',s) \quad (5.16)$$

and

$$\mathcal{T}(k,s;v,v')=A(v,s)\delta(v-v') +h_0(v)A(v,s)\mathcal{W}(k,s;v,v')\,, \quad (5.17)$$

where $\mathcal{W}(k,s;v,v')$ is defined by Eq. (2.17), we find that $\mathcal{H}$ satisfies the equation

$$\mathcal{H}(k,s;v,v')=\mathcal{W}(k,s;v,v') -\int dw\,\mathcal{W}(k,s;v,w)h_0(w)\mathcal{H}(k,s;w,v'). \quad (5.18)$$

We may now attempt to expand the collision operator $\mathcal{B}$ in powers of the density (or of n):

$$\mathcal{B}(k,s;v,v')=n\mathcal{B}_1(k,s;v,v')+O(n^2)$$

or

$$B(r-r',t-t',v,v') =nB_1(r-r',t-t';v,v')+O(n^2). \quad (5.19)$$

We then find that the "Boltzmann" collision operator $\mathcal{B}_1$ is independent of s, which means that in the time domain B_1 has a δ function in $t-t'$ and hence the equation corresponding to (5.9) with $\mathcal{B}_1$ instead of $\mathcal{B}$ is Markoffian. More explicitly,

$$\mathcal{B}_1(k,s;v,v')=h_0(v)[|v-v'|(1-\cos ka)+i(v-v')\sin ka] -\delta(v-v')[\mu(v)(1-\cos ka)-iv\sin ka]. \quad (5.20)$$

It is seen from (5.20) that $\mathcal{B}_1$ (and this is true also for $\mathcal{B}$) vanishes when $k=0$, i.e., when η is spatially homogeneous.

Since the operator (5.20) represents the instantaneous collision or Boltzmann approximation, it can be obtained in an alternative manner. To emphasize the short-time development in (5.6), we may write it instead as

$$\partial\eta(t)/\partial t=\dot{T}(t)f_s(0)\,, \quad (5.21)$$

thereby implying the kinetic equation

$$\partial\eta(t)/\partial t=[\dot{T}(t)T^{-1}(t)]\eta(t). \quad (5.22)$$

The kinetic operator is thus local in time but time-dependent. If the approximation is made that the ensemble maintains the special form of the initial ensemble, then the transition operator may be replaced by $\dot{T}(0)T^{-1}(0)$, where $T^{-1}(0)$ is given by (5.8). We call this approximate collision operator $\bar{B}$:

$$\bar{B}=\dot{T}(0)T^{-1}(0)-ikv. \quad (5.23)$$

$\bar{B}$ will, of course, be exact at $t=0$, and since B_1 is independent of t, we must have

$$\bar{B}=n\bar{B}_1+O(n^2)$$

and

$$\bar{B}_1=B_1.$$

Carrying out explicitly the time derivative of T in (5.5), we find

$$\dot{T}(k,v,0/v')=[i\beta v-\alpha\mu(v)]\delta(v-v') +nh_0(v)[(e^{-ika}-1)v'-(e^{ika}-1)v +(e^{ika}-1)(e^{-ika}-1)\epsilon(v'-v)(v'-v)]. \quad (5.24)$$

[10] See, for example, Ref. 3, p. II-58.

Inserting (5.8), we have

$$\bar{B}(k|vv')+ikv\delta(v-v')=\delta(v-v')[i\beta v-\alpha\mu(v)] +h_0(v)\{\alpha|v-v'|-i(\beta-k)v'-i[\beta-k+k\rho C(k)]v\}. \tag{5.25}$$

Observing that

$$k^2[1-\rho C(k)]=\alpha^2+\beta^2, \tag{5.26}$$

we see that (5.20) and (5.25) are indeed identical to first order in n.

The solution of (5.11) with either the kernel $n\mathcal{B}_1$ of (5.20) or with (5.25) can now be found without difficulty. Equation (5.11) becomes explicitly

$$[s-i\beta v+\alpha\mu(v)]\tilde{\eta}(k,v,s)=\eta(k,v,0)+h_0(v)\int_{-\infty}^{\infty}dv' \times[\alpha|v-v'|+i\lambda v-i(\beta-k)v']\tilde{\eta}(k,v',s), \tag{5.27}$$

where $\lambda=\beta-k$ or $k[1-\rho C(k)]-\beta$ in the two cases. Dividing by $h_0(v)$, differentiating twice, and using $h_0(v)=\frac{1}{2}\mu''(v)$, we obtain

