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Scaling Dynamics of a Massive Piston in a Cube Filled with Ideal Gas: Exact Results

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We continue the study of the time evolution of a system consisting of a piston in a cubical container of large size L filled with an ideal gas. The piston has mass $M \sim L^2$ and undergoes elastic collisions with $N \sim L^3$ gas particles of mass m. In a previous paper, Lebowitz *et al.*[(1)] considered a scaling regime, with time and space scaled by L, in which they argued heuristically that the motion of the piston and the one particle distribution of the gas satisfy autonomous coupled differential equations. Here we state exact results and sketch proofs for this behavior.

KEY WORDS: Piston; ideal gas; hydrodynamic limit.

1. THE MODEL AND MAIN RESULTS

This paper is a continuation of ref. 1, where deterministic scaled equations describing the dynamics of a massive piston in a cubical container filled with ideal gas were given. Here we state exact conditions on the validity of those equations and outline the arguments. Full proofs will be published in a separate paper.[(2)]

We refer the reader to refs. 1 and 2 as well as refs. 3–7 for a detailed description of the problem of a massive piston moving in a cylinder. Here we just recall necessary facts.

Consider a cubical domain Λ_L of size L separated into two parts by a wall (piston), which can move freely without friction inside Λ_L. Each part of Λ_L is filled by a noninteracting gas of particles, each of mass m. The

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Dedicated to Robert Dorfman on the occasion of his 65th birthday.

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piston has mass $M = M_L$ and moves along the x-axis under the action of elastic collisions with the particles. The size L of the cube is a large parameter of the model, and we are interested in the limit behavior as $L \to \infty$. The mass m of gas particles is fixed. We will assume that M grows proportionally to L^2 and the number of gas particles N is proportional to L^3, while their velocities remain of order one.

The position of the piston at time t is specified by a single coordinate $X = X_L(t)$, $0 \leqslant X \leqslant L$, its velocity is then given by $V = V_L(t) = \dot{X}_L(t)$. Since the components of the particle velocities perpendicular to the x-axis play no role in the dynamics, we may assume that each particle has only one coordinate, x, and one component of velocity, v, directed along the x-axis.

When a particle with velocity v hits the piston with velocity V, their velocities after the collision, v' and V', respectively, are given by

$$V' = (1-\varepsilon)\, V + \varepsilon v \tag{1.1}$$

$$v' = -(1-\varepsilon)\, v + (2-\varepsilon)\, V \tag{1.2}$$

where $\varepsilon = 2m/(M+m)$. We assume that $M+m = 2mL^2/a$, where $a > 0$ is a constant, so that

$$\varepsilon = \frac{2m}{M+m} = \frac{a}{L^2} \tag{1.3}$$

When a particle collides with a wall at $x = 0$ or $x = L$, its velocity just changes sign.

The evolution of the system is completely deterministic, but one needs to specify initial conditions. We shall assume that the piston starts at the midpoint $X_L(0) = L/2$ with zero velocity $V_L(0) = 0$. The initial configuration of gas particles is chosen at random as a realization of a (two-dimensional) Poisson process on the (x, v)-plane (restricted to $0 \leqslant x \leqslant L$) with density $L^2 p_L(x, v)$, where $p_L(x, v)$ is a function satisfying certain conditions, see below, and the factor of L^2 is the cross-sectional area of the container. In other words, for any domain $D \subset [0, L] \times \mathbb{R}^1$ the number of gas particles $(x, v) \in D$ at time $t = 0$ has a Poisson distribution with parameter $\lambda_D = L^2 \iint_D p_L(x, v)\, dx\, dv$.

Let Ω_L denote the space of all possible configurations of gas particles in Λ_L. For each realization $\omega \in \Omega_L$ the piston trajectory will be denoted by $X_L(t, \omega)$ and its velocity by $V_L(t, \omega)$.

As $L \to \infty$, space and time are rescaled as

$$y = x/L \qquad \text{and} \qquad \tau = t/L. \tag{1.4}$$

which is a typical rescaling for the hydrodynamic limit transition (see refs. 1 and 2 for motivation and physical discussion). We call y and τ the *macroscopic* ("slow") variable, as opposed to the original *microscopic* ("fast") x and t. Now let

$$Y_L(\tau, \omega) = X_L(\tau L, \omega)/L, \qquad W_L(\tau, \omega) = V_L(\tau L, \omega) \tag{1.5}$$

be the position and velocity of the piston in the macroscopic variables. The initial density $p_L(x, v)$ satisfies

$$p_L(x, v) = \pi_0(x/L, v)$$

where the function $\pi_0(y, v)$ is independent of L. Without loss of generality, assume that π_0 is normalized so that

$$\int_0^1 \int_{-\infty}^{\infty} \pi_0(y, v)\, dv\, dy = 1$$

Then the mean number of particles in the entire container Λ_L is equal exactly to

$$E(N) = \int_0^L \int_{-\infty}^{\infty} L^2 p_L(x, v)\, dv\, dx = L^3$$

In order to describe the dynamics by differential equations, we assume that the function $\pi_0(y, v)$ satisfies several technical requirements stated below.

(P1) *Smoothness.* $\pi_0(y, v)$ is a piecewise C^1 function with uniformly bounded partial derivatives, i.e., $|\partial \pi_0/\partial y| \leqslant D_1$ and $|\partial \pi_0/\partial v| \leqslant D_1$ for some $D_1 > 0$.

(P2) *Discontinuity lines.* $\pi_0(y, v)$ may be discontinuous on the line $y = Y_L(0)$ (i.e., "on the piston"). In addition, it may have a finite number ($\leqslant K_1$) of other discontinuity lines in the (y, v)-plane with strictly positive slopes (each line is given by an equation $v = f(y)$ where $f(y)$ is C^1 and $0 < c_1 < f'(y) < c_2 < \infty$).

