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Correlation Inequalities for Spin Glasses

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Abstract. We prove a correlation type inequality for spin systems with quenched symmetric random interactions. This gives monotonicity of the pressure with respect to the strength of the interaction for a class of spin glass models. Consequences include existence of the thermodynamic limit for the pressure and bounds on the surface pressure. We also describe other conjectured inequalities for such systems.

1. Introduction

Correlation inequalities have played and continue to play an important role in many areas of statistical mechanics. In addition to describing microscopic structure they also provide information about macroscopic properties: for ferromagnetic spin systems they give monotonicity of the critical temperature with respect to the lattice dimensions, inequalities for critical exponents, etc.

It would clearly be desirable to obtain similar results also for disordered systems, like spin glasses, in which both ferromagnetic and anti-ferromagnetic interactions are present with assigned probabilities. In this note we obtain some correlation type inequalities for such systems. These yield various results usually obtained for ferromagnetic systems from the first GKS inequalities [6–8], e.g., monotonicity and hence existence of the thermodynamic limit of the pressure, and bounds on the surface pressure. We also discuss intriguing examples of versions of the second type GKS inequalities, but at the moment a general proof is still lacking.

2. Definitions and results

Let $\sigma_n = \pm 1$, $n \in \Lambda \subset \mathbb{Z}^d$, and denote by Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n \in \Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. To each $X \subset \Lambda$ we associate the spin product $\sigma_X = \prod_{i \in X} \sigma_i$ and random variable J_X. The J's, denoted collectively by $\mathbf{J}$, are a family of independent

unit symmetric random variables:

$$\mathrm{Av}\,(J_X) = 0\,, \tag{2.1}$$

$$\mathrm{Av}\,(J_X J_Y) = \delta_{X,Y}\,, \tag{2.2}$$

where the $\mathrm{Av}\,()$ denotes the average over the distributions of the J's. The energy of a configuration σ is given by

$$U_\Lambda(\sigma) = -\sum_{X \subset \Lambda} \lambda_X J_X \sigma_X + \bar{U}_\Lambda(\sigma)\,, \tag{2.3}$$

where $\bar{U}_\Lambda$ is a non-random (reference) interaction which we need not specify in any detail for the moment. It can in particular be set equal to zero. The $\{\lambda_X\}$, denoted collectively by λ, are auxiliary parameters which are used to tune the strength of the different interactions. Some can be set (at the end) equal to zero.

The probability distribution for specified λ and $\mathbf{J}$ is given by $\mu = Z_\Lambda^{-1} e^{-U_\Lambda}$ with random partition function

$$Z_\Lambda = \sum_{\sigma \in \Sigma_\Lambda} e^{-U_\Lambda(\sigma)}\,. \tag{2.4}$$

The quenched pressure is given by

$$P_\Lambda = \mathrm{Av}\,(\log Z_\Lambda)\,. \tag{2.5}$$

We define the thermal correlations for a given $\mathbf{J}$ and an $A \subset \Lambda$ by

$$\omega_A = \frac{\sum_\sigma \sigma_A e^{-U_\Lambda(\sigma)}}{\sum_\sigma e^{-U_\Lambda(\sigma)}}\,. \tag{2.6}$$

The quenched correlations $\bar{\omega}_A$ are given by $\mathrm{Av}\,(\omega_A)$ and we also define (dropping the subscript Λ)

$$\rho_A = \mathrm{Av}\,(J_A \omega_A) = \frac{\partial P}{\partial \lambda_A}\,. \tag{2.7}$$

The pressure (and the $\bar{\omega}_A$) is clearly an even function of each λ_X and hence ρ_A is an odd function of λ_A and an even function of λ_X, $X \neq A$.