$$\frac{\partial^2}{\partial v^2}\left([s-i\beta v+\alpha\mu(v)]\frac{\tilde{\eta}(k,v,s)}{h_0(v)}\right)=\frac{\partial^2}{\partial v^2}\left(\frac{\eta(k,v,0)}{h_0(v)}\right) +\left(\frac{\partial^2}{\partial v^2}[s-i\beta v+\alpha\mu(v)]\right)\frac{\tilde{\eta}(k,v,s)}{h_0(v)}, \tag{5.28}$$

which in the form

$$\frac{\partial}{\partial v}\left[[s-i\beta v+\alpha\mu(v)]^2\frac{\partial}{\partial v}\left(\frac{\tilde{\eta}(k,v,s)}{h_0(v)}\right)\right] =[s-i\beta v+\alpha\mu(v)]\frac{\partial^2}{\partial v^2}\left(\frac{\eta(k,v,0)}{h_0(v)}\right) \tag{5.29}$$

can be solved at once:

$$\frac{\tilde{\eta}(k,v,s)}{h_0(v)}=c_1+c_2\int_{-\infty}^{v}dv'[s-i\beta v'+\alpha\mu(v')]^{-2} +\int_{-\infty}^{v}dv'[s-i\beta v'+\alpha\mu(v')]^{-2}\int_{-\infty}^{v'}dv'' \times[s-i\beta v''+\alpha\mu(v'')]\frac{\partial^2}{\partial v''^2}\left(\frac{\eta(k,v'',0)}{h_0(v'')}\right). \tag{5.30}$$

The constants c_1 and c_2 can be determined from (5.27).

Consider the special case of the development of space correlations:

$$\tilde{\chi}(k,s)=\int_{-\infty}^{\infty}dv\ \tilde{\eta}(k,v,s), \tag{5.31}$$

with

$$\eta(k,v,0)=[k^2/(\alpha^2+\beta^2)]h_0(v)=\chi(k,0)h_0(v). \tag{5.32}$$

We have

$$\tilde{\eta}(k,v,s)/h_0(v)=c_1+c_2\int_{-\infty}^{v}dv'[s-i\beta v'+\alpha\mu(v')]^{-2}. \tag{5.33}$$

By integrating by parts, it follows that

$$\int_{-\infty}^{\infty}dv\ \tilde{\eta}(k,v,s)=c_1+c_2\frac{(\alpha-i\beta)}{2\alpha}\int_{-\infty}^{\infty}dv \times[s-i\beta v+\alpha\mu(v)]^{-2} \tag{5.34}$$

and

$$\int_{-\infty}^{\infty}dv\ v\tilde{\eta}(k,v,s)=\frac{c_2}{\alpha^2+\beta^2}-\frac{c_2 s}{2\alpha}\int_{-\infty}^{\infty}dv \times[s-i\beta v+\alpha\mu(v)]^{-2}. \tag{5.35}$$

Now, in order to determine c_1 and c_2, we let $v\to\infty$ in (5.33), obtaining

$$[s-i\beta v+\alpha\mu(v)]\tilde{\eta}(k,v,s)/h_0(v)\to[s+(\alpha-i\beta)v] \times\left(c_1+c_2\int_{-\infty}^{\infty}dv[s-i\beta v+\alpha\mu(v)]^{-2}\right)-c_2/(\alpha-i\beta), \tag{5.36}$$

to be compared with

$$[s-i\beta v+\alpha\mu(v)]\tilde{\eta}(k,v,s)/h_0(v)\to\chi(k,0) +v(\alpha+i\lambda)\int_{-\infty}^{\infty}dv\ \tilde{\eta}(k,v,s) -[\alpha+i(\beta-k)]\int_{-\infty}^{\infty}dv\ v\tilde{\eta}(k,v,s). \tag{5.37}$$

Hence

$$[c_1+c_2 z]s-c_2/(\alpha-i\beta)=\chi(k,0)-[\alpha+i(\beta-k)] \times\int_{-\infty}^{\infty}dv\ v\tilde{\eta}(k,v,s) \tag{5.38}$$

and

$$[c_1+c_2 z](\alpha-i\beta)=(\alpha+i\lambda)\int_{-\infty}^{\infty}dv\ \tilde{\eta}(k,v,s),$$

where

$$z=\int_{-\infty}^{\infty}dv[s-i\beta v+\alpha\mu(v)]^{-2}. \tag{5.39}$$

The equations (5.38) can be solved for c_1 and c_2 using (5.34) and (5.35):

$$c_1=\frac{\chi(k,0)}{s}+\frac{c_2}{s}\left(\frac{ik}{\alpha^2+\beta^2}-\frac{zs}{2\alpha}[\alpha-i(\beta-k)]\right) \tag{5.40}$$

and

$$c_2\left(\frac{k(\lambda+\beta)}{\alpha^2+\beta^2}+\frac{zs}{2\alpha}[(\alpha-i\beta)(\alpha+i\beta-ik)+ik(\alpha+i\lambda)]\right) =i(\lambda+\beta)\chi(k,0). \tag{5.41}$$

