Generation of primordial cosmological perturbations from statistical mechanical models

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The initial conditions describing seed fluctuations for the formation of structure in standard cosmological models, i.e. the Harrison-Zeldovich distribution, have very characteristic "super-homogeneous" properties: they are statistically translation invariant and isotropic, and the variance of the mass fluctuations in a region of volume V grows more slowly than V. We discuss the geometrical construction of distributions of points in $\mathbf{R}^3$ with similar properties encountered in tiling and in statistical physics, e.g. the Gibbs distribution of a one-component system of charged particles in a uniform background [one-component plasma (OCP)]. Modifications of the OCP can produce equilibrium correlations of the kind assumed in the cosmological context. We then describe how such systems can be used for the generation of initial conditions in gravitational N-body simulations.

I. INTRODUCTION

A central problem in contemporary cosmology is the quantitative explanation of the inhomogeneity observed in the universe at large scales (see e.g. [1,2]). Such inhomogeneity is probed both indirectly for early times via the fluctuations in temperature in the microwave sky, and directly at present in the distribution of matter in space. All currently standard cosmological models work within a paradigm in which these fluctuations are the result of (mostly) gravitational evolution operating on some initially very small fluctuations. Common to all such models is the assumption—supported in particular by observations of the microwave background [3]—of a very specific form of these initial perturbations at large scale, i.e. Gaussian fluctuations with a Harrison-Zeldovich (HZ, often referred to as "scale-invariant") power spectrum [4,5]. In a recent paper some of us [6] have discussed how, in a simple classification of correlated systems, this spectrum corresponds to highly ordered glasslike or latticelike distributions. In particular such systems are characterized in real space by the striking behavior of the variance in the mass, $\langle \Delta M^2 \rangle$, contained in a given volume, e.g. in spheres of radius R, $\langle \Delta M^2 \rangle \propto R^2$, or $R^2 \log R$; i.e., it is essentially proportional to the *surface area* of the sphere. This is to be contrasted with the Poisson behavior (proportional to the volume R^3) characteristic of many equilibrium systems with short range interactions away from phase transitions or that characteristic of long-range correlations at a critical point in many systems (for which $\langle \Delta M^2 \rangle \propto R^\alpha$ with $\alpha > 3$). Actually R^2 is the slowest growth possible for *any* isotropic translationally invariant distribution of points [7].

In this paper we examine the nature of the correlations in such super-homogeneous systems and compare them with equilibrium correlations in systems with long range interactions studied in statistical mechanics [8–10]. More precisely we consider the one component plasma (OCP), which is simply a system of charged point particles interacting through a repulsive $1/r$ potential, in a uniform background which gives overall charge neutrality (for a review, see [11]). We discuss how the OCP can be modified so that its equilibrium correlations are precisely those considered in cosmology. This involves an appropriately designed attractive short range potential as well as a repulsive $1/r^2$ potential at large scales. The latter corresponds to a four dimensional Coulomb potential with the particles confined to three dimensions.

Our analysis of these parallels has more than theoretical interest. It also provides a new method for generating initial conditions (IC) for numerical studies of the formation of structure in cosmology [12,13]. We explain that this method avoids the problems associated with currently used algorithms which involve superimposing small perturbations on a lattice or glasslike distribution. The latter is understood to be a "sufficiently uniform" discretization of a constant density background, not as representing the perturbed distribution. The problem with this choice of initial state is that the *unperturbed* distribution has its own inherent fluctuations/correlations, which can play an important role in the evolution of the system [14–17].

II. THE HZ SPECTRUM IN COSMOLOGY

Let us recall first the necessary essentials of modeling perturbations from a uniform density used in cosmology. These are described as a *continuous* stationary stochastic process (SSP), with the property that the volume averages of observables, i.e. empirical averages, approach the ensemble averages in the large volume limit. The stationarity refers in this context to statistical invariance under spatial translation.