Theorem 1. *Let $\lambda_A \geq 0$, then $\rho_A \geq 0$. Consequently the quenched pressure P is monotone increasing with respect to each λ_A, i.e.,*

$$\frac{\partial P}{\partial \lambda_A} = \rho_A \geq 0\,, \quad \forall A \subset \Lambda \quad with \quad \lambda_A \geq 0\,. \tag{2.8}$$

Remark. Theorem 1 immediately implies that the pressure $P(\lambda)$ has a minimum for $\lambda_A = 0$, for any $A \in \Lambda$. E.g., the pressure of a system with symmetric nearest-neighbor random interactions is smaller than the one with added next nearest neighbor interactions. Another consequence is that in the case when $\bar{U}_\Lambda = 0$ or containing only one site potentials, e.g., for standard spin glasses, the pressure for periodic boundary conditions is larger than the one with free boundary conditions.

Proof. The function P is symmetric and convex with respect to λ_A, i.e.,

$$\rho_A = \frac{\partial P}{\partial \lambda_A} = 0 \quad \text{for} \quad \lambda_A = 0 \,, \tag{2.9}$$

and

$$\frac{\partial^2 P}{\partial \lambda_A^2} = \text{Av}\left(J_A^2[1 - \omega_A^2]\right) \geq 0 \,. \tag{2.10}$$

The first equality follows from symmetry and the second from the fact that $\omega_A^2 \leq 1$. Consequently from (2.10)

$$\rho_A(\lambda_A^{(1)}) \geq \rho_A(\lambda_A^{(2)}), \quad \forall \lambda_A^{(1)} \geq \lambda_A^{(2)} \geq 0 \tag{2.11}$$

and the theorem follows by choosing $\lambda_A^{(2)} = 0$ and from (2.9). $\qquad\square$

Remark. The previous theorem holds also with the only condition that $\text{Av}(J_A) = 0$ without the hypotheses of symmetry for any distributions. Convexity of P is in fact a general property thanks to (2.10). This implies monotonicity of the first derivative

$$\frac{\partial P}{\partial \lambda_A} \geq \frac{\partial P}{\partial \lambda_A}|_{\lambda_A=0} = \text{Av}\left(J_A \omega_A|_{\lambda_A=0}\right) \,.$$

Since the two random variables J_A and $\omega_A|_{\lambda_A=0}$ are indipendent one has

$$\text{Av}\left(J_A \omega_A|_{\lambda_A=0}\right) = \text{Av}\left(J_A\right) \text{Av}\left(\omega_A|_{\lambda_A=0}\right) = 0 \,.$$

which immediately gives the inequality (2.8). It is also clear that the condition $Av(J_A) = 0$ is also necessary for the theorem to hold in the generality, i.e., independent of what the rest of the Hamiltonian is.

Theorem 2. *When $\bar{U}_\Lambda$ is just a sum of one site potentials, i.e., $\bar{U}_\Lambda = \sum_{i \in \Lambda} h_i \sigma_i$, the pressure P is super-additive, i.e., for all disjoint decompositions $\Lambda = \cup_{h=1}^K \Lambda_h$, $\Lambda_l \cap \Lambda_m = \emptyset \; l \neq m$*

$$P_\Lambda \geq \sum_{h=1}^K P_{\Lambda_h} \,. \tag{2.12}$$

Remark. These properties of P are similar to those for the ferromagnetic case [7] with the $\rho_A \geq 0$ playing the role of the first GKS inequality. This result generalizes the one obtained for the Gaussian case in [1]. See also [10–12].

Corollary 2.1. *Assuming thermodynamic stability,*

$$\text{Av}\left(e^{-U}\right) \leq e^{c|\Lambda|} \,, \tag{2.13}$$

the super-additivity provided by Theorem 2 implies, by standard methods, monotonicity and existence of the thermodynamic limit for the pressure with free boundary conditions. In particular

$$p = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{P_\Lambda}{|\Lambda|} = \sup_\Lambda \frac{P_\Lambda}{|\Lambda|} \,. \tag{2.14}$$