(P3) *Density bounds.* Let

$$\pi_0(y, v) > \pi_{\min} > 0 \qquad \text{for} \quad v_1 < |v| < v_2 \tag{1.6}$$

for some $0 < v_1 < v_2 < \infty$, and

$$\sup_{y, v} \pi_0(y, v) = \pi_{\max} < \infty \tag{1.7}$$

The requirements (1.6) and (1.7) basically mean that $\pi_0(y, v)$ takes values of order one.

(P4) *Velocity "cutoff."* Let

$$\pi_0(y, v) = 0, \quad \text{if} \quad |v| \leqslant v_{\min} \quad \text{or} \quad |v| \geqslant v_{\max} \tag{1.8}$$

with some $0 < v_{\min} < v_{\max} < \infty$. This means that the speed of gas particles is bounded from above by $v_{\max}$ and from below by $v_{\min}$.

(P5) *Approximate pressure balance.* $\pi_0(y, v)$ must be nearly symmetric about the piston, i.e.,

$$|\pi_0(y, v) - \pi_0(1-y, -v)| < \varepsilon_0 \tag{1.9}$$

for all $0 < y < 1$ and some sufficiently small $\varepsilon_0 > 0$.

The requirements (P4) and (P5) are made to ensure that the piston velocity $|V_L(t, \omega)|$ will be smaller than the minimum speed of the particles, with probability close to one, for times $t = O(L)$. Such assumptions were first made in ref. 1.

We think of D_1, K_1, c_1, c_2, v_1, v_2, $v_{\min}$, $v_{\max}$, $\pi_{\min}$ and $\pi_{\max}$ in (P1)–(P4) as fixed (global) constants and ε_0 in (P5) as an adjustable small parameter. We will assume throughout the paper that ε_0 is small enough, meaning that

$$\varepsilon_0 < \bar{\varepsilon}_0(D_1, K_1, c_1, c_2, v_1, v_2, v_{\min}, v_{\max}, \pi_{\min}, \pi_{\max})$$

It is important to note that the hydrodynamic limit does *not* require that $\varepsilon_0 \to 0$. The parameter ε_0 stays positive and fixed as $L \to \infty$.

Here is our main result:

Theorem 1.1. There is an L-independent function $Y(\tau)$ defined for all $\tau \geqslant 0$ and a positive $\tau_* \approx 2/v_{\max}$ (actually, $\tau_* \to 2/v_{\max}$ as $\varepsilon_0 \to 0$), such that

$$\sup_{0 \leqslant \tau \leqslant \tau_*} |Y_L(\tau, \omega) - Y(\tau)| \to 0 \tag{1.10}$$

and

$$\sup_{0 \leqslant \tau \leqslant \tau_*} |W_L(\tau, \omega) - W(\tau)| \to 0 \tag{1.11}$$

in probability, as $L \to \infty$. Here $W(\tau) = \dot{Y}(\tau)$.

This theorem establishes the convergence in probability of the random functions $Y_L(\tau, \omega)$, $W(\tau, \omega)$ characterizing the mechanical evolution of the piston to the deterministic functions $Y(\tau)$, $W(\tau)$, in the hydrodynamic limit $L \to \infty$.

The functions $Y(\tau)$ and $W(\tau)$ satisfy certain (Euler-type) differential equations stated in the next section. Those equations have solutions for all $\tau \geqslant 0$, but we can only guarantee the convergence (1.10) and (1.11) for $\tau < \tau_*$. What happens for $\tau > \tau_*$, especially as $\tau \to \infty$, remains an open problem. Some experimental results and heuristic observations in this direction are presented in ref. 8 and discussed in Section 4.

Remark 1. The function $Y(\tau)$ is at least C^1 and, furthermore, piecewise C^2 on the interval $(0, \tau_*)$. Its first derivative $W = \dot{Y}$ (velocity) and its second derivative $A = \ddot{Y}$ (acceleration) remain ε_0-small: $\sup_\tau |W(\tau)| \leqslant \text{const} \cdot \varepsilon_0$ and $\sup_\tau |A(\tau)| \leqslant \text{const} \cdot \varepsilon_0$, see the next section.

Remark 2. We also estimate the speed of convergence in (1.10) and (1.11): there is a $\tau_1 > 0$ ($\tau_1 \approx 1/v_{\max}$) such that $|Y_L(\tau, \omega) - Y(\tau)| = O(\ln L/L)$ for $0 < \tau < \tau_1$ and $|Y_L(\tau, \omega) - Y(\tau)| = O(\ln L/L^{1/7})$ for $\tau_1 < \tau < \tau_*$. The same bounds are valid for $|W_L(\tau, \omega) - W(\tau)|$. These estimates hold with "overwhelming" probability, specifically they hold for all $\omega \in \Omega_L^* \subset \Omega_L$ such that $P(\Omega_L^*) = 1 - O(L^{-\ln L})$.

2. HYDRODYNAMICAL EQUATIONS

The equations describing the deterministic function $Y(\tau)$ involve another deterministic function—the density of the gas $\pi(y, v, \tau)$. Initially, $\pi(y, v, 0) = \pi_0(y, v)$, and for $\tau > 0$ the density $\pi(y, v, \tau)$ evolves according to the following rules.

(H1) *Free motion.* Inside the container the density satisfies the standard continuity equation for a noninteracting particle system without external forces:

$$\left(\frac{\partial}{\partial \tau} + v \frac{\partial}{\partial y} \right) \pi(y, v, \tau) = 0 \tag{2.1}$$

for all y except $y = 0$, $y = 1$ and $y = Y(\tau)$.

Equation (2.1) has a simple solution

$$\pi(y, v, \tau) = \pi(y - vs, v, \tau - s) \tag{2.2}$$

for $0 < s < \tau$ such that $y - vr \notin \{0, Y(\tau - r), 1\}$ for all $r \in (0, s)$. Equation (2.2) has one advantage over (2.1): it applies to all points (y, v), including those where the function π is not differentiable.