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Moreover one assumes also statistical isotropy of the stochastic process, i.e. invariance under rotation of the ensemble quantities. Thus if $\rho(\vec{r})$ is the density field, we have

$$\langle \rho(\vec{r}) \rangle = \rho_0 > 0, \tag{1}$$

and we can define a *reduced* 2-point correlation function $\tilde{\xi}(r)$ by

$$\langle \rho(\vec{r}_1)\rho(\vec{r}_2) \rangle \equiv \rho_0^2 [1 + \tilde{\xi}(r)] \tag{2}$$

where $r = |\vec{r}_1 - \vec{r}_2|$. Corresponding isotropies are assumed to hold for the higher order correlation functions. Further, the probability distribution for the fluctuations in the initial uniform density are generally assumed to be Gaussian.

Much more used in cosmology than $\tilde{\xi}(r)$ is the equivalent k-space quantity, the power spectrum $P(\vec{k})$ which is defined as

$$P(\vec{k}) = \lim_{V \to \infty} \frac{\langle |\delta_\rho(\vec{k})|^2 \rangle}{V} \tag{3}$$

where $\delta_\rho(\vec{k}) = \int_V d^3 r\, e^{-i\vec{k}\cdot\vec{r}} \delta(\vec{r})$ is the Fourier integral in the volume V of the normalized fluctuation field $\delta(\vec{r}) = [\rho(\vec{r}) - \rho_0]/\rho_0$. In a statistically isotropic SSP this depends only on $|\vec{k}|$ with

$$P(\vec{k}) \equiv P(k) = \int_0^\infty \tilde{\xi}(r) \frac{\sin(kr)}{kr} 4\pi r^2 dr. \tag{4}$$

It follows from its definition that $P(k) \geq 0$, and, in the case we are considering, its integral, which is equal to $\tilde{\xi}(0)$, is assumed to be finite.

In current cosmological models it is generally assumed that the power spectrum $P(k) \propto k$ at small k. This is known as the HZ or "scale-invariant" power spectrum. It is believed to describe the "primordial" fluctuations at very early times, the putative remnants of a period of "inflation" prior to the ordinary big bang phase [1,2]. The linear k behavior is cut off by a short distance scale (needed to ensure integrability) i.e. for k larger than some cutoff, k_c, $P(k)$ decreases faster than k^{-3}. The reason for the appearance of this particular spectrum is tied to considerations about the cosmological model (see [1,2]). Thus in the homogeneous and isotropic Friedmann-Robertson-Walker cosmology, the only characteristic length scale is the size of the horizon $R_H(t)$ (the region causally connected at a time t). The HZ spectrum then corresponds to the choice which gives (as discussed in [6] this condition is in fact satisfied only for a spherical Gaussian window function, and not in a real physical sphere)

$$\sigma_M^2(R_H(t)) = \text{const}, \tag{5}$$

independent of t, where $\sigma_M^2(R)$ is the normalized mass variance

$$\sigma_M^2(R) = \langle \Delta M^2 \rangle / \langle M \rangle^2. \tag{6}$$

The HZ spectrum then follows from a consistency criterion for the treatment of the perturbations in the model: any other spectrum will give mass fluctuations which dominate over the homogeneous background at some time in either the future or past.

All current cosmological models share the HZ spectrum (or something very close to it) as the "primordial" form of their perturbations. This spectrum evolves, at larger and larger scales, until the characteristic time at which the densities of matter and radiation energy are equal. This evolution which depends strongly on the particular model used may change completely the form of the spectrum for distances smaller than a characteristic scale k_{eq}^{-1} (i.e. $k > k_{eq}$). For larger physical scales ($k < k_{eq}$), however, the evolution is almost exactly the same in all models, and it leaves the primordial HZ form intact.

Viewed in the general framework of correlated processes a very crucial property of the HZ type power spectrum is that

$$\lim_{k \to 0} P(k) = 0 \tag{7}$$

which in real space implies that

$$\lim_{R \to \infty} \frac{\langle \Delta M^2 \rangle}{V(R)} = \lim_{R \to \infty} \int_{V(R)} d^3 r\, \tilde{\xi}(r) = 0 \tag{8}$$

where $V(R)$ is the volume of a sphere of radius R (with arbitrary origin). Equation (8) is to be contrasted with the behavior of this integral in a Poisson distribution, when it yields a finite positive constant, and with that associated with power-law correlations in critical systems [6] for which the same integral of $\tilde{\xi}(r)$ diverges.