Remark. Decomposing a d-dimensional hypercube Λ into $(d-1)$-dimensional hypercubes (2.12) immediately implies that the pressure in dimension d, $p^{(d)}$ is an increasing function of d

$$p^{(d+1)} \geq p^{(d)} \,. \tag{2.15}$$

Theorem 2 admits the following

Proof. To each partition of Λ we associate the interpolating potential for $0 \leq t \leq 1$

$$U_\Lambda(t) = \sum_{h=0}^{K} t_h U_{\Lambda_h}^{(h)}\,, \quad \Lambda_0 = \Lambda\,, \tag{2.16}$$

with $t_0 = t$ and $t_h = (1-t)$ for $1 \leq h \leq K$,

$$U_{\Lambda_h}^{(h)} = -\sum_{X \subset \Lambda_h} \lambda_X J_X^{(h)} \sigma_X \tag{2.17}$$

and we have dropped the argument σ and set $\bar{U}_n = 0$, for notational convenience. The $J_X^{(h)}$ are centered independent unit random variables

$$\mathrm{Av}\left(J_X^{(l)} J_Y^{(m)}\right) = \delta_{l,m} \delta_{X,Y}\,. \tag{2.18}$$

We define the interpolating partition function

$$Z_\Lambda(t) = \sum_{\sigma \in \Sigma_N} e^{-U_\Lambda(t)}\,, \tag{2.19}$$

and we observe that

$$Z_\Lambda(0) = \prod_{h=1}^{K} Z_{\Lambda_h}(J^{(h)})\,, \quad Z_\Lambda(1) = Z_\Lambda(J)\,. \tag{2.20}$$

We consider the interpolating pressure

$$P_\Lambda(t) := \mathrm{Av}\left(\ln Z_\Lambda(t)\right)\,. \tag{2.21}$$

Using (2.20) we get

$$P_\Lambda(0) = \sum_{h=1}^{K} P_{\Lambda_h}\,, \quad P_\Lambda(1) = P_\Lambda\,. \tag{2.22}$$

Considering $\mathcal{C}_\Lambda$ the set of all subsets of Λ with non empty intersection with more than one Λ_h a straightforward computation gives

$$\frac{d}{dt} P_\Lambda(t) = \sum_{X \in \mathcal{C}_\Lambda} \lambda_X^2 \rho_X \geq 0\,, \tag{2.23}$$

where the last inequality comes from Theorem 1. Hence (2.22) and (2.23), imply (2.12). $\qquad\square$

Theorem 3. *For finite range interactions, i.e., $\lambda_X = 0$ for $|X| \geq r$ (e.g., the nearest neighbor case) the first correction to the leading term of the pressure, T_Λ defined by:*

$$P_\Lambda = p|\Lambda| + T_\Lambda \tag{2.24}$$

has a non positive value and is, at most, of surface size:

$$c|\partial\Lambda| \leq T_\Lambda \leq 0 \,. \tag{2.25}$$

Remark. The result (2.25) is analogous to that for the ferromagnetic case in [4, 5]. The theorem generalizes to all symmetric distributions of the J_X the result obtained for the Gaussian case in [2].

Proof. We give the proof for nearest-neighbor interactions, $r = 2$, $U_\Lambda(\sigma) = \sum_b \lambda_b J_b \sigma_b$ where b represents a bond of the lattice $b = (n, n')$, $|n - n'| = 1$. The extension to the general case is straightforward and only changes the value of the constant of (2.25). Given the d-dimensional cube Λ, $|\Lambda| = L^d$ consider its magnification $|\Lambda_k| = (kL)^d$. Clearly Λ_k can be partitioned into k^d disjoint cubes Λ_s all congruent to Λ. We call $\mathcal{C}_\Lambda$ the set of bonds connecting the different Λ_s. In finite volume and with free boundary conditions we have by definition

$$P_\Lambda = \mathrm{Av}\left(\ln Z_\Lambda\right) = k^{-d}\mathrm{Av}\left(\ln Z_\Lambda^{k^d}\right) . \tag{2.26}$$