(H2) *Collisions with the walls.* At the walls $y = 0$ and $y = 1$ we have

$$\pi(0, v, \tau) = \pi(0, -v, \tau) \tag{2.3}$$

$$\pi(1, v, \tau) = \pi(1, -v, \tau) \tag{2.4}$$

(H3) *Collisions with the piston.* At the piston $y = Y(\tau)$ we have

$$\begin{aligned} \pi(Y(\tau)-0, v, \tau) &= \pi(Y(\tau)-0, 2W(\tau)-v, \tau) \qquad \text{for} \quad v < W(\tau) \\ \pi(Y(\tau)+0, v, \tau) &= \pi(Y(\tau)+0, 2W(\tau)-v, \tau) \qquad \text{for} \quad v > W(\tau) \end{aligned} \tag{2.5}$$

where v represents the velocity after the collision and $2W(\tau) - v$ that before the collision; here

$$W(\tau) = \frac{d}{d\tau} Y(\tau) \tag{2.6}$$

is the (deterministic) velocity of the piston.

It remains to describe the evolution of $W(\tau)$. Suppose the piston's position at time τ is Y and its velocity W. The piston is affected by the particles (y, v) hitting it from the right (such that $y = Y + 0$ and $v < W$) and from the left (such that $y = Y - 0$ and $v > W$). Accordingly, we define the density of the particles colliding with the piston ("density on the piston") by

$$q(v, \tau; Y, W) = \begin{cases} \pi(Y+0, v, \tau) & \text{if} \quad v < W \\ \pi(Y-0, v, \tau) & \text{if} \quad v > W \end{cases} \tag{2.7}$$

(H4) *Piston's velocity.* The velocity $W = W(\tau)$ of the piston must satisfy the equation

$$\int_{-\infty}^{\infty} (v - W)^2 \operatorname{sgn}(v - W)\, q(v, \tau; Y, W)\, dv = 0 \tag{2.8}$$

We also remark that for $\tau > 0$, when (2.5) holds,

$$W(\tau) = \frac{\int v\pi(Y-0, v, \tau)\, dv}{\int \pi(Y-0, v, \tau)\, dv} = \frac{\int v\pi(Y+0, v, \tau)\, dv}{\int \pi(Y+0, v, \tau)\, dv}$$

i.e., the piston's velocity is the average of the nearby particle velocities on each side.

The system of (hydrodynamical) equations (H1)–(H4) is now closed and, given appropriate initial conditions, should completely determine the functions $Y(\tau)$, $W(\tau)$ and $p(y, v, \tau)$ for $\tau > 0$. To specify the initial conditions, we set $p(y, v, 0) = \pi(y, v)$ and $Y(0) = 0.5$. The initial velocity $W(0)$ does not have to be specified, it comes "for free" as the solution of the equation (2.8) at time $\tau = 0$. It is easy to check that the initial speed $|W(0)|$ will be smaller than $v_{\min}$, in fact $W(0) \to 0$ as $\varepsilon_0 \to 0$ in (P5). Note that if the initial conditions at $\tau = 0$ do not satisfy (2.3)–(2.5), there will be a discontinuity in p as $\tau \to 0$ (see also Remark 4 later).

Equation (2.8) has a unique solution W as long as the piston interacts with some gas particles on both sides, i.e., as long as

$$\inf\{v \colon \pi(Y+0, v, \tau) > 0\} \leqslant \sup\{v \colon \pi(Y-0, v, \tau) > 0\}$$

Indeed, the left hand side of (2.8) is a continuous and strictly monotonically decreasing function of W, and it takes both positive and negative values. The solution $W(\tau)$ may not be continuous in τ, though. But if $\pi(y, v, \tau)$ is piecewise C^1 and has a finite number of discontinuity lines with positive slopes (as we require of $\pi_0(y, v)$ in Section 1), then $W(\tau)$ will be continuous and piecewise differentiable.

Remark 3. One can easily check that the total mass $\mathscr{M} = \iint \pi(y, v, \tau)\, dv\, dy$ and the total kinetic energy $2E = \iint v^2 \pi(y, v, \tau)\, dv\, dy$ remain constant along any solution of our system of equations (H1)–(H4). Also, the mass in the left and right part of Λ_L separately remains constant. Equation (2.8) also preserves the total momentum of the gas $\iint v\pi(y, v, \tau)\, dv\, dy$, but it changes due to collisions with the walls.

Remark 4. Previously, Lebowitz *et al.*[1] studied the piston dynamics under essentially the same initial conditions as our (P1)–(P5). They argued heuristically that the piston dynamics could be approximated by certain deterministic equations in the original (microscopic) variables x and t. The deterministic equations found in ref. 1 correspond to our (2.2)–(2.6) with obvious transformation back to the variables x, t, but our main equation (2.8) has a different counterpart in the context of ref. 1, which reads

$$\frac{d}{dt} V(t) = a \left[\int_V^\infty (v - V(t))^2\, \pi(Y-0, v, t)\, dv - \int_{-\infty}^V (v - V(t))^2\, \pi(Y+0, v, t)\, dv \right] \tag{2.9}$$

Here $X = X(t)$ and $V = V(t) = \dot{X}(t)$ denote the deterministic position and velocity of the piston and $\pi(x, v, t)$ the density of the gas (the constant a appeared in (1.3)). We refer to ref. 1 for more details and a heuristic derivation of (2.9). Since (2.9), unlike our (2.8), is a differential equation, the initial velocity $V(0)$ has to be specified separately, and it is customary to set $V(0) = 0$. Alternatively, one can set $V(0) = W(0)$, see ref. 2. Equation (2.9) can be reduced to (2.8) in the limit $L \to \infty$ as follows. One can show (we omit details) that (2.9) is a dissipative equation whose solution with any (small enough) initial condition $V(0)$ converges to the solution of (2.8) during a t-time interval of length $\sim \ln L$. That interval has length $\sim L^{-1} \ln L$ on the τ axis, and so it vanishes as $L \to \infty$, this is why we replace (2.9) with (2.8) and ignore the initial condition $V(0)$ when working with the thermodynamic variables τ and y. Equation (2.9) is not used in this paper.