Systems satisfying Eq. (8) are thus "more homogeneous" than a Poisson type system, as can be seen when one considers the behavior of the mass variance in spheres. For a spectrum such that $P(k) \sim k^n$ for $k \to 0$ and appropriately cut off at large k we have for large R

$$\sigma_M^2(R) \propto \begin{cases} 1/R^{3+n} & \text{if } n < 1, \\ \log(R)/R^4 & \text{if } n = 1, \\ 1/R^4 & \text{if } n > 1. \end{cases} \tag{9}$$

In terms of the non-normalized quantity $\langle (\Delta M)^2 \rangle \propto \sigma_M^2(R)R^6$, Eq. (8) says that for $n > 0$ we have a slower increase of the variance as a function of R than for Poisson fluctuations, corresponding to $n = 0$, with the limiting behavior corresponding to a variance which is proportional to the *surface area* of the sphere. These systems are thus characterized by surface fluctuations, ordered (or homogeneous enough, one could say) to give this very particular behavior. In Fig. 1 are shown, in two dimensions, a Poissonian distribution and a super-homogeneous distribution. The relatively greater uniformity of the latter is clearly identifiable.
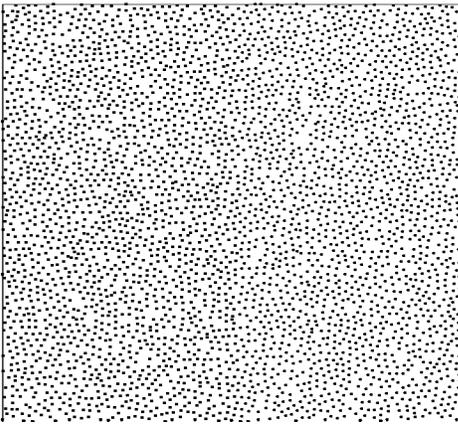
FIG. 1. A super-homogeneous distribution (bottom) and a Poisson distribution (top) with approximately the same number of points. Both are projections of thin slices of three dimensional distributions. The super-homogeneous one is a ''pre-initial'' configuration used in setting up the configuration representing the initial conditions for the cosmological *N*-body simulations of [18]. As described in Sec. VI the dynamics used to generate this ''pre-initial'' configuration gives rise to surface fluctuations as it is essentially the one component plasma which we discuss below.

III. DISTRIBUTIONS OF POINTS WITH SURFACE GROWTH OF THE SQUARED VARIANCE

A simple cubic lattice is the simplest example of a discrete set of points which shows this limiting behavior ($\sigma_M^2 \propto 1/R^4$) of the variance [19]. The result is not hard to understand: The standard deviation σ_M is that measured by averaging over spheres of radius R centered at a randomly chosen point of the unit cell. It is proportional to the typical variation of points in the volume, which in the case of a lattice is the square root of the average number of points in the last spherical shell of thickness equal to the lattice unit. A better (more statistically uniform) example of the same kind is the so-called *shuffled lattice* [6]; this is a lattice whose sites are independently randomly displaced by a distance x in all directions from their initial position according to some distribution $p(x)$ which has a finite second moment. In this case we find $P(k) \sim k^2$ at small k and, consequently, again $\sigma_M^2(R) \propto 1/R^4$ at large R. The simple lattice, however, is not a SSP, and even the shuffled lattice, though it can be defined as
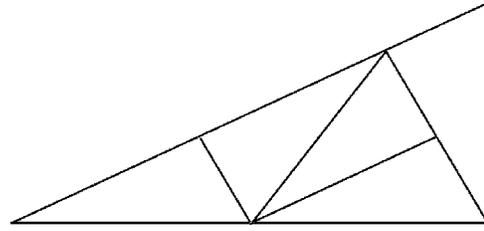

FIG. 2. Fragmentation step for the ''pinwheel'' tiling of the plane.

a SSP, is not statistically isotropic because the underlying lattice structure is not completely erased by the shuffling [20].