Then from (5.31) and (5.34),

$$\tilde{\chi}(k,s)=\frac{\chi(k,0)(\alpha^2+\beta^2)^2 z}{\{2\alpha k(\lambda+\beta)+sz(\alpha^2+\beta^2)[(\alpha-i\beta)(\alpha+i\beta-ik)+ik(\alpha+i\lambda)]\}}. \tag{5.42}$$

Choosing the full Boltzmann operator $\bar{B}$,

$$\lambda=k[1-\rho C(k)]-\beta$$

and

$$(k,s)=(k^2/2\alpha)z,$$

which is the exact result.

Note added in proof. It turns out that the kinetic equation with the "short-time" collision operator $\bar{B}$ [Eq. (5.23)] is *exact* for arbitrary initial condition $\eta(0)$. We are now investigating a similar equation in three dimensions.

ACKNOWLEDGMENT

We would like to thank J. Steadman for help in computing Figs. 1–3.

APPENDIX

For the discrete velocity distribution

$$h_0(v)=\sum_{j=1}^{n} A_j \tfrac{1}{2}[\delta(v-c_j)+\delta(v+c_j)], \tag{A1}$$

with

$$c_n>\cdots>c_{m+1}>c_m>\cdots>c_1>0$$

and

$$\sum_{j=1}^{n} A_j=1, \tag{A2}$$

we find

$$\mu(v)=\int_{-\infty}^{\infty} dw\, h_0(w)|w-v|=\sum_{j=1}^{n} A_j \max[c_j,|v|]. \tag{A3}$$

For $t>0$, from Eq. (3.7),

$$\chi(k,t)=\frac{k^2 t}{\alpha(k)}\mathrm{Re}\int_0^{\infty} dw\, e^{iwt\beta(k)-\mu(w)t\alpha(k)}. \tag{A4}$$

Since

$$\begin{aligned}\mu(w)&=\sum_{j=1}^{n} A_j c_j && \text{for } 0\leq w\leq c_1\\ &=w\sum_{j=1}^{m} A_j+\sum_{j=m+1}^{n} A_j c_j && \text{for } c_m\leq w\leq c_{m+1}\\ &=w && \text{for } c_n\leq w,\end{aligned}$$

the w integration in (A4) can be done to give

$$\chi(k,t)=\mathrm{Re}\sum_{m=1}^{n} D_m(k)e^{iw_m(k)t-\lambda_m(k)t}, \tag{A5}$$

where

$$w_m(k)=c_m\beta(k);\quad \lambda_m(k)=\mu(c_m)\alpha(k) \tag{A6}$$

and

$$D_m(k)=-A_m k^2\Big[\alpha(k)\sum_{j=1}^{m-1} A_j-i\beta(k)\Big]^{-1} \times\Big[\alpha(k)\sum_{j=1}^{m} A_j-i\beta(k)\Big]^{-1} \tag{A7}$$

for $1<m\leq n$, with

$$D_1(k)=-A_1k^2[-i\beta(k)]^{-1}[\alpha(k)A_1-i\beta(k)]^{-1}.$$

$\chi(k,t)$ is thus a superposition of damped oscillations and satisfies a linear differential equation in time.

The prototype δ-function distribution (3.8) is obtained by putting $A_1=1$, $c_1=c$ so that $\lambda_1(k)=c\alpha(k)$ and $w_1(k)=c\beta(k)$. Another interesting case is

$$h_0(v)=A_1\delta(v)+A_2\tfrac{1}{2}[\delta(v-c)+\delta(v+c)], \tag{A8}$$

with

$$c_1=0,\quad c_2=c,\ \text{and}\ A_1+A_2=1.$$

Then

$$\lambda_1=A_2 c\alpha(k),\quad w_1=0$$

and

$$\lambda_2=c\alpha(k),\quad w_2=c\beta(k),$$

so that

$$\chi(k,t)=\mathrm{Re}[D_1(k)e^{-\lambda_1 t}+D_2(k)e^{iw_2 t-\lambda_2 t}]. \tag{A9}$$