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Evolution of a Model Quantum System Under Time Periodic Forcing: Conditions for Complete Ionization

O. Costin, R. D. Costin, J. L. Lebowitz, A. Rokhlenko

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

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Abstract: We analyze the time evolution of a one-dimensional quantum system with an attractive delta function potential whose strength is subjected to a time periodic (zero mean) parametric variation $\eta(t)$. We show that for generic $\eta(t)$, which includes the sum of any finite number of harmonics, the system, started in a bound state will get fully ionized as $t \to \infty$. This is irrespective of the magnitude or frequency (resonant or not) of $\eta(t)$. There are however exceptional, very non-generic $\eta(t)$, that do not lead to full ionization, which include rather simple explicit periodic functions. For these $\eta(t)$ the system evolves to a nontrivial localized stationary state which is related to eigenfunctions of the Floquet operator.

1. Introduction and Results

We are interested in the qualitative long time behavior of a quantum system evolving under a time dependent Hamiltonian $H(t) = H_0 + H_1(t)$, i.e. in the nature of the solutions of the Schrödinger equation

$$i\hbar\partial_t\psi = [H_0 + H_1(t)]\psi. \tag{1}$$

Here ψ is the wavefunction of the system, belonging to some Hilbert space $\mathcal{H}$, H_0 and H_1 are Hermitian operators and Eq. (1) is to be solved subject to some initial condition ψ_0. Such questions about the solutions of (1) belong to what Simon [1] calls "second level foundation" problems of quantum mechanics. They are of particular practical interest for the ionization of atoms and/or dissociation of molecules, in the case when H_0 has both a discrete and a continuous spectrum corresponding respectively to spatially localized (bound) and scattering (free) states in $\mathbb{R}^d$. Starting at time zero with the system in a bound state and then "switching on" at $t = 0$ an external potential $H_1(t)$, we want to know the "probability of survival", $P(t)$, of the bound states, at times $t > 0$: $P(t) = \sum_j |\langle\psi(t), u_j\rangle|^2$, where the sum is over all the bound states u_j [2–6, 8, 9].

This problem has been investigated both analytically and numerically for the case $H_1(t) = \eta(t)V_1(x)$ with $\eta(t) = r\sin(\omega t + \theta)$ and V_1 a time independent potential, $x \in \mathbb{R}^d$. When ω is sufficiently large for "one photon" ionization to take place, i.e., when $\hbar\omega > -E_0$, E_0 the energy of the bound (e.g. ground) state of H_0 and r is "small enough" for H_1 to be treated as a perturbation of H_0 then this is a problem discussed extensively in the literature ([8,9]). Starting with the system in its ground state the long time behavior of $P(t)$ is there asserted to be given by the $P(t) \sim \exp[-\Gamma_F t]$. The rate constant Γ_F is computed from first order perturbation theory according to Fermi's golden rule. It is proportional to the square of the matrix element between the bound and free states, multiplied by the appropriate density of continuum states in the vicinity of the final state which will have energy $\hbar\omega - E_0$ [6,8–10].

Going from perturbation theory to an exponential decay involves heuristics based on deep physical insights requiring assumptions which seem very hard to prove. It is therefore very gratifying that many features of this scenario have been recently made mathematically rigorous by Soffer and Weinstein [6] (their analysis was generalized by Soffer and Costin [7]). They considered the case when $H_0 = -\nabla^2 + V_0(x)$, $x \in \mathbb{R}^3$, V_0 compactly supported and such that there is exactly one bound state with energy $-\omega_0$ (from now on we use units in which $\hbar = 2m = 1$) and a continuum of quasi-energy states with energies k^2 for all $k \in \mathbb{R}^3$. The perturbing potential is $H_1(t) = r\cos(\omega t)V_1(x)$ with $V_1(x)$ also of compact support and satisfying some technical conditions. They then showed that for $\omega > \omega_0$ and r small enough there is indeed an intermediate time regime where $P(t)$ has a dominant exponential form with the Fermi exponent Γ_F. This regime is followed for longer times by an inverse power law decay. Some of these restrictions can presumably be relaxed but the requirement that r be small is crucial to their method which is essentially perturbative.

The behavior of $P(t)$ becomes much more difficult to analyze when the strength of $H_1(t)$ is not small and perturbation theory is no longer a useful guide. This became clear in the seventies with the beautiful experiments by Bayfield and Koch, cf. [11] for a review, on the ionization of highly excited Rydberg (e.g. hydrogen atoms) by intense microwave electric fields. These experiments showed quite unexpected nonlinear behavior of $P(t)$ as a function of the initial state, field strength E and the frequency ω. These results as well as other multiphoton ionizations of hydrogen atoms have been (and continue to be) analyzed by various authors using a variety of methods. Prominent among these are semi-classical phase-space analysis, numerical integration of the Schrödinger equation, Floquet theory, complex dilation, etc. While the results obtained so far are not rigorous, they do give physical insights and quite good agreement with experiments although many questions still remain open even on the physical level [11–15].

In addition to the above experiments on Rydberg atoms there are also many experiments which use strong laser fields to produce multiphoton ($\omega < -E_0$) ionization of multielectron atoms and/or dissociation of molecules [16,17]. These systems are more complex than Rydberg atoms and their analysis is correspondingly less developed. One unexpected result of certain studies is that an increase in the intensity of the field may reduce the degree of ionization, i.e., $P(t)$ can be non-monotone in the field strength E at large values of E. This phenomenon, which is often called "stabilization", can be observed in some numerical simulations, analyzed rigorously in some models and is claimed to have been seen experimentally cf. [5] and [18–21].

It turns out that many features observed for Rydberg atoms and also stabilization are already present in a simple model system which we have recently begun to investigate analytically [22–24]. This somewhat surprising finding is based on comparisons between experimental and model results described in detail in [23]. In fact the phenomenon of ionization by periodic fields is very complex indeed once one goes beyond the perturbative regime even in the most simple model. This will become clear from the new results about this model presented here.

2. The Model

We consider a very simple quantum system where we can analyze rigorously many of the phenomena expected to occur in more realistic systems described by (1). This is a one dimensional system with an attractive delta function potential. The unperturbed Hamiltonian H_0 has, in suitable units, the form

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\,\delta(x), \quad -\infty < x < \infty. \tag{2}$$

The zero range (delta-function) attractive potential is much used in the literature to model short range attractive potentials [25–28]. It belongs, in one dimension, to the class K_1 [2]. H_0 has a single bound state $u_b(x) = e^{-|x|}$ with energy $-\omega_0 = -1$. It also has continuous uniform spectrum on the positive real line, with generalized eigenfunctions

$$u(k,x) = \frac{1}{\sqrt{2\pi}}\left(e^{ikx} - \frac{1}{1+i|k|}e^{i|kx|}\right), \quad -\infty < k < \infty$$

and energies k^2.