To construct a statistically isotropic and homogeneous particle distribution with such a behavior of $\sigma^2(R)$ is nontrivial. A particular example is the so-called ''pinwheel'' tiling of the plane [21,22]. The generation algorithm for it is defined by taking a right angled triangle with sides of respective lengths one and two (and hypotenuse $\sqrt{5}$) and, at the first step, forming five similar square triangle of sides $1/\sqrt{5}$ and $2/\sqrt{5}$ respectively as shown in Fig. 2. At the second step we expand these new triangles, to the size of the original triangle, and repeat the procedure *ad infinitum*, so that they cover the plane completely. Finally, placing a point randomly inside each elementary triangle will give a superhomogeneous point distribution which is statistically isotropic (with a continuous power spectrum).

IV. A PHYSICAL EXAMPLE: THE ONE-COMPONENT PLASMA

In the study of the one component plasma one considers the equilibrium statistical mechanics (at inverse temperature β) of a system of charged point particles interacting through a Coulomb potential, in a continuous uniform background giving overall charge neutrality [11]. Taking the particles to carry unit mass and charge the microscopic mass density of the particles is given by

$$\rho(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i) \tag{10}$$

where $\vec{r}_i$ is the position of the *i*th particle. The total microscopic charge density is

$$\rho_c(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i) - n \tag{11}$$

where

$$n = \langle \rho(\vec{r}) \rangle. \tag{12}$$

The two-point correlation function is defined as

$$\tilde{\xi}(\vec{r} - \vec{r}') = \frac{\langle \rho(\vec{r})\rho(\vec{r}') \rangle}{\langle \rho \rangle^2} - 1 = \frac{\delta(\vec{r} - \vec{r}')}{n} + h(\vec{r} - \vec{r}'). \tag{13}$$

The first term comes from the diagonal terms $\vec{r}=\vec{r}'$ in $\langle\rho(\vec{r})\rho(\vec{r}')\rangle$ and is characteristic of any point distribution.

It can be proven quite generally [8] that at high temperatures the OCP has a translation invariant isotropic distribution with $\tilde{\xi}(\vec{r})$ satisfying Eq. (8) with $r=|\vec{r}|$. In terms of $h(r)$ this gives

$$n\int h(r)d^3\vec{r}=-1. \tag{14}$$

Equation (14) is usually referred to as a sum rule and originates from the perfect screening of each charge caused by the long-range nature of the Coulomb potential. To see this let us suppose that an external infinitesimal charge density $\rho_{\text{ext}}=\epsilon e^{i\vec{k}\cdot\vec{r}}$ of very long wavelength is applied to the system. It creates an external electric potential

$$\Phi(\vec{r})=\int\frac{\rho_{\text{ext}}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'=\frac{4\pi}{k^2}\epsilon e^{i\vec{k}\cdot\vec{r}} \tag{15}$$

and a perturbation to the Hamiltonian

$$H_{\text{ext}}=\int\rho_c(\vec{r})\Phi(\vec{r})d^3\vec{r}=\epsilon\frac{4\pi}{k^2}\int\rho_c(\vec{r})e^{i\vec{k}\cdot\vec{r}}d^3\vec{r}. \tag{16}$$

Using linear response theory the induced charge in the system is given by

$$\rho_{\text{ind}}(\vec{r}')=-\beta\langle\rho_c(\vec{r}')H_{\text{ext}}\rangle \tag{17}$$

where the average is over the unperturbed statistical distribution. Thus, assuming that the applied charge is perfectly screened i.e. the system responds with an induced charge density $\rho_{\text{ind}}=-\rho_{\text{ext}}$ we have, in the limit $k\to0$,

$$-\epsilon e^{i\vec{k}\cdot\vec{r}'}\sim-\beta\epsilon\frac{4\pi}{k^2}\int\langle\rho_c(\vec{r}')\rho_c(\vec{r})\rangle e^{i\vec{k}\cdot\vec{r}}d^3\vec{r} \tag{18}$$

and therefore, since

$$P(k)=\frac{1}{n}+\tilde{h}(k) \tag{19}$$

where $\tilde{h}(k)$ is the Fourier transform of h, we find that

$$P(k)\sim\frac{k^2}{4\pi n^2\beta} \tag{20}$$

for small k.