Since the limiting pressure per particle is independent of the boundary conditions we have, indicating by Π the periodic boundary conditions (see [4])

$$p|\Lambda| = \lim_{k\to\infty} k^{-d}\mathrm{Av}\left(\ln Z_{k\Lambda}^{(\Pi)}\right) . \tag{2.27}$$

By (2.26) and (2.27) we obtain

$$
\begin{aligned}
T_\Lambda &= (P_\Lambda - p|\Lambda|) \\
&= \lim_{k\to\infty} k^{-d}\mathrm{Av}\left(\ln Z_\Lambda^{k^d} - \ln Z_{k\Lambda}^{(\Pi)}\right) .
\end{aligned}
\tag{2.28}
$$

We chose now the λ's in such a way that

$$\lambda_b = \begin{cases} \lambda, & \text{if } b \in \mathcal{C}_\Lambda \,, \\ 1, & \text{otherwise} \,. \end{cases} \tag{2.29}$$

Equation (2.28) then becomes, using the fundamental theorem of calculus,

$$T_\Lambda = \lim_{k\to\infty} k^{-d}\left[P_\Lambda^{(\Pi)}(0) - P_\Lambda^{(\Pi)}(1)\right] = -\lim_{k\to\infty} k^{-d}\int_0^1 \frac{d}{d\lambda}P_\Lambda^{(\Pi)}(\lambda)d\lambda . \tag{2.30}$$

By a simple computation

$$\frac{d}{d\lambda}P_\Lambda^{(\Pi)}(\lambda) = \sum_{b\in\mathcal{C}_\Lambda} \rho_b^{(\Pi)}(\lambda) = |\mathcal{C}_\Lambda|\rho_b^{(\Pi)}(\lambda) , \tag{2.31}$$

where we have used the translation symmetry over the torus. By the identity

$$2|\mathcal{C}_\Lambda| = k^d|\partial\Lambda| \tag{2.32}$$

we obtain

$$T_\Lambda = -\frac{|\partial\Lambda|}{2} \lim_{k\to\infty} \int_0^1 \rho_b^{(\Pi)}(\lambda)d\lambda\,, \qquad (2.33)$$

which immediately entails the theorem.   □

3. Other inequalities

We have seen that Theorem 1 expresses for spin glasses a monotonicity property that entails the same consequences that the first GKS inequality does for ferromagnetic systems. From this perspective it is interesting to examine whether the quantity

$$\frac{\partial^2 P}{\partial\lambda_A\partial\lambda_B} = \mathrm{Av}\big(J_A J_B[\omega_{AB} - \omega_A\omega_B]\big) \qquad (3.34)$$

has a definite sign when $A \neq B$ (for $A = B$, $\omega_{AB} = 1$ and the sign is positive). We have some evidence based on explicit calculations (work in progress with F. Unguendoli) that for $A \neq B$ with the J's satisfying (2.2), the sign is negative, but are unable at this time to produce a general proof or find a counter example. A simple computation based on integration by parts shows that such an inequality would imply that the overlap expectations in Gaussian spin glasses would be monotonic non decreasing with the volume, similar to spin expectations in ferromagnetic systems.

Another interesting question is the case where the J's are not centered random variables but with $\mathrm{Av}\,(J_A) = \mu_A > 0$. We then ask whether $\mathrm{Av}\,(\omega_A) \geq 0$. This is essentially the same question as whether $\mathrm{Av}\,(\omega_A) \geq 0$ when $\bar{U}_\Lambda$ is ferromagnetic. In that case $\omega_A \geq 0$ when $\Lambda_X = 0$ for all $X \subset \Lambda$ and it seems very reasonable to expect that this will be preserved when we add symmetric random interactions. Also in this case we have evidence that the inequality hold for specific cases (work in progress with F.Unguendoli) but unfortunately we don't have a general proof or a counter example. It is interesting to note that in a subspace of the random parameter space in which variance and mean of the J's are chosen to be identical (Nishimori line) a suitable version of the GKS inequalities can be proved with consequences similar to those obtained here for the surface pressure (see [3,9]).

Acknowledgements

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