We now describe the solution of our equations (H1)–(H4) in more detail. Assume that for some $\tau > 0$ the gas density $\pi(y, v, \tau)$ satisfies the same requirements (P1)–(P4) as those imposed on the initial function $\pi_0(y, v)$ in Section 1, with constants D'_1, K'_1, c'_1, c'_2, v'_1, v'_2, $v'_{\min}$, $v'_{\max}$, $\pi'_{\min}$ and $\pi'_{\max}$, whose values are not essential, but are independent of τ.

We also assume an analogue of (P5), but this one is not so straightforward, since the piston does not have to stay at the middle point $y = 0.5$ at any time $\tau > 0$. We require that

$$|Y(\tau) - 0.5| < \varepsilon'_0 \tag{2.10}$$

and for any point (y, v) with $v'_{\min} \leqslant |v| \leqslant v'_{\max}$ there is another point (y_*, v_*) "across the piston" (i.e., such that $(y - Y(\tau))(y_* - Y(\tau)) < 0$) satisfying

$$|y + y_* - 1| < \varepsilon'_0, \qquad |v + v_*| < \varepsilon'_0 \tag{2.11}$$

and

$$|\pi(y, v, \tau) - \pi(y_*, v_*, \tau)| < \varepsilon'_0 \tag{2.12}$$

for some sufficiently small $\varepsilon'_0 > 0$. Actually, the map $(y, v) \mapsto (y_*, v_*)$, which we denote by R_τ, is one-to-one and will be explicitly constructed below. The constant ε'_0 here, just like ε_0 in (P5), is assumed to be small enough, and moreover

$$\varepsilon'_0 < C'_0 \varepsilon_0 \tag{2.13}$$

with some constant $C'_0 > 0$.

We now derive elementary but important consequences of the above assumptions. Since the density $\pi(y, v, \tau)$ vanishes for $|v| < v'_{\min}$, so does the function $q(v, \tau; Y, W)$ defined by (2.7). Moreover, for all $|W| < v'_{\min}$, the function $q(v, \tau; Y, W)$ will be independent of W, and so we can write it as $q(v, \tau; Y)$. Also, Eq. (2.8) can be simplified: the factor $\operatorname{sgn}(v-W)$ can be replaced by $\operatorname{sgn} v$. Then, expanding the square in (2.8) reduces it to a quadratic equation for W:

$$Q_0 W^2 - 2Q_1 W + Q_2 = 0 \tag{2.14}$$

where

$$Q_0 = \int \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.15}$$

$$Q_1 = \int v \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.16}$$

$$Q_2 = \int v^2 \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.17}$$

with $Y = Y(\tau)$. The integrals Q_0, Q_1, Q_2 have the following physical meaning:

$$\begin{aligned} mQ_0 &= m_L - m_R \\ mQ_1 &= p_L - p_R \\ mQ_2 &= 2(e_L - e_R) \end{aligned}$$

where m_L, p_L, e_L represent the total mass, momentum and energy of the incoming gas particles (per unit length) on the left hand side of the piston, and m_R, p_R, e_R—those on the right hand side of it. The value Q_2 also represents the net pressure exerted on the piston by the gas as if the piston did not move. Of course, if $Q_2(\tau) = 0$, then we must have $W(\tau) = 0$, which agrees with (2.14).

Next, under the above requirements on $\pi(y, v, \tau)$, the function $q(v, \tau; Y)$ is, in a certain sense, nearly symmetric in v about $v = 0$ (see ref. 2 for details). This fact implies that Q_0 and Q_2 are small, more precisely

$$\max\{|Q_0|, |Q_2|\} \leqslant C' \varepsilon_0 \tag{2.18}$$

where $C' > 0$ is a constant depending on the parameters D'_1, K'_1, etc., but not on ε_0. At the same time, the assumption (P3) guarantees that

$$Q_1 \geqslant Q_{1,\min} > 0 \tag{2.19}$$

where $Q_{1,\min}$ is a constant depending on $\pi'_{\min}, v'_1, v'_2$, etc., but not on ε_0.

If ε_0 is small enough, there is a unique root of the quadratic polynomial (2.14) on the interval $(-v'_{\min}, v'_{\min})$, which corresponds to the only solution of (2.8). Since this root is smaller, in absolute value, than the other root of (2.14), it can be expressed by

$$W(\tau) = \frac{Q_1 - \sqrt{Q_1^2 - Q_0 Q_2}}{Q_0} \tag{2.20}$$

where the sign before the radical is "$-$," not "$+$." Of course, (2.20) applies whenever $Q_0 \neq 0$, while for $Q_0 = 0$ we simply have $W(\tau) = Q_2/2Q_1$.

Equations (2.18)–(2.20) imply an upper bound on the piston velocity: $|W(\tau)| \leqslant B'\varepsilon_0$ for some constant $B' > 0$ depending on D'_1, K'_1, etc., but not on ε_0. A similar bound holds for the piston acceleration $A(\tau) = \dot{W}(\tau)$, since

$$A(\tau) = \frac{(dQ_0/d\tau)\, W^2 - 2(dQ_1/d\tau)\, W + (dQ_2/d\tau)}{2(Q_1 - Q_0 W)}$$

and $|dQ_i/d\tau| = |(dQ_i/dY)\, W| \leqslant \text{const} \cdot \varepsilon_0$, see ref. 2 for more details.

Next we consider the evolution of a point (y, v) in the domain $G := \{(y, v) \colon 0 \leqslant y \leqslant 1\}$ under the rules (H1)–(H3), i.e., as it moves freely with constant velocity and collides elastically with the walls and the piston. Denote by (y_τ, v_τ) its position and velocity at time $\tau \geqslant 0$. Then (H1) translates into $\dot{y}_\tau = v_\tau$ and $\dot{v}_\tau = 0$ whenever $y_\tau \notin \{0, 1, Y(\tau)\}$, (H2) becomes $(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, -v_{\tau-0})$ whenever $y_{\tau-0} \in \{0, 1\}$, and (H3) gives

$$(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, 2W(\tau) - v_{\tau-0}) \tag{2.21}$$

whenever $y_{\tau-0} = Y(\tau)$. Note that (2.21) corresponds to a special case of the mechanical collision rules (1.1)–(1.2) with $\varepsilon = 0$ (equivalently, $m = 0$). Hence the point (y, v) moves in G as if it was a gas particle with zero mass.