The behavior of the power spectrum at small k is traceable through Eq. (15) as being simply, up to a factor β, the inverse of the Fourier transform (in the sense of distributions) of the repulsive $1/r$ potential. It is evident that what one needs to obtain the same kind of behavior but with $P(k)\sim k$ at small k is a repulsive potential of which the Fourier transform behaves as $1/k$, i.e. a repulsive $1/r^2$ potential. Be-

fore considering this in more detail we discuss some issues relevant to the cosmological context which we have passed over without comment so far.

V. DISCRETE VS CONTINUOUS

An essential difference between the case of cosmological perturbations and the system just discussed is that the former refers to a continuous density field while the latter describes the correlation properties of a set of discrete points. To interpret the latter as giving a realization of the former we obviously need to specify explicitly how to relate the two. There is no unique prescription to pass from a discrete to a continuous field. A prescription simply corresponds to a regularization of the Dirac delta with a function $W_L(\vec{r})$ with the property

$$W_L(\vec{r})=L^{-3}W_o\left(\frac{\vec{r}}{L}\right), \quad \int W_o(\vec{r})d^3\vec{r}=1, \tag{21}$$

where L is the characteristic scale introduced by the regularization e.g. the Gaussian

$$W_L(\vec{r})=\left(\frac{1}{\sqrt{2\pi}L}\right)^3\exp\left(-\frac{r^2}{2L^2}\right). \tag{22}$$

For any finite value of L we can define a continuous density field $\rho_L(\vec{r})$ as the convolution of this function with the discrete density field. The pair correlation function of the continuous field can then be written also as a double convolution integral of $W_L(\vec{r})$ and the correlation function in the discrete case. The singularity in the latter is thus also removed by the regularization. The power spectrum of this continuous field is then simply given as

$$P_L(\vec{k})=|W_L(\vec{k})|^2P_D(\vec{k}) \tag{23}$$

where $W_L(\vec{k})$ is the Fourier transform of the regularization $W_L(\vec{r})$ and $P_D(\vec{k})$ is the power spectrum of the discrete field. In particular for the Gaussian smoothing function we have

$$P_L(\vec{k})=\exp(-k^2L^2)P_D(\vec{k}). \tag{24}$$

For the OCP discussed above the correlation function, in the weak coupling limit, is approximately the Debye-Hückel formula

$$h(r)=-\beta\frac{e^{-\kappa r}}{r}, \quad \kappa^2=4\pi\beta n, \tag{25}$$

so that

$$P_D(\vec{k})=\frac{1}{n}\left[\frac{k^2}{\kappa^2+k^2}\right] \tag{26}$$

which is a non-integrable function of k. Integrability is ensured by the cutoff imposed at large k in the power spectrum through the smoothing function. An assumed property of the primordial density field in standard cosmological models is

that it is Gaussian distributed. Since the density field is inherently positive this assumption of Gaussianity is more properly attributed to fluctuations (expected to be small) around the uniform density. Small fluctuations in the discrete OCP are in fact Gaussian [9].

VI. INITIAL CONDITIONS IN *N*-BODY SIMULATIONS

A context in which it is necessary to have a concrete realization of the density $\rho(\vec{r})$ in cosmological models is that of the *N*-body approach to the problem of structure formation [12,13]. In these numerical simulations an initial configuration, which should represent the universe at early times, is evolved under its self-gravity. The goal of these simulations is an understanding of how structures grow in such models, and whether they are compatible with observations of the distribution of matter (as probed, primarily, by the distribution of galaxies and galaxy clusters). The initial continuous density field is thus necessarily represented by a discrete set of points, which, because the discretization scale does not represent the real underlying physical one, must be considered just as we have discussed above as representing the continuous density field via some appropriate smoothing.