Beginning at $t = 0$, we apply a parametric perturbing potential, i.e. for $t > 0$ we have

$$H(t) = H_0 - 2\,\eta(t)\delta(x) \tag{3}$$

and solve the time dependent Schrödinger equation (1) for $\psi(x,t)$, with $\psi(x,0) = \psi_0(x)$. Expanding ψ in eigenstates of H_0 we write

$$\begin{aligned}\psi(x,t) &= \theta(t)u_b(x)e^{it} \\ &\quad + \int_{-\infty}^{\infty}\Theta(k,t)u(k,x)e^{-ik^2t}dk \;\; (t \ge 0)\end{aligned} \tag{4}$$

with initial values $\theta(0) = \theta_0,\ \Theta(k,0) = \Theta_0(k)$ suitably normalized,

$$\langle\psi_0, \psi_0\rangle = |\theta_0|^2 + \int_{-\infty}^{\infty}|\Theta_0(k)|^2 dk = 1. \tag{5}$$

We then have that the survival probability of the bound state is $P(t) = |\theta(t)|^2$, while $|\Theta(k,t)|^2 dk$ gives the “fraction of ejected particles” with (quasi-) momentum in the interval dk.

This problem can be reduced to the solution of an integral equation in a single variable [22,23]. Setting

$$Y(t) = \psi(x=0,t)\eta(t)e^{it} \tag{6}$$

we have

$$\theta(t) = \theta_0 + 2i \int_0^t Y(s)ds, \tag{7}$$

$$\Theta(k,t) = \Theta_0(k) + 2|k|/\big[\sqrt{2\pi}(1 - i|k|)\big] \int_0^t Y(s)e^{i(1+k^2)s}ds. \tag{8}$$

$Y(t)$ satisfies the integral equation

$$\begin{aligned} Y(t) &= \eta(t)\left\{I(t) + \int_0^t [2i + M(t-t')]Y(t')dt'\right\} \\ &= \eta(t)\Big(I(t) + (2i + M) * Y\Big), \end{aligned} \tag{9}$$

where the inhomogeneous term is

$$I(t) = \theta_0 + \frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{\Theta_0(k) + \Theta_0(-k)}{1 + ik} e^{-i(k^2+1)t} dk,$$

and

$$M(s) = \frac{2i}{\pi} \int_0^\infty \frac{u^2 e^{-is(1+u^2)}}{1+u^2} du = \frac{1+i}{2\sqrt{2\pi}} \int_s^\infty \frac{e^{-iu}}{u^{3/2}} du$$

with

$$f * g = \int_0^t f(s)g(t-s)ds.$$

In our previous works we considered the case where $\Theta_0(k) = 0$ and $\eta(t)$ is a finite sum of harmonics with period $2\pi\omega^{-1}$. In particular, we showed in [23] how to compute the survival probability $P(t)$ as a function of the strength r and frequency ω when $\eta(t) = r \sin \omega t$. Here we study the general periodic case and write

$$\eta = \sum_{j=0}^{\infty} \Big(C_j e^{i\omega j t} + C_{-j} e^{-i\omega j t}\Big).$$

Our assumptions on the C_j are

(a) $0 \not\equiv \eta \in L^\infty(\mathbb{T})$,
(b) $C_0 = 0$,
(c) $C_{-j} = \overline{C_j}$.

Genericity condition (g). Consider the right shift operator T on $l_2(\mathbb{N})$ given by

$$T(C_1, C_2, \ldots, C_n, \ldots) = (C_2, C_3, \ldots, C_{n+1}, \ldots).$$

We say that $\mathbf{C} \in l_2(\mathbb{N})$ is *generic with respect to* T if the Hilbert space generated by all the translates of $\mathbf{C}$ contains the vector $e_1 = (1, 0, 0 \ldots,)$ (which is the kernel of T):

$$e_1 \in \bigvee_{n=0}^{\infty} T^n \mathbf{C} \tag{10}$$

(where the right side of (10) denotes the closure of the space generated by the $T^n\mathbf{C}$ with $n \geq 0$). This condition is generically satisfied, and is obviously weaker than the "cyclicity" condition $l_2(\mathbb{N}) \ominus \bigvee_{n=0}^{\infty} T^n \mathbf{C} = \{\mathbf{0}\}$, which is also generic [29] (Appendix B discusses in more detail the rather subtle cyclicity condition).

An important case, which satisfies (10), (but fails the cyclicity condition) corresponds to η being a trigonometric polynomial, namely $\mathbf{C} \not\equiv \mathbf{0}$ but $C_n = 0$ for all large enough n. (We can in fact replace $\mathbf{e}_1$ in (10) by $\mathbf{e}_k$ with any $k \geq 1$.) A simple example which fails (10) is

$$\eta(t) = 2r\lambda \frac{\lambda - \cos(\omega t)}{1 + \lambda^2 - 2\lambda \cos(\omega t)} \tag{11}$$

for some $\lambda \in (0, 1)$, for which $C_n = -r\lambda^n$ for $n \geq 1$. In this case the space generated by $T^n \mathbf{C}$ is one-dimensional. We will prove that there are values of r and λ for which the ionization is incomplete, i.e. $\theta(t)$ does not go to zero for large t.

3. Results and Remarks

Theorem 1. *Under assumptions (a) ... (c) and (g), the survival probability $P(t)$ of the bound state u_b, $|\theta(t)|^2$ tends to zero as $t \to \infty$.*

Theorem 2. *For $\psi_0(x) = u_b(x)$ there exist values of λ, ω and r in (11), for which $|\theta(t)| \not\to 0$ as $t \to \infty$.*

Remarks. 1. Theorem 1 can be extended to show that $\int_D |\psi(x,t)|^2 dx \to 0$ for any compact interval $D \subset \mathbb{R}$. This means that the initially localized particle really wanders off to infinity since by unitarity of the evolution $\int_{\mathbb{R}} |\psi(x,t)|^2 dx = 1$. Theorem 2 can be extended to show that for some fixed r and ω in (11) there are infinitely many λ, accumulating at 1, for which $\theta(t) \not\to 0$. In these cases, it can also be shown that for large t, θ approaches a quasiperiodic function.

2. While Theorem 1 holds for arbitrary ψ_0, care has to be taken with the initial conditions for Theorem 2. In particular we cannot have an initial state such that in (9) $I(t) = 0$ for all t. This would occur, for example, if $\psi_0(x)$ is an odd function of x. In that case the evolution takes place as if the particle was entirely free – never feeling the delta function potential. There may also be other special ψ_0 for which $\theta_0 \neq 0$ but for which $\theta(t) \to 0$ as $t \to \infty$. We have therefore stated Theorem 2 for the case $\psi_0 = u_b$. We shall also, for simplicity, use this choice of ψ_0 in the proofs of Theorem 1. For this case, which is natural from the physical point of view, $I(t) = 1$ in (9). The extension to general ψ_0 is immediate and is given at the end of Sect. 5.

3. In [23] we gave a detailed picture of how the decay of $\theta(t)$ depends on r and ω when $\eta(t) = r\sin(\omega t)$, $\theta_0 = 1$. For small r and ω^{-1} not too close to an integer we get an exponential decay with a decay rate $\Gamma(r,\omega) \sim r^{2(1+\lfloor \omega^{-1} \rfloor)}$, where $\lfloor \omega^{-1} \rfloor$ is the integer part of ω^{-1}. (For $\omega > 1$, this corresponds to $\Gamma \sim \Gamma_F$). At times large compared to Γ^{-1}, $|\theta(t)|$ decays as $t^{-3/2}$. The picture becomes much more complicated when r is large and/or ω^{-1} is an integer. In particular there is no monotonicity in $|\theta(t)|$ as a function of r. In [24] we proved complete ionization for the case where $C_n = 0$ for $n > N$, $N \geq 1$.