The motion of points in (y, v) is described by a one-parameter family of transformations $F^\tau \colon G \to G$ defined by $F^\tau(y_0, v_0) = (y_\tau, v_\tau)$ for $\tau > 0$. We will also write $F^{-\tau}(y_\tau, v_\tau) = (y_0, v_0)$. According to (H1)–(H3), the density $\pi(y, v, \tau)$ satisfies a simple equation

$$\pi(y_\tau, v_\tau, \tau) = \pi(F^{-\tau}(y_\tau, v_\tau), 0) = \pi_0(y_0, v_0) \tag{2.22}$$

for all $\tau \geqslant 0$. Also, it is easy to see that for each $\tau > 0$ the map F^τ is one-to-one and preserves area, i.e., $\det |DF^\tau(y, v)| = 1$.

Now, because of (P4), the initial density $\pi_0(y, v)$ can only be positive in the region

$$G^+ := \{(y, v) \colon 0 \leqslant y \leqslant 1,\ v_{\min} \leqslant |v| \leqslant v_{\max}\}$$

hence we will restrict ourselves to points $(y, v) \in G^+$ only. At any time $\tau > 0$, the images of those points will be confined to the region $G^+(\tau) := F^\tau(G^+)$. In particular, $\pi(y, v, \tau) = 0$ for $(y, v) \notin G^+(\tau)$.

The map $R_\tau: (y, v) \mapsto (y_*, v_*)$ involved in (2.11) and (2.12) can now be defined as $R_\tau = F^\tau \circ R_0 \circ F^{-\tau}$, where $R_0(y, v) = (1-y, -v)$ is a simple reflection "across the piston" at time $\tau = 0$.

We now make an important observation. If a fast point (y_τ, v_τ) collides with a slow piston, $|W(\tau)| \ll |v_\tau|$, they cannot recollide too soon: the point must travel to a wall, bounce off it, and then travel back to the piston before it hits it again.

Therefore, as long as (P1)–(P4) hold, the collisions of each moving point $(y_\tau, v_\tau) \in G^+(\tau)$ with the piston occur at well separated time moments, which allows us to effectively count them. For $(x, v) \in G^+$

$$N(y, v, \tau) = \#\{s \in (0, \tau) : y_s = Y(s), v_s \neq W(s)\}$$

is the number of collisions of the point (y, v) with the piston during the interval $(0, \tau)$. For each $\tau > 0$, we partition the region $G^+(\tau)$ into subregions

$$G_n^+(\tau) := \{F^\tau(y, v) : (y, v) \in G^+ \ \& \ N(y, v, \tau) = n\}$$

so $G_n^+(\tau)$ is occupied by the points that at time τ have experienced exactly n collisions with the piston during the interval $(0, \tau)$.

Now, for each $n \geqslant 1$ we define $\tau_n > 0$ to be the first time when a point $(y_\tau, v_\tau) \in G^+(\tau)$ experiences its $(n+1)$-st collision with the piston, i.e.,

$$\tau_n = \sup\{\tau > 0 : G_{n+1}^+(\tau) = \varnothing\}$$

In particular, $\tau_1 > 0$ is the earliest time when a point $(y_\tau, v_\tau) \in G^+(\tau)$ experiences its first recollision with the piston. Hence, no recollisions occur on the interval $[0, \tau_1)$, and we call it the *zero-recollision interval*. Similarly, on the interval (τ_1, τ_2) no more than one recollision with the piston is possible for any point, and we call it the *one-recollision interval*.

The time moment τ_* mentioned in Theorem 1.1 is the earliest time when a point $(y_\tau, v_\tau) \in G^+(\tau)$ either experiences its third collision with the piston or has its second collision with the piston given that the first one occurred after τ_1. Hence, $\tau_* \leqslant \tau_2$, and actually τ_* is very close to τ_2, see below.

The following theorem summarizes the properties of the solutions of the hydrodynamical equations (H1)–(H4).

Theorem 2.1 (ref. 2). Let $T>0$ be given. If the initial density $\pi_0(y, v)$ satisfies (P1)–(P5) with a sufficiently small ε_0, then

(a) the solution of our hydrodynamical equations (H1)–(H4) exists and is unique on the interval $(0, T)$;

(b) the density $\pi(y, v, \tau)$ satisfies conditions similar to (P1)–(P4) for all $0<\tau<T$, it also satisfies (2.10)–(2.13);

(c) the piston velocity and acceleration remain small, $|W(\tau)|=O(\varepsilon_0)$ and $|A(\tau)|=O(\varepsilon_0)$;

(d) we have $|\tau_k - k/v_{\max}| = O(\varepsilon_0)$ for all $1 \leqslant k < Tv_{\max}$, and if $Tv_{\max} > 2$, then also $|\tau_* - \tau_2| = O(\varepsilon_0)$.

Corollary 2.2. If $\varepsilon_0 = 0$, so that the initial density $\pi_0(y, v)$ is completely symmetric about the piston, the solution is trivial: $Y(\tau) \equiv 0.5$ and $W(\tau) \equiv 0$ for all $\tau > 0$.

Lastly, we demonstrate the reason for our assumption that all the discontinuity curves of the initial density $\pi_0(y, v)$ must have positive slopes. It would be quite tempting to let $\pi_0(y, v)$ have more general discontinuity lines, e.g., allow it be smooth for $v_{\min} < |v| < v_{\max}$ and abruptly drop to 0 at $v = v_{\min}$ and $v = v_{\max}$. The following example shows why this is not acceptable.