In practice such a discretization is always generated in a very particular way [13,23]: points of equal mass are displaced from a "pre-initial" configuration in a way prescribed by the correlation field one wants to set up. This prescription can be understood as follows. Superimposing an infinitesimal displacement field $\vec{u}(\vec{r})$ on a uniform density ρ_o one has

$$\frac{\rho(\vec{r})-\rho_o}{\rho_o} \simeq -\vec{\nabla}\cdot\vec{u}(\vec{r}) \tag{27}$$

and therefore in *k* space the power spectrum of the correlated density field $\rho(\vec{r})$ produced is roughly proportional to $k^2 P_u(k)$, where $P_u(k)$ is that of the displacement field. The "pre-initial" configuration used, which gives the initial unperturbed positions of the particles, is a discretization of the initial uniform density field ρ_o and the displacement field is specified by the power spectrum of the continuous field which is desired as the IC in the simulation. In practice the former is taken to be either an exact simple lattice or a "glassy" configuration [13]. When this latter configuration is used it is generated by evolving the system with the *N*-body code, but with the sign of gravity reversed. This corresponds to a system essentially like the OCP (at low temperature), and the long-time evolution brings it to a highly uniform configuration with normalized variance decaying at large scales as $1/R^4$ [15].

In the generation of the IC in this way the fluctuations associated with the "pre-initial" configuration are neglected, or rather implicitly assumed to be negligible with respect to those which are introduced by the perturbations to it. The problem with this procedure is that it neglects precisely these "pre-initial" fluctuations. [If we assume the initial continuous field to have a power spectrum $P_i(k)$ which results from the infinitesimal shuffling is a power spectrum $P(k)=P_i(k)+k^2 P_u(k)$.] Such a neglect is justified if these fluc-

tuations play no role in the evolution of the system, as one wants to see only the growth of structure coming from the perturbations superimposed on the uniform density field. Indeed the reason the lattice or "glassy" configuration is used is that they are configurations in which the gravitational force is (effectively) zero, whereas in a Poisson distribution for example (which is *a priori* as good a discretization of the uniform background) it is not. While the Poisson distribution will thus evolve and form structures even without additional perturbation, the perfect lattice will not. That the discretized lattice or glass solves this problem is however far from clear: they both represent unstable point configurations with respect to gravity. A small applied perturbation will thus in general give an evolution which depends on the correlations and/or fluctuations in the initial unperturbed configurations. In principle such effects can be kept under control by making the discretization of the original density sufficiently fine. In practice in *N*-body simulations the "non-linear regime" in which structure formation takes place is comparable to the discretization scale, and the role of such effects seems, at the very least, problematic [14–17].

One way in which this difficulty could be got around is by generating directly a discrete distribution whose regularization is exactly the desired continuous field. Our observations on the OCP provide in principle a way of realizing this possibility.

VII. IC AS EQUILIBRIUM OF A MODIFIED OCP

We have seen that the OCP equilibrium correlations give surface fluctuations ($\langle \Delta M^2 \rangle \sim R^2$), but with a power spectrum at small *k* which varies like k^2. By considering instead a repulsive $1/r^2$ potential, whose Fourier transform is $2\pi^2/k$, we obtain from Eq. (19)

$$P(k)\sim \frac{k}{2\pi^2 n^2 \beta}, \text{ for } k\sim 0 \tag{28}$$

and

$$h(r)\sim -\frac{1}{2\pi^4\beta n^2}\frac{1}{r^4} \tag{29}$$

for $r\gg(2\pi^2\beta n)^{-1}$. The change from the exponential decay of Eq. (25) to a power-law decay is a result of the different analyticity properties of the two power spectra: the k^2 behavior is analytic at the origin, guaranteeing a rapidly decaying behavior of its Fourier transform, while the *k* spectrum is not.