4. We note here that Pillet [3] proved complete ionization for quite general H_0 under the assumption that $H_1(t)$ is "very random", in fact a Markov process. Our results are not only consistent with this but support the expectation that generic perturbations will lead to complete ionization for general H_0. This is what we expect from entropic considerations – there is just too much phase space "out there". The surprising thing is that even for our simple example one can readily find exceptions to the rule.

We should also mention here the work of Martin et al. [31,32] who consider the case where H_0 has an isolated eigenvalue E_0 plus an absolutely continuous spectrum in the interval $[0, E_{\max}]$. They show that if the frequency ω of the periodic, small, perturbation $H_1(t)$ is larger than E_0 then the bound state is stable. This can be understood in terms of Fermi's golden rule by noting that the density of states at the energy $E_0 + \omega > E_{max}$ is zero so that Γ_F would be zero.

5. There is a direct connection between our results and Floquet theory where, for a time-periodic Hamiltonian $H(t)$ with period $T = 2\pi/\omega$, one constructs a quasienergy operator (QEO) [2,33,34]

$$K = -i\frac{\partial}{\partial\theta} + H(\theta).$$

K acts on functions of x and θ , periodic in θ, i.e. on the extended Hilbert space $\mathcal{H} \otimes L_2(S, T^{-1}d\theta)$. Let now $\phi(x,\theta)$ be an eigenfunction satisfying

$$K\phi = \mu\phi, \quad \phi(x, \theta + T) = \phi(x, \theta) \tag{12}$$

then,

$$\psi(x,t) = e^{-i\mu t}\phi(x,t)$$

is a solution of the Schrödinger equation $i\frac{\partial\psi}{\partial t} = H(t)\psi$.

The existence of a real eigenvalue μ of the QEO with an associated $\phi(x,\theta) \in L_2(\mathbb{R}^d \otimes S)$ is thus seen to imply the existence of a solution of the time-dependent Schrödinger equation which is, in absolute value, periodic. This shows that for appropriate initial conditions, the particle has a nonvanishing probability of staying in a compact domain and thus, for the case considered here, that ionization is incomplete. We also note that for each such μ there is actually a whole set $\mu_n = \mu + n\omega$ of eigenvalues of K.

For the specific model considered here, (12) takes the form

$$K\phi = -\frac{\partial^2\phi(x,\theta)}{\partial x^2} - 2(1+\eta(\theta))\delta(x)\phi - i\frac{\partial\phi}{\partial\theta} = \mu\phi. \tag{13}$$

We can now look for solutions of (13) in the form

$$\phi_\mu(x,\theta) = \sum_{n\in\mathbb{Z}} y_n e^{in\omega\theta} e^{\alpha_n x}$$

with $\alpha_n^\pm = \pm\sqrt{\mu - n\omega}$. Such a solution is in L^2 only if $\Re(\alpha_n x) < 0$, a condition which obviously selects different roots λ_n depending on whether $x > 0$ or $x < 0$. The requirement that ϕ_μ be in $L^2(\mathbb{R})$ leads to a set of matching conditions which determine whether such eigenvalues μ can exist. It is easy to see that ϕ_μ has to be continuous at zero and satisfy the condition

$$2\phi_\mu(0^-,\theta) - \phi_\mu(0^+,\theta) = 2(1+\eta(\theta))\phi_\mu(0,\theta).$$

This implies, after taking the Fourier coefficients of both sides of the above equality, the recurrence relation

$$y_n(2 - \alpha_n^+ + \alpha_n^-) = 2\sum_{j\neq 0} C_j y_{n-j} \tag{14}$$

for which a (nontrivial) solution $y_n \in l^2$ is sought. This is effectively the same equation as (20) below which is at the core of our analysis. Complete ionization thus corresponds to the absence of a discrete spectrum of the QEO operator and conversely stabilization implies the existence of such a discrete spectrum. In fact, an extension of Theorem 2 shows that for the initial condition $\psi_0 = u_b$, ψ_t approaches such a function with $\mu = -s_0$. More details about Floquet theory and stability can be found in [33,34].

6. We are currently investigating extensions of our results to the case where $H_0 = -\nabla^2 + V_0(x)$, $x \in \mathbb{R}^d$, has a finite number of bound states and the perturbation is of the form $\eta(t)V_1(x)$ and both V_0 and V_1 have compact support. Preliminary results indicate that, with much labor, we shall be able to generalize Theorem 1, to generic $V_1(x)$. The definition of genericity will, however, depend strongly on V_0.

The physically important case of an external electric dipole field, $V_1(x) = -Ex$ can be transformed into the solution of a Schrödinger equation of the form $H(t) = -\nabla^2 + V_0(x - g(t))$, see [2]. This should, in principle, also be amenable to our methods but so far we have no results for that case.

Outline of the technical strategy. The method of proof relies on the properties of the Laplace transform of Y, $y(p) = \mathcal{L}Y(p) = \int_0^\infty e^{-pt}Y(t)dt$.

Since the time evolution of ψ is unitary, $|\theta(t)| \le 1$. This gives some a priori control on Y. For our purposes however it is useful to characterize directly the solution of the convolution equation (9). (We restrict ourselves to $\Theta_0(k) = 0$ and $I(t) = 1$ there.) We show that this equation has a unique solution in suitable norms. This solution is Laplace transformable and the Laplace transform y satisfies a linear functional equation.

The solution of the functional equation satisfied by the transform of Y is unique in the right half plane provided it satisfies the additional property that $y(p_0 + is)$ is square integrable in s for any $p_0 > 0$. Any such solution y transforms back (by the standard properties of the inverse Laplace transform) into a solution of our integral equation with no faster than exponential growth; however there is a unique locally integrable solution of this equation, and this solution is exponentially bounded. This must thus be our Y. We can thus use the functional equation to determine the analytic properties of $y(p)$.

This is done using (appropriately refined versions of) the Fredholm alternative. After some transformations, the functional equation reduces to a linear inhomogeneous recurrence equation in l_2, involving a compact operator depending parametrically on p, see e.g. (17). The dependence is analytic except for a finite set of poles and square-root branch-points on the imaginary axis and we show that the associated homogeneous equation has no nontrivial solution. We then show that the poles in the coefficients do not create poles of y, while the branch points are inherited by y. The decay of $y(p)$ when $|\Im(p)| \to \infty$, and the degree of regularity on the imaginary axis give us the needed information about the decay of $Y(t)$ for large t.

4. Behavior of $y(p)$ in the Open Right Half Plane $\mathbb{H}$

Lemma 3. (i) *Equation (9) has a unique solution $Y \in L^1_{\text{loc}}(\mathbb{R}^+)$ and $|Y(t)| < Ke^{Bt}$ for some $K, B \in \mathbb{R}$.*

(ii) *The function $y(p) = \mathcal{L}Y$ exists and is analytic in $\mathbb{H}_B = \{p : \Re(p) > B\}$.*

(iii) *In $\mathbb{H}_B$, the function $y(p)$ satisfies the functional equation*

$$y = \sum_{j=-\infty}^{\infty} C_j \mathcal{T}^j \Big(h + by\Big) \tag{15}$$

with

$$\Big(\mathcal{T}f\Big)(p) = f(p + i\omega), \quad h(p) = -p^{-1} \quad \text{and } b(p) = -\frac{i}{p}\Big(1 + \sqrt{1 - ip}\Big).$$

The branch of the square root is such that for $p \in \mathbb{H} = \{p : \Re(p) > 0\}$*, the real part of* $\sqrt{1 - ip}$ *is nonnegative and the imaginary part nonpositive.*

The straightforward proofs of this lemma are done in Appendix A. (Some of the results can also be gotten directly from standard results on the Schrödinger operators and on integral equations.)