Example. Suppose the initial density $\pi_0(y, v)$ has a horizontal discontinuity line $v = v_0$ (say, $v_0 = v_{\min}$ or $v_0 = v_{\max}$). After one interaction with the piston the image of this discontinuity line can oscillate up and down, due to the fluctuations of the piston acceleration (Fig. 1). As time goes on, this oscillating curve will "travel" to the wall and come back to the piston, experiencing some distortions on its way, caused by the differences in

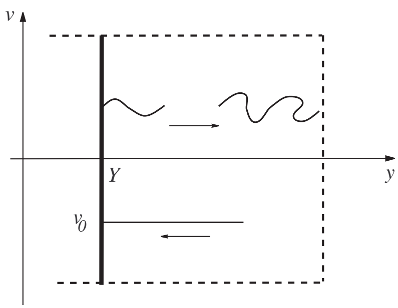

Fig. 1. A horizontal discontinuity line (bottom) comes off the piston as an oscillating curve (top).

velocities of its points (Fig. 1). When this curve comes back to the piston again, it may well have "turning points" where its tangent line is vertical, or even contain vertical segments of positive length. This produces unwanted singularities or even discontinuities of the piston velocity and acceleration. The same phenomena can also occur when a discontinuity line of the initial density $\pi_0(y, v)$ has a negative slope.

3. SKETCH OF THE ARGUMENT

Our proof of Theorem 1.1 is based on large deviation estimates for the Poisson random variable:

Lemma 3.1 (ref. 2). Let X be a Poisson random variable with parameter $\lambda > 0$. For any $b > 0$ there is a $c > 0$ such that for all $0 < B < b\sqrt{\lambda}$ we have

$$P(|X-\lambda| > B\sqrt{\lambda}) \leqslant 2e^{-cB^2}$$

This shows that the probabilities of large deviations rapidly decay, as they do for the Gaussian distribution.

The principal step in our proof of Theorem 1.1 is the velocity decomposition scheme described next. Let $V_L(t, \omega)$ be the velocity of the piston at time $t \geqslant 0$ for a random configuration of particles $\omega \in \Omega_L$. Let $\Delta t > 0$ be a small time increment. Then the law of elastic collision (1.1) implies

$$V_L(t+\Delta t, \omega) = (1-\varepsilon)^k V_L(t, \omega) + \varepsilon \sum_{j=1}^{k} (1-\varepsilon)^{k-j} \cdot v_j \tag{3.1}$$

Here $k = k(t, \Delta t, \omega)$ is the number of particles colliding with the piston during the time interval $(t, t+\Delta t)$, and v_j are their velocities numbered in the order in which the particles collide.

We rearrange the formula (3.1) as follows:

$$V_L(t+\Delta t, \omega) = (1-\varepsilon k)\, V_L(t, \omega) + \varepsilon \sum_{j=1}^{k} v_j + \chi^{(1)} + \chi^{(2)} \tag{3.2}$$

where

$$\chi^{(1)} = V_L(t, \omega)[(1-\varepsilon)^k - 1 + \varepsilon k]$$

and

$$\chi^{(2)} = \varepsilon \sum_{j=1}^{k} v_j [(1-\varepsilon)^{k-j} - 1]$$

Let us assume that the fluctuations of the velocity $V_L(s, \omega)$ on the interval $(t, t+\Delta t)$, are bounded by some quantity δV:

$$\sup_{s \in (t, t+\Delta t)} |V_L(s, \omega) - V_L(t, \omega)| \leqslant \delta V \tag{3.3}$$

Consider two regions on the x, v plane:

$$D_1 = \left\{ (x, v) \colon \frac{v - V_L(t, \omega) - (\operatorname{sgn} v)\, \delta V}{x - X_L(t, \omega)} < -\frac{1}{\Delta t},\ v_{\min} < |v| < v_{\max} \right\} \tag{3.4}$$

and

$$D_2 = \left\{ (x, v) \colon \frac{v - V_L(t, \omega) + (\operatorname{sgn} v)\, \delta V}{x - X_L(t, \omega)} < -\frac{1}{\Delta t},\ v_{\min} < |v| < v_{\max} \right\} \tag{3.5}$$

Each of them is a union of two trapezoids $D_i = D_i^+ \cup D_i^-$, $i = 1, 2$, where D_i^- denotes the upper and D_i^+ the lower trapezoid, see Fig. 2.

Note that $D_1 \subset D_2$. The bound (3.3) implies that all the particles in the region D_1 necessarily collide with the piston during the time interval $(t, t+\Delta t)$. Moreover, the trajectory of every point $(x, v) \in D_1$ hits the piston

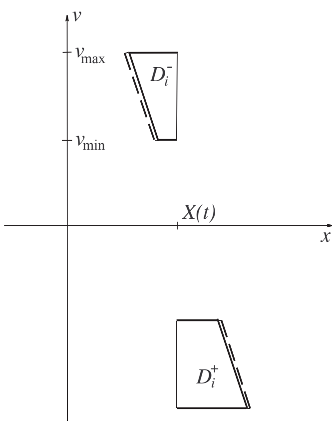

Fig. 2. Region D_1 is bounded by solid lines. Region D_2 is bounded by a dashed line.

within time Δt. The bound (3.3) also implies that all the particles actually colliding with the piston during the interval $(t, t+\Delta t)$ are contained in D_2.

Let us denote by $k_r^\pm$ the number of particles in the regions $D_r^\pm$ for $r=1, 2$ at time t. We also denote by k^- the number of particles actually colliding with the piston "on the left," and by k^+ that number "on the right" (of course, $k^- + k^+ = k$). Due to the above observations, $k_1^\pm \leqslant k^\pm \leqslant k_2^\pm$.