In the context of cosmological *N*-body simulations what one needs is not simply the primordial HZ power spectrum with some appropriate small scale cutoff: what is simulated is only a part of the cosmological evolution, starting from a time at which the initial spectrum of fluctuations is already significantly modified from its primordial form. While purely gravitational evolution at these early times does not modify the HZ spectrum, non-gravitational effects, present until the time when the universe becomes dominated by matter, do so significantly. The nature of these modifications depends on
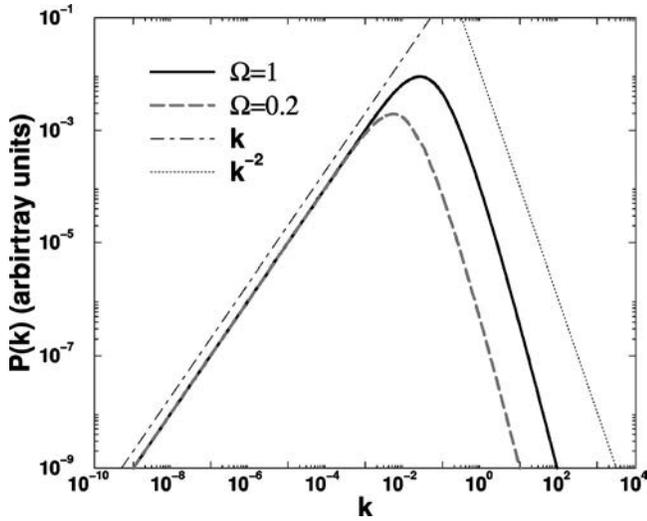
FIG. 3. Power spectra of two different cosmological models, both of the CDM (cold dark matter) type but with a differing density of matter (parametrized by Ω). The plot is in log-log units so that the behavior $P(k) \sim k$ and $P(k) \sim k^{-2}$ correspond to straight lines of slope $+1$ and -2 respectively.

the details of the cosmological model, but in all cases they affect only k larger than a characteristic k_{eq} corresponding to the causal horizon at the "time of equality" (when matter starts to dominate over radiation). One then has a power spectrum of the form

$$P(k) = k^n f(k) \qquad (30)$$

with n exactly (or very close to) unity and $f(k)$ a model-dependent form for the spectrum with $f(0) = a > 0$. In Fig. 3 are shown the power spectra of two representative cosmological models [1,2]. Both are CDM ("cold dark matter") type. Hot dark matter models, now strongly disfavored, have a more abrupt (typically exponential) cutoff at larger k, due to the "wash-out" of small scale structure associated with their higher velocity dispersions.

To produce such a spectrum as the equilibrium one of an OCP-like system requires further modification of the form of the potential. Just as in the standard OCP, an unmodified $1/r^2$ potential will give, in the weak-coupling limit, a spectrum which becomes flat (i.e. Poisson-like) at large k. This is simply due to the fact that in this limit the thermal fluctuations

dominate over the potential at small scales, effectively randomizing the particle positions up to some scale. A crude guess of what potential would produce the behavior of a typical cosmological model can be obtained by supposing that at small scales

$$1 + h(r) \approx e^{-\beta V(r)}, \qquad (31)$$

which corresponds to completely neglecting collective effects ($1 + h$ represents the relative probability compared to random of finding a particle at distance r from a given one). Given that the desired fluctuations always have small amplitudes ($|h| \ll 1$), we would then need to be in the regime of temperature and scales such that $|\beta V(r)| \ll 1$, so that $h(r) \sim -\beta V(r)$. The potential should thus be attractive at smaller scales, as the system is positively correlated at those scales. A k^{-2} behavior at large k, which is often used (see Fig. 3) as an initial condition approximating cosmological models (CDM type) in this regime (beyond the "turnover"), may be obtained from an attractive $1/r$ potential, e.g.

$$V(r) = \frac{1}{r^2} - \eta \frac{e^{-\mu r}}{r}. \qquad (32)$$

By modifying the parameters μ and η, as well as the temperature, both the amplitude of the $P(k)$ and the location of a change from a $P(k) \sim k^{-2}$ to a behavior $P(k) \sim k$ for small k can be controlled. The potential (32) is repulsive at short distances, but it may be necessary to make it more strongly repulsive in order to ensure that the system is not unstable to collapse [24]. Such issues, as well as the practical numerical implementation of these observations, will be investigated in forthcoming work. Once this method has been implemented to set up the initial conditions, it will naturally be interesting to understand the effect of this change in initial conditions on the dynamical evolution of gravitational N-body simulations.

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