Remark 4. It is clear that the functional equation (15) only links points on the one dimensional lattice $\{p + i\mathbb{Z}\omega\}$. It is convenient to take p_0 such that $p = p_0 + in\omega$ with $\Re(p_0) = \Re(p)$ and

$$\Im(p_0) \in [0, \omega). \tag{16}$$

The functions y, h, b in (15) will now depend parametrically on p_0. We set $y = \{y_j\}_{j\in\mathbb{Z}}$, $h = \{h_j\}_{j\in\mathbb{Z}}$, $b = \{b_j\}_{j\in\mathbb{Z}}$ with $y_n = y(p_0 + in\omega) = y(p)$ (and similarly for $h(p)$ and $b(p)$). It is convenient to define the operator $(\hat{H}y)_n = b_n y_n$. Let $(\mathcal{T}y)_n = y_{n+1}$ be the right shift on $l_2(\mathbb{Z})$ (which we denote for simplicity by l_2) and rewrite (15) as

$$y = \sum_{j=-\infty}^{\infty} C_j \mathcal{T}^j h + \sum_{j=-\infty}^{\infty} C_j \mathcal{T}^j \hat{H} y \equiv f + \mathcal{J} y. \tag{17}$$

Proposition 5. *For* $\Re(p_0) > 0$ *there exists a unique solution of (17) in* l_2*. This solution is analytic in* $p_0, \Re(p_0) > 0$*. Thus* $y(p)$ *is analytic in* $p \in \mathbb{H}$ *and inverse Laplace transformable there with* $\mathcal{L}^{-1}(y) = Y$*.*

Proof. The proof uses the Fredholm alternative. We first prove the following results.

Lemma 6. *The operator* $\mathcal{J}$ *is compact on* l_2 *if* $p_0 \neq 0$*.*

Proof. The proof uses standard compact operator results, see e.g. [30]. First note that the operator $\hat{H}$ is compact. This is straightforward: since $b_j \to 0$ as $j \to \infty$, it follows that $\hat{H}$ is the norm limit as $N \to \infty$ of the finite rank operators defined by $(\hat{H}_N y)_j = b_j y_j$ for $|j| \le N$ and $(\hat{H}_N y)_j = 0$ otherwise, and thus is compact. The operator $\mathcal{J}$ is the composition between the "convolution" operator $\mathcal{C}$ given by $(\mathcal{C}v)_n := (C * v)_n := \sum_{j\in\mathbb{Z}} C_j v_{n+j}$, which is continuous on l_2, and the compact operator $\hat{H}$. Thus $\mathcal{J}$ is compact. □

Remarks. 1. Note that $f \in l_2$ if $p_0 \neq 0$ (a straightforward consequence of the fact that **C** and h in (17) are in l_2).

2. The operator $\mathcal{J}$ is analytic in p_0, except for $p_0 = 0$, where the coefficients have poles, and for an additional value on the imaginary axis (possibly also 0), where the coefficients have square root branch points.

Remark 7. Setting, for $p_0 \neq 0$,

$$y_l = (\sqrt{1 - i(p_0 + il\omega)} - 1)z_l \tag{18}$$

the homogeneous equation

$$y = \mathcal{J} y \tag{19}$$

clearly has a (nontrivial) l_2 solution y only if

$$\left(\sqrt{1 - ip_0 + l\omega} - 1\right)z_l = -\sum_{k=1}^{\infty}\left(C_k z_{l+k} + \overline{C}_k z_{l-k}\right) \tag{20}$$

has a (nontrivial) l_2 solution z with

$$\left\{\left(\sqrt{1 - ip_0 + j\omega} - 1\right)z_j\right\}_{j\in\mathbb{Z}} \in l_2. \tag{21}$$

Lemma 8. *For any η under assumptions (a) to (c), if $p_0 \in \mathbb{H}$ there is no nonzero l_2 solution of (20) such that (21) holds.*

Proof. To get a contradiction, assume $z \in l_2$, $z \not\equiv 0$, satisfying (21), is a solution of (20). Multiplying (20) by $\overline{z_l}$, and summing with respect to l from $-\infty$ to $+\infty$ we get

$$\begin{aligned}\sum_{l=-\infty}^{\infty}\left(\sqrt{1 - ip_0 + l\omega} - 1\right)|z|_l^2 &= -\sum_{l=-\infty}^{\infty}\sum_{k=1}^{\infty}\left(C_k z_{l+k}\overline{z_l} + \overline{C}_k z_{l-k}\overline{z_l}\right)\\ &= -\sum_{l=-\infty}^{\infty}\sum_{k=1}^{\infty}\left(C_k z_l\overline{z_{l-k}} + \overline{C}_k z_{l-k}\overline{z_l}\right)\\ &= -\sum_{l=-\infty}^{\infty}\sum_{k=1}^{\infty} 2\Re\left(C_k z_l\overline{z_{l-k}}\right).\end{aligned} \tag{22}$$

If $p_0 \in \mathbb{H}$ the imaginary part of $\sqrt{1 - ip_0 + l\omega}$ is negative (see Remark 24) and thus, if some z_l is nonzero then the left side of (22) has strictly negative imaginary part, which is impossible since the right side is real. □

Proof of Proposition 5. The existence of the analytic solution follows now immediately from the analytic Fredholm alternative and the analyticity of the coefficients, for $p_0 \in \mathbb{H}$. The fact that $\{y_n\} \in l_2$ together with the stated analyticity imply that the function $\mathcal{L}^{-1}y(p)$ exists and satisfies the integral equation of Y, and thus coincides with Y. □

5. Behavior of $y(p)$ in the Neighborhood of $\Im(p) = 0$ in the Generic Case

Discussion of methods. We start again from relation (17). This has the form

$$y_n = i\sum_j \frac{C_j}{-ip_0 + (n+j)\omega} - \sum_j C_j q_{n+j} y_{n+j}, \quad C_0 = 0, \tag{23}$$

where

$$q_n = \frac{\left[1 + \sqrt{1 - ip_0 + n\omega}\right]}{-ip_0 + n\omega}. \tag{24}$$

As the imaginary axis $\Re(p_0) = 0$ is approached, two types of potential singularities in the coefficients need attention: the poles in the coefficients due to the presence of p^{-1}, and the square root singularities. It will turn out that by cancellation effects, the poles play no role, generically. The square root singularities will be manifested in the solution y. The study of these questions requires further regularization of the functional Eq. (23).

It is convenient to separate out the terms in (23) which are singular at $p_0 = 0$. Using (from now on) the notation $s_0 = -ip_0$ we have

$$\begin{aligned} y_n &= i\frac{C_{-n}}{s_0} - \frac{C_{-n}(1+\sqrt{1+s_0})}{s_0} y_0 + i \sum_{j \neq -n} \frac{C_j}{s_0 + (n+j)\omega} \\ &\quad - \sum_{j\neq -n} C_j q_{n+j} y_{n+j}, \quad n \neq 0, \\ y_0 &= i \sum_{j\neq 0} \frac{C_j}{s_0 + j\omega} - \sum_{j\neq 0} C_j q_j y_j. \end{aligned} \tag{25}$$

We break up the proof into two parts, the non-resonant and resonant case. We start with the former.