Now, suppose that $t+\Delta t < \tau_1 L$. Then we show that for typical configurations ω the particles in each domain D_r, $r=1, 2$, have never collided with the piston before. Therefore, their number, $k_r^\pm$, $r=1, 2$, satisfies the laws of Poisson distribution, in particular, the large deviation estimate in Lemma 3.1 applies. This gives the bound (for typical ω)

$$\lambda_1^\pm - \Delta k_1^\pm \leqslant k^\pm \leqslant \lambda_2^\pm + \Delta k_2^\pm \tag{3.6}$$

where

$$\lambda_r^\pm = E(k_r^\pm) = L^2 \int_{F_L^{-t}(D_r^\pm)} p_L(x, v)\, dx\, dv$$

and F_L^{-t} corresponds to the action of $F^{-\tau} = F^{-t/L}$ in the original time-space coordinates x, t. The deviations $\Delta k_r^\pm$ in (3.6) can be adjusted by using Lemma 3.1. The difference $\lambda_2^\pm - \lambda_1^\pm$ is estimated by

$$\lambda_2^\pm - \lambda_1^\pm = L^2 \int_{F_L^{-t}(D_2^\pm \setminus D_1^\pm)} p_L(x, v)\, dx\, dv \leqslant \text{const} \cdot L^2\, \delta V\, \Delta t$$

By putting all these estimates together we get tight bounds on k in (3.2). Similarly we get bounds on $\sum_{j=1}^{k} v_j$ in (3.2). The following is the final result of this analysis:

$$V_L(t+\Delta t, \omega) - V_L(t, \omega) = \mathscr{D}(t, \omega)\, \Delta t + \chi_3 \tag{3.7}$$

Here

$$\mathscr{D}(t, \omega) = a[Q_0 V_L^2(t, \omega) - 2Q_1 V_L(t, \omega) + Q_2] \tag{3.8}$$

and Q_0, Q_1, Q_2 are defined similarly to (2.15)–(2.17), in which $Y(\tau)$ must be replaced by the actual piston position $X_L(t, \omega)/L$. The error term χ_3 in (3.7) is bounded by

$$|\chi_3| \leqslant \text{const} \cdot \frac{\ln L \sqrt{\Delta t}}{L} \tag{3.9}$$

which corresponds to Brownian motion-type random fluctuations.

The term $\mathscr{D}(t, \omega)$ in (3.7) represents the main ("deterministic") force acting on the piston. The term χ_3 describes random fluctuations of that force. When the piston velocity stabilizes, then the main force $\mathscr{D}$ should vanish, and an "equilibrium" velocity $\bar{V}_L(t, \omega)$ will be established. The latter is the root of the equation $\mathscr{D}(t, \omega) = 0$, which is

$$\bar{V}_L(t, \omega) = \frac{Q_1 - \sqrt{Q_1^2 - Q_0 Q_2}}{Q_0} \tag{3.10}$$

The reason why $V_L(t, \omega)$ converges to $\bar{V}_L$ is that $\mathscr{D}$ is almost proportional to $\bar{V}_L - V_L$, i.e., $0 < E_1 < \mathscr{D}/(\bar{V}_L - V_L) < E_2 < \infty$ for some constants E_1, E_2. In fact, $\bar{V}_L(t, \omega)$ is a very slowly changing function of t, whose derivative is small: $|d\bar{V}_L(t, \omega)/dt| \leqslant \text{const} \cdot L^{-1}\varepsilon_0$. As a result, V_L will always stay close to $\bar{V}_L$, more precisely

$$|V_L(t, \omega) - \bar{V}_L(t, \omega)| < \text{const} \cdot L^{-1} \ln L \tag{3.11}$$

on the entire zero-recollision interval $0 < t < \tau_1 L$.

Now, the piston coordinate $Y_L(\tau, \omega) = X_L(\tau L, \omega)/L$ is the solution of the differential equation

$$\dot{Y}_L = V_L = \bar{V}_L + \chi_4$$

where $|\chi_4| < \text{const} \cdot L^{-1} \ln L$ by (3.11). On the other hand, the deterministic piston coordinate $Y(\tau)$ is the solution of the equation $\dot{Y} = W$, and both $\bar{V}$ and W are given by the same radical expression, cf. (2.20) and (3.10). Lastly, a simple application of Gronwall's inequality completes the proof of Theorem 1.1 on the zero-recollision interval $(0, \tau_1)$.

The proof on the one-recollision interval (τ_1, τ_*) goes along the same lines. One major difference is that the number of particles $k_r^{\pm}$, $r = 1, 2$, in the domain $D_r^{\pm}$ constructed in the velocity decomposition scheme is no longer a Poisson variable, so Lemma 3.1 does not apply directly.

To handle this new situation, we pull the domain $D_r^{\pm}$ back in time, as we did before. But now that pullback involves one interaction with the piston (corresponding to the first collision of the particles in $D_r^{\pm}$ with the piston, which occurs during the zero-recollision interval $0 < t < \tau_1 L$).

Since the piston position and velocity at the moment of that first collision are random, the preimage of $D_r^{\pm}$ will be a random domain. Its shape will depend on the piston velocity $V(t, \omega)$ during the zero-recollision interval $0 < t < \tau_1 L$. We observe that the boundary of the preimage of $D_r^{\pm}$ is described by a random, yet Hölder continuous function, and its Hölder exponent is 0.5 due to (3.9). Then we pick a small $d > 0$ and construct a d-dense set in the space of all Hölder continuous functions in the spirit of a work by Kolmogorov and Tihomirov.[9] The elements of that d-dense set can be used to construct a finite collection of (nonrandom) domains, so that one of them will approximate the (random) preimage of our $D_r^{\pm}$ (we need to select the small $d > 0$ carefully to ensure sufficient accuracy of the approximation). Now the number of particles in our random domain (the preimage of $D_r^{\pm}$) can be approximated by the number of particles in the corresponding nonrandom domain. The latter has Poisson distribution, and finally we can apply Lemma 3.1. This trick gives necessary estimates on $k_r^{\pm}$.

A full proof of Theorem 1.1 is given in ref. 2. At present, we do not know if this theorem can be extended beyond the critical time τ_*, this is an open question. Some other open problems are discussed in the next section.

4. DISCUSSION AND OPEN PROBLEMS

1. The main goal of this work is to prove that under suitable initial conditions random fluctuations in the motion of a massive piston are small and vanish in the thermodynamic limit. We are, however, able to control those fluctuations effectively only as long as the surrounding gas particles can be described by a Poisson process, i.e., during the zero-recollision interval $0 < \tau < \tau_1$. In that case the random fluctuations are bounded by $\text{const} \cdot L^{-1} \ln L$, see Remark 2 after Theorem 1.1. Up to the logarithmic factor, this bound is optimal, see ref. 8 and earlier estimates by Holley,[10] Dürr *et al.*[11]

During the one-recollision interval $\tau_1 < \tau < \tau_2$, the situation is different. The probability distribution of gas particles that have experienced one collision with the piston is no longer a Poisson process, it has intricate correlations. We are only able to show that random fluctuations remain bounded by $L^{-1/7}$, see again Remark 2. Perhaps, our bound is far from optimal, but our numerical experiments reported in ref. 8 show that random fluctuations indeed grow during the one-recollision interval.

We have tested numerically whether random fluctuations remained small after more than one recollision, i.e., at times $\tau > \tau_2$. We found that for some initial π_0 they actually increased very rapidly, and we conjectured that the rate of increase was exponential in τ. We found, indeed, that at times $\tau \sim \log L$ the fluctuations became large even on a macroscopic scale, and then many unexpected phenomena occurred.[8]

Interestingly, the exponential growth of random fluctuations seems to be related to the instability of our hydrodynamical equations. We found that small perturbations of the initial density π_0 can grow exponentially in

τ under certain conditions, matching the growth of random fluctuations of the piston motion in the mechanical model. We refer the reader to ref. 8 for further discussion and to our work in progress.[12]

2. It is clear that in our model recollisions of gas particles with the piston have a very "destructive" effect on the dynamics in the system. However, we need to distinguish between two types of recollisions.

We say that a recollision of a gas particle with the piston is *long* if the particle hits a wall $x = 0$ or $x = L$ between the two consecutive collisions with the piston. Otherwise a recollision is said to be *short*. Long recollisions require some time, as the particle has to travel all the way to a wall, bounce off it, and then travel back to the piston before it hits it again. Short recollisions can occur in rapid succession.

We have imposed the velocity cut-off (P4) in order to avoid any recollisions for at least some initial period of time (which we call the zero-recollision interval). More precisely, the upper bound $v_{\max}$ guarantees the absence of long recollisions. Without it, we would have to deal with arbitrarily fast particles that dash between the piston and the wall very many times in any interval $(0, \tau)$. On the other hand, the lower bound $v_{\min}$ was assumed to exclude short recollisions.

There are good reasons to believe, though, that short recollisions may not be so destructive for the piston dynamics. Indeed, let a particle experience two or more collisions with the piston in rapid succession (i.e., without hitting a wall in between). This can occur in two cases: (i) the particle's velocity is very close to that of the piston, or (ii) the piston's velocity changes very rapidly. The latter should be very unlikely, since the deterministic acceleration of the piston is very small, cf. Theorem 2.1(c). In case (i), the recollisions should have very little effect on the velocity of the piston according to the rule (1.1), so that they may be safely ignored, as it was done already in earlier studies.[10, 11]

We therefore expect that our results can be extended to velocity distributions without a cut-off from zero, i.e., allowing $v_{\min} = 0$.

3. In our paper, L plays a dual role: it parameterizes the mass of the piston ($M \sim L^2$), and it represents the length of the container ($0 \leqslant x \leqslant L$). This duality comes from our assumption that the container is a cube.

However, our model is essentially one-dimensional, and the mass of the piston M and the length of the interval $0 \leqslant x \leqslant L$ can be treated as two independent parameters. In particular, we can assume that the container is infinitely long in the x direction (so, *that* L is infinite), but the mass of the piston is still finite and given by $M \sim L^2$. In this case there are no recollisions with the piston, as long as its velocity remains small. Hence, our zero-recollision interval is effectively infinite. As a result, Theorem 1.1 can be extended to arbitrarily large times. Precisely, for any $T>0$ we can prove the convergence in probability:

$$P(\sup_{0\leqslant\tau\leqslant T} |Y_L(\tau,\omega)-Y(\tau)|\leqslant C_T \ln L/L)\to 1$$

and

$$P(\sup_{0\leqslant\tau\leqslant T} |W_L(\tau,\omega)-W(\tau)|\leqslant C_T \ln L/L)\to 1$$

as $L\to\infty$, where $C_T>0$ is a constant and $Y(\tau)$ and $W(\tau)=\dot{Y}(\tau)$ are the solutions of the hydrodynamical equations described in Section 2.

4. Along the same lines as above, we can assume that the container is d-dimensional with $d\geqslant 3$. Then the mass of the piston and the density of the particles are proportional to L^{d-1} rather than L^2.

When d is large, the gas particles are very dense on the x, v plane. This leads to a much better control over fluctuations of the particle distribution and the piston trajectory. As a result, Theorem 1.1 can be extended to the k-recollision interval (τ_k, τ_{k+1}), where $k\geqslant 1$ depends on d. It can be shown that for any $k\geqslant 1$ there is a $d_k\geqslant 3$ such that for all $d\geqslant d_k$ the convergence (1.10) and (1.11) holds with $\tau_*=\tau_k$. Therefore, a higher dimensional piston is more stable than a lower dimensional one.

It would be interesting to investigate other modifications of our model that lead to more stable regimes. For example, let the initial density $\pi_0(y,v)$ of the gas depend on the factor $a=\varepsilon L^2$ in such a way that $\pi_0(y,v)=a^{-1}\rho(y,v)$, where $\rho(y,v)$ is a fixed function. Then the particle density grows as $a\to 0$. This is another way to increase the density of the particles, but without changing the dimension. One may expect a better control over random fluctuations in this case, too.

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