5.1. The non-resonant case, $\omega^{-1} \notin \mathbb{N}$.

Proposition 9. *If condition (g) is satisfied, and $\omega^{-1} \notin \mathbb{N}$, then the solution y of (25) is analytic in a small neighborhood of $s_0 = 0$.*

For the proof we write $y_0 = i/2 + s_0 u_0$, and for $n \neq 0$ we make the substitution $y_n = v_n + d_n u_0$, where we will choose d_n according to (26) in order to eliminate u_0 from all equations with $n \neq 0$.

Lemma 10. (i) *For $s_0 \in \mathbb{R}$ there exists a unique solution $d \in l_2(\mathbb{Z} \setminus \{0\})$ of the system*

$$d_n = -C_{-n}(1+\sqrt{1+s_0}) - \sum_{k\neq 0} C_{k-n} q_k d_k, \quad n \neq 0. \tag{26}$$

This solution is analytic at $s_0 = 0$.

(ii) *With this choice of d, the system (25) becomes*

$$\begin{aligned} v_n &= f_n - \sum_{k\neq 0} C_{k-n} q_k v_k, \\ \left(s_0 + \sum_{j\neq 0} C_j q_j d_j\right) u_0 &= f_0 - \sum_{j \neq 0} C_j q_j v_j, \end{aligned} \tag{27}$$

where

$$f_0 = -\frac{i}{2} + i \sum_{j\neq 0} \frac{C_j}{s_0 + j\omega}, \quad f_n = iC_{-n}\frac{1-\sqrt{1+s_0}}{2s_0} + i\sum_{k\neq 0}\frac{C_{k-n}}{s_0 + k\omega}. \tag{28}$$

(iii) *For small* s_0 *we have* $\sum_{j\neq 0} C_j q_j d_j \neq 0$, *and the system (27) has a unique solution with* $v \in l_2(\mathbb{Z}\setminus\{0\})$, *and* v_n, u_0 *are analytic at* $s_0 = 0$.

Proof. (i) Equation (26) is of the form $(\mathbb{I} - \mathcal{J}')d = c'$ in $l_2(\mathbb{Z}\setminus\{0\})$, where $c'_n = -(1+\sqrt{1+s_0})\overline{C}_n$ and

$$(\mathcal{J}'d)_n = -\sum_{k\neq 0} C_{k-n} q_k d_k, \quad (n \neq 0).$$

We show first that $\operatorname{Ker}(\mathbb{I} - \mathcal{J}') = \{0\}$. Indeed, assume $d = \mathcal{J}'d$ and set $D_k = q_k d_k$. Then we see that

$$q_n{}^{-1} D_n + \sum_{k\neq 0} C_{k-n} D_k = 0 \tag{29}$$

and, by multiplying with $\overline{D}_n$ and summing over n we get

$$\sum_{n\neq 0} q_n^{-1}|D_n|^2 + \sum_{n,k\neq 0} C_{k-n} D_k \overline{D_n} = 0. \tag{30}$$

Note that, because $C_{-n} = \overline{C}_n$, the following quantity is real:

$$\overline{\sum_{n,k\neq 0} C_{k-n} D_k \overline{D}_n} = \sum_{n,k\neq 0} C_{n-k}\overline{D_k} D_n = \sum_{n,k\neq 0} C_{k-n} D_k \overline{D}_n, \tag{31}$$

implying that

$$\sum_{n\neq 0} q_n^{-1}|D_n|^2 \in \mathbb{R}$$

with (cf. (24))

$$q_n^{-1} = -1 + \sqrt{1 + s_0 + n\omega}.$$

Let $N_0 = -(1+s_0)\omega^{-1} \in \mathbb{R}$. Obviously $q_n^{-1} \in \mathbb{R}$ for $n \geq N_0$ while for $n < N_0$ we have, by Remark 24

$$\Im(q_n^{-1}) < 0.$$

Thus it is necessary that $D_n = 0$ for all $n < N_0$.

Assume $D \neq 0$. Let $N \in \mathbb{N}$ be such that $D_n = 0$ for all $n < N$ and $D_N \neq 0$ (thus $N_0 \leq N$). Then from (29),

$$\sum_{k\geq N; k\neq 0} C_{k-n} D_k = 0 \quad \text{for any } n < N$$

or, setting $k = N - 1 + j$,

$$\sum_{j\geq 1, j\neq 1-N} C_{j+n} D_{N-1+j} = 0 \quad \text{for } n \geq 0. \tag{32}$$

It is here that we use the genericity condition on $\mathbf{C}$. In fact we will show that (32) implies $D = 0$ if condition (g) is satisfied. To see this define $\tilde{D} \in l_2(\mathbb{N})$ as $\tilde{D}_j = D_{N-1+j}$ if $j \geq 1$, $j \neq 1 - N$ and, if $1 - N \geq 1$, $\tilde{D}_{1-N} = 0$. Then by (32) $\tilde{D}$ is orthogonal in

$l_2(\mathbb{N})$ to all T^nC, $n \geq 0$. By the genericity condition (g) then $< \tilde{D}, e_1 >= D_N = 0$, which is a contradiction. Thus $D = 0$.

Since $\mathcal{J}'$ is analytic in s_0 for small enough s_0, and compact by the same simple arguments as in Lemma 6, it follows that $(\mathbb{I} - \mathcal{J}')^{-1}$ exists and is analytic in s_0 at $s_0 = 0$.

(ii) This part is an immediate calculation.

(iii) Note first that $f \in l_2(\mathbb{Z} \setminus \{0\})$, because

$$\|f\| \leq \left| \frac{1-\sqrt{1+s_0}}{2s_0} \right| \|c\| + \left(\sum_{n\neq 0} \left| \sum_{k\neq 0} \frac{C_{k-n}}{s_0 + k\omega} \right|^2 \right)^{1/2}$$
$$\leq \|c\| \sum_{k\neq 0} \frac{1}{|s_0 + k\omega|^2} < \infty.$$

Also, formula (28) expresses f in terms of a discrete measure integral with respect to k of a function which depends analytically on the (small) parameter s_0, and which is uniformly in l_1. Therefore f depends analytically on s_0.

The rest of the proof of (iii) closely follows that of part (i), using the following result.

Lemma 11. *For $s_0 = 0$ we have* $\sum_{j\neq 0} C_j q_j d_j \neq 0$.

Proof. Assume the contrary was true. At $s_0 = 0$, with $D_n^0 = D_n|_{s_0=0}$ and $q_n^0 = q_n|_{s_0=0}$, relation (29), using (26), gives

$$0 = \frac{D_n^0}{q_n^0} = -\sum_{k\neq 0} C_{k-n} D_k^0 - 2C_{-n} \quad (n \neq 0). \tag{33}$$

Multiplying with $\overline{D_n^0}$ and summing over $n \neq 0$ we would get

$$\sum_{n\neq 0} (-1 + \sqrt{1+n\omega})|D_n^0|^2 = -\sum_{k,n\neq 0} C_{k-n} D_k^0 \overline{D_n^0} - \sum_{n\neq 0} 2C_{-n} \overline{D_n^0}, \tag{34}$$

and since we assumed $\sum_n C_n D_n^0 = 0$ then, as in the proof of Lemma 10 (i), it follows that $D_n^0 = 0$ for all $n < N_0 = -\omega^{-1}$. This gives, using (33), that

$$\sum_{k\geq N_0; k\neq 0} C_{k-n} D_k^0 + 2C_{-n} = 0. \tag{35}$$

Denote by $D^1 \in l_2$ the sequence $D_k^1 = D_k^0$ if $k \neq 0$ and $D_0^1 = 2$. As in the proof of Lemma 10 (i), using the genericity condition (g), we get $D^1 = 0$, an obvious contradiction. $\square$

This concludes the proof of Proposition 9: for generic η the solution y of (17) has, for $\omega^{-1} \notin \mathbb{N}$, analytic components y_n when $p = 0$.

Square root singularities. We now study the behavior at the square root singularities of the coefficients of the equation of y.

Let k_0 be the unique integer such that for some $s_r \in [0, \omega)$ we have $1 + s_r + k_0\omega = 0$ (then s_r is a branch point in the coefficient q).

The following proposition describes the analytic structure of $y(p)$ near the imaginary axis.

Proposition 12. *We have the decomposition* $y_n = u_n + (\sqrt{s_0 - s_r})v_n$, *where* u_n *and* v_n *are analytic in* s_0 *in a* complex *neighborhood of the segment* $[0, \omega)$.

Proof. The substitution $y_n = u_n + (\sqrt{s_0 - s_r})v_n$, and

$$U_k = q_k u_k;\ V_k = q_k v_k \quad (k \neq k_0) \quad \text{and } U_{k_0} = \frac{u_{k_0}}{s_0 + k_0\omega}; \quad V_{k_0} = \frac{v_{k_0}}{s_0 + k_0\omega}$$

leads to the following system of equations for U_n and V_n:

$$\begin{aligned}
q_n^{-1}U_n &= ri\sum_k \frac{C_{k-n}}{s_0 + k\omega} - \sum_k C_{k-n}U_k - C_{k_0-n}(s_0 - s_r)V_{k_0} \quad (n \neq k_0),\\
q_n^{-1}V_n &= -\sum_k C_{k-n}V_k - C_{k_0-n}(s_0 - s_r)V_{k_0} - C_{k_0-n}U_{k_0} \quad (n \neq k_0), \qquad (36)\\
(s_0 + k_0\omega)U_{k_0} &= i\sum_k \frac{C_{k-k_0}}{s_0 + k\omega},\\
(s_0 + k_0\omega)V_{k_0} &= -\sum_k C_{k-k_0}V_k.
\end{aligned}$$

We now let $Q_{k_0} = s_0 + k_0\omega$ and, for $n \neq k_0$, $Q_n = q_n^{-1} = -1 + \sqrt{1 + s_0 + k\omega}$. We use again the Fredholm alternative and, as in the previous proofs, we need only to show the absence of a solution of the homogeneous equation *at* $s_0 = s_r$. We thus multiply the homogeneous equations associated to (36) in the following manner: the equation for U_j by $\overline{U_j}$ and the equation for V_j by $\overline{V_j}$, then sum over all j. As in the previous proofs, from the reality of the r.h.s. and then from the genericity condition (g) $U \equiv 0$. Then, similarly, $V \equiv 0$. The rest is immediate. □

5.2. The resonant case: $\omega^{-1} = M \in \mathbb{N}$. In this case when $s_0 = 0$ there are poles in the coefficients of (23) when $n + j = 0$ and branch points when $n + j = -M$. The proof is a combination of the two regularization techniques used in the previous case.

Proposition 13. *We can set* $y(s_0) = A(s_0) + B(s_0)\sqrt{s_0}$ *with* A *and* B *analytic in a complex neighborhood of the segment* $[0, \omega)$.

Proof. Special care is only needed near $s_0 = 0$. The system (26)–(28) now reads

$$\begin{aligned}
d_n &= -C_{-n}(1 + \sqrt{1 + s_0}) - \sum_{k \notin \{0, -M\}} C_{k-n}q_k d_k - C_{-M-n}\frac{1 + \sqrt{s_0}}{s_0 - 1}d_{-M},\\
v_n &= f_n - \sum_{k \notin \{0, -M\}} C_{k-n}q_k v_k - C_{-M-n}\frac{1 + \sqrt{s_0}}{s_0 - 1}v_{-M}. \qquad (37)
\end{aligned}$$

We take $d_n = \alpha_n + \sqrt{s_0}\beta_n$ and $v_n = \gamma_n + \sqrt{s_0}\delta_n$. The system becomes

$$\alpha_n = -C_{-n}(1+\sqrt{1+s_0}) - \sum_{k\notin\{0,-M\}} C_{k-n}q_k\alpha_k - C_{-M-n}\frac{1}{s_0-1}(\alpha_{-M}+s_0\beta_{-M}),$$

$$\beta_n = -\sum_{k\notin\{0,-M\}} C_{k-n}q_k\beta_k - C_{-M-n}\frac{1}{s_0-1}(\alpha_{-M}+\beta_{-M}), \tag{38}$$

$$\gamma_n = f_n - \sum_{k\notin\{0,-M\}} C_{k-n}q_k\gamma_k - C_{-M-n}\frac{1}{s_0-1}(\gamma_{-M}+s_0\delta_{-M}),$$

$$\delta_n = -\sum_{k\notin\{0,-M\}} C_{k-n}q_k\delta_k - C_{-M-n}\frac{1}{s_0-1}(\delta_{-M}+\gamma_{-M}). \tag{39}$$

The system (38) is of the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = S(s_0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \begin{pmatrix}F_1\\ F_2\end{pmatrix},$$

where α, β, F_1, F_2 are in l_2. We prove that the homogeneous equation has no nontrivial solutions:

Lemma 14. $(\mathbb{I} - S(0))\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ *implies* $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$.

Proof. Let $Q_n = q_n$, $A_n = q_n\alpha_n$, $B_n = q_n\beta_n$ for $n \neq 0, -M$ and $Q_{-M} = -1$, $A_{-M} = -\alpha_{-M}$ and $B_{-M} = -\beta_{-M}$. The system (38) becomes

$$Q_n^{-1}A_n = -\sum_{k\neq 0} C_{k-n}A_k,$$
$$Q_n^{-1}B_n = -\sum_{k\neq 0} C_{k-n}B_k - C_{-M-n}A_{-M}. \tag{40}$$

As in the proofs in Case I, multiplying the first equation by $\overline{A_n}$, summing over n we first get from the reality of the r.h.s. that $A_n = 0$ for $n < -M$ and then by the condition (g) we get that $A \equiv 0$. The conclusion $B \equiv 0$ now follows in the same way. □

End of proof of Proposition 13. The operator S is compact on $l_2 \oplus l_2$ and S and (F_1, F_2) are analytic in a complex neighborhood of 0. We saw in Lemma 14 that the kernel of $\mathbb{I} - S(0)$ is trivial and by the analytic Fredholm alternative it follows that $(\mathbb{I} - S(0))^{-1}$ exists and is analytic in a small neighborhood of $s_0 = 0$. Hence (α, β) are analytic. Similarly, γ, δ are analytic in the same region. □

5.3. Proof of Theorem 1 Combining the above results we have the following conclusion:

Proposition 15. *If condition (g) is fulfilled, then $y(p)$ is analytic in a neighborhood of $i\mathbb{R} \setminus \{is_r + i\omega\mathbb{Z}\}$. For any $j \in \mathbb{Z}$, in a neighborhood of $p = is_r + ij\omega$ ($s_r \in \mathbb{R}$) y has the form $y(p) = A_j(p) + B_j(p)\sqrt{-ip - s_r - ij\omega}$, where A_j and B_j are analytic. In particular, y is Lipschitz continuous of exponent $1/2$ in the closed right half plane. Thus $Y(t) = O(t^{-3/2})$ for large t.*

Proof. All but the last claim has already been shown. The last statement is a standard Tauberian theorem (note that $\mathcal{L}^{-1}$ is the Fourier transform along the imaginary line). □

Proposition 16. *We have* $\theta(t) \to 0$ *as* $t \to \infty$.

Proof. We can write (9) (with $I(t) = 1$) as

$$Y = \eta(\theta + M * Y). \tag{41}$$

It is easy to check, in view of the fact that M and Y are $O(t^{-3/2})$, that $M * Y \to 0$. Furthermore $1 + 2i \int_0^t Y(s)ds$ is convergent as $t \to \infty$. Thus $\theta(t) \to const$ as $t \to \infty$. Since now the l.h.s. of (9) converges to zero and η does not, the equality (41) is only consistent if $\theta(t) \to 0$. □

This completes the proof of Theorem 1 for the case $\psi_0 = u_b = e^{-|x|}$.

The general case follows by noting that the inhomogeneous term does not affect the main argument, using the Fredholm alternative. Hence we will still have $|\theta(t)| \to 0$ but the rate of decay may be different.

6. A Nongeneric Example

Let η be given by (11), for which

$$C_n = -r\lambda^n \text{ for } n \geq 1, \quad C_n = C_{-n}. \tag{42}$$

As in Sect. 5 set $-ip_0 = s_0$ and let q_n be given by (24). Denote

$$a_n = a_n(s_0) = \frac{1}{r}\frac{1}{q_n} = \frac{1}{r}\left(\sqrt{1 + s_0 + n\omega} - 1\right). \tag{43}$$

For $r \in (0, 1)$, $\omega > 1$, $\omega^{-1} \notin \mathbb{N}$ such that $(1 - r)^2 < \omega - 1$, let s_r and s_p be the unique numbers in $(0, \omega)$ so that $1 + s_r \in \omega\mathbb{Z}$ and $1 + a_{-1}(s_p) = 0$. We choose r, ω such that $s_r \neq s_p$.

6.1. The homogeneous equation.

Lemma 17. *Let* $s_{0,0}$ *be a point in* $(0, s_r) \cup (s_r, \omega)$. *Consider* s_0 *in a small enough neighborhood of* $s_{0,0}$. *The linear operator* $\mathcal{J} = \mathcal{J}(s_0)$ *of (17) depends analytically on* s_0, *and is compact on* l_2. *For* $s_0 \neq s_p$, $(I - \mathcal{J}(s_0))^{-1}$ *exists and is analytic.*

Lemma 18. *Denote for short* $\mathcal{J}_0 = \mathcal{J}(s_r)$.

There exists a value $\lambda = \lambda_s \in (0, 1)$ *such that*

$$\dim \operatorname{Ker} (I - \mathcal{J}_0) = 1. \tag{44}$$

Denote by A the diagonal (unbounded) operator $(Az)_n = a_n z_n$ in l_2; A^{-1} is bounded.

Lemma 19. *For* $\lambda = \lambda_s$ *as in Lemma 18 we have*

$$\operatorname{Ker} (I - \mathcal{J}_0) = A\left[\operatorname{Ker}\left(I - \mathcal{J}_0^*\right)\right]. \tag{45}$$

6.2. Proof of Lemma 17. The operator $\mathcal{J}$ is compact by Lemma 6. To show that $I-\mathcal{J}$ is invertible we prove this for any points $s_0 \in (0,\omega)$, $s_0 \neq s_p, s_r$; by the analytic Fredholm theorem it will follow that $I-\mathcal{J}$ is invertible in a small enough neighborhood of any such point, thus proving the lemma.

Let $s_0 \in (0,\omega)$, $s_0 \neq s_p, s_r$. As in Remark 7 in Sect. 5, the substitution $y_n = a_n z_n$ ($n \in \mathbb{Z}$) transforms the homogeneous equation (19) to

$$a_n z_n = \sum_{j=1}^{\infty} \lambda^j \left(z_{n+j} + z_{n-j}\right), \quad n \in \mathbb{Z}. \tag{46}$$

Note that $\Im a_n < 0$ for $n < -1$ for $s_0 \in [\omega-1,\omega)$ and $\Im a_n < 0$ for $n < 0$ for $s_0 \in (0,\omega-1)$. We will discuss only the first case, $s_0 \geq \omega-1$, since the second one is completely analogous.

As in the proof of Lemma 8, it follows that

$$z_n = 0 \quad \text{for } n < -1. \tag{47}$$

Then Eqs. (46) for $n < -1$ become

$$\sum_{k=1}^{\infty} \lambda^k z_{k-2} = 0. \tag{48}$$

For $n = -1$ (46) gives

$$(a_{-1}+1) z_{-1} = 0, \tag{49}$$

and for $n \geq 0$, using (48), we get

$$(1+a_n) z_n = \sum_{j=1}^{n+1} (\lambda^j - \lambda^{-j}) z_{n-j}\ ,\ n \geq -1. \tag{50}$$

Since $s_0 \neq s_p$, (49) gives $z_{-1} = 0$, and it follows by induction, from (50), that $z_n = 0$ for all n. By the Fredholm alternative theorem then $I - \mathcal{J}(s_0)$ is invertible.

6.3. Proof of Lemma 18. In what follows $s_0 = s_r$.

6.3.1. An auxiliary lemma. We show that if $z \in l_2$ then Eq. (48) is redundant.

Lemma 20. *If z is an l_2 solution of (50) with $z_n = 0$ for $n < -1$ then z satisfies (48).*

Proof. Let $z \in l_2$ be a solution of (50). Then

$$Z^{[n+1]} \equiv \sum_{k=1}^{n} \lambda^k z_{k-2} \tag{51}$$

is the truncation of a convergent series, since there is a constant M with $|z_n| < M$ for all n. Note that

$$1 + a_n) z_n = \sum_{j=1}^{n+1} \lambda^j z_{n-j} - \lambda^{-n-2} Z^{[n+1]},$$

hence

$$Z^{[n+1]} = \lambda^{n+2}\sum_{j=1}^{n+1}\lambda^j z_{n-j} - \lambda^{n+2}(1+a_n)z_n,$$

so that

$$\left|Z^{[n+1]}\right| \le \lambda^{n+2}\frac{M\lambda}{1-\lambda} + \lambda^{n+2}M\left(1+\text{ const}\sqrt{n}\right) \longrightarrow 0 \quad \text{as } n\to\infty. \tag{52}$$

Since (51) are truncations of the series in the LHS of (48), then (52) implies (48). □

6.3.2. Behavior of the general solution of (50). A direct calculation shows that the sequence z_n satisfying the infinite order recurrence (50) and the initial condition $z_{-1}=1$ satisfies, in fact, the three step recurrence

$$(1+a_{n+1})z_{n+1} + (1+a_{n-1})z_{n-1} = [\lambda(1+a_n)+\lambda+a_n\lambda^{-1}]z_n \quad (n\ge 0) \tag{53}$$

with the initial condition

$$z_{-1}=1, \quad z_0 = \frac{\lambda-\lambda^{-1}}{1+a_0}. \tag{54}$$

Denote

$$z_n = \frac{\lambda-\lambda^{-1}}{1+a_n}V_{n-1}, \tag{55}$$

then (53) becomes

$$V_n + V_{n-2} = \left[\lambda + \frac{\lambda^2+a_n}{\lambda(1+a_n)}\right]V_{n-1} \quad n\ge 1. \tag{56}$$

We are looking for l_2 solutions. Recent rigorous WKB estimates (see e.g. [35]) would imply there are solutions of the discrete equation (56) behaving like $\lambda^{-n}e^{-\sqrt{n/\omega}}$ and like $\lambda^n e^{\sqrt{n/\omega}}$. We will prove this in our context and find special values of λ for which the solution decaying for large n satisfies the initial condition. We will show that this solution is obtained by taking

$$V_{n-2} = g_{n-1}V_{n-1} \tag{57}$$

in (56) and iterating:

$$g_{n-1} = G_n - \frac{1}{g_n} \quad \text{with } G_n = \lambda + \frac{\lambda^2+a_n}{\lambda(1+a_n)}, \tag{58}$$

i.e., g_0 is given by the continued fraction:

$$g_0 = G_1 - \cfrac{1}{G_2 - \cfrac{1}{G_3}\cdots},$$

which needs to match the initial condition (see (54):

$$g_0 = g_0(\lambda) = \frac{1}{\lambda+\lambda^{-1}+(1+a_0)^{-1}(\lambda-\lambda^{-1})}\,. \tag{59}$$

Lemma 21. (i) *Let $\lambda \in (0,1)$. The recurrence (58) has a solution such that $g_n \to \lambda^{-1}$ as $n \to \infty$.*
(ii) *g_0 is meromorphic in λ on $[0,1)$ and has poles.*
(iii) *There exists $\lambda_s \in (0,1)$ such that $g_0(\lambda_s)$ satisfies (59).*
(iv) *Let $\lambda = \lambda_s$. To the solution of (i) there corresponds a solution $V^{[s]}$ of the recurrence (56) such that $V_n^{[s]} \sim \lambda_s^{n+o(n)}$ as $n \to \infty$. The corresponding solution $z^{[s]}$ of (50) satisfies $z_n \to 0$ as $n \to \infty$.*
(v) *Let $\lambda = \lambda_s$. There exists a solution of (56) with the asymptotic behavior $V_n^{[l]} \sim \lambda_s^{-n+o(n)}$.*

Thus, for $\lambda = \lambda_s$, there exists a unique (up to a multiplicative constant) "small" solution of (56), with the behavior $V_n^{[s]} \sim \lambda_s^{n+o(n)}$ for large n, while the general solution behaves like $V_n \sim \lambda_s^{-n+o(n)}$. As a consequence, a similar statement holds for the recurrence (53).

Remark. The proof of (iii) can be refined to show that, in fact, there is a countable set of points λ_s for which g_0 satisfies the initial condition, and that these values accumulate to 1.

Proof. (i) With the substitution

$$g_n = G_{n+1} - \lambda + \delta_n, \tag{60}$$

the recurrence (58) becomes

$$\delta_n = \lambda - \frac{1}{G_{n+2} - \lambda + \delta_{n+1}} \equiv (\mathcal{S}\delta)_n\,, \quad n \geq 0. \tag{61}$$

For $n_0 \geq 0$ and ϵ small, positive, define $\lambda_{n_0} = a_{n_0+2}\left(2 + a_{n_0+2}\right)^{-1} - \epsilon$. Let $\mathcal{N}_{n_0}$ be a small neighborhood of the interval $I_{n_0} = [0, \lambda_{n_0}]$. Consider the Banach space $\mathcal{B}_{n_0}$ of sequences $\{\delta_n\}_{n\geq n_0}$ with $\delta_n = \delta_n(\lambda)$ analytic on $\mathcal{N}_{n_0}$ and continuous up to the boundary, with the norm $\|\delta\| = \sup_{n\geq n_0}\sup_{\lambda\in\mathcal{N}_{n_0}} |\delta_n(\lambda)|$. Direct estimates show that the operator $\mathcal{S}$ defined by (61) takes the ball of radius $\rho_{n_0} = 2/(2 + a_{n_0+2}) + \epsilon'$ in $\mathcal{B}_{n_0}$ into itself (if ϵ, ϵ' and $\mathcal{N}_{n_0}$ are small enough), and is a contraction in this ball. Therefore the equation $\delta = \mathcal{S}(\delta)$ has a unique solution in $\mathcal{B}_{n_0}$, of norm less than ρ_{n_0}. Then $|\delta_n(\lambda)| < \text{const}(n+2)^{-1/2}$ for all $\lambda \in I_n$ and all $n \geq 0$. Since the sequence λ_n increases to 1, (i) follows.

(ii) *Step I: All g_n are meromorphic on* $[0,1)$. Since δ_n is analytic on I_n, then from (60), g_n is analytic on $I_n \setminus \{0\}$, having a pole at $\lambda = 0$: $g_n \sim \lambda^{-1}a_{n+1}(1 + a_{n+1})^{-1}$ $(\lambda \to 0)$. Iterating (58) it follows that $g_{n-1}, g_{n-2}, \ldots, g_0$ are meromorphic on I_n. Since the intervals I_n increase toward $[0,1)$ it follows that $g_0, g_1, \ldots g_n \ldots$ are meromorphic on $[0,1)$.

Step II: There exists n_1 and $\lambda_0 \in (0,1)$ such that $g_{n_1}(\lambda_0) \leq 0$. Define $\epsilon_n = (1 + a_n)^{-1}$; we have (see (43))

$$\epsilon_{n_0} \sim r(n_0\omega)^{-1/2}, \quad n_0 \to \infty. \tag{62}$$

Let n_0 be large and denote $\lambda_0 = 1 - \epsilon_{n_0}$. Let N_0 be large enough so that λ_0 is in the domain of analyticity of g_{N_0}. Iterating (58) starting from N_0 (and decreasing indices) we get the value $g_{n_0}(\lambda_0)$. If for some $n \in \{n_0, n_0+1, \ldots, N_0\}$ we get $g_n(\lambda_0) \leq 0$, *Step II* is proved. Then assume that $g_{n_0}(\lambda_0) > 0$.

Consider the recurrence

$$\tilde{r}_{n-1} = G_{n_0}(\lambda_0) - \frac{1}{\tilde{r}_n} \quad \text{for } n \le n_0, \quad \tilde{r}_{n_0} = g_{n_0}(\lambda_0), \tag{63}$$

where, in fact, $G_{n_0}(\lambda_0) = 2 - \epsilon_{n_0}^2$.

The recurrence (63) can be solved explicitly (it is a discrete Riccati equation and a substitution $\tilde{r}_{n-1} = x_{n-1}/x_n$ transforms it into a linear recurrence with constant coefficients). It has the solution

$$\tilde{r}_n = \frac{\cos((n-n_0)\phi + \theta)}{\cos((n+1-n_0)\phi + \theta)}, \tag{64}$$

where $\cos\phi = 1-\epsilon_{n_0}^2/2$, $\sin\phi > 0$, and $\tan\theta = (\cos\phi - \lambda)/\sin\phi$ so that $\theta \sim \frac{\pi}{4} - \frac{1}{4}\epsilon_{n_0}$ ($\epsilon_{n_0} \to 0$).

We assume, to get a contradiction, that $g_n(\lambda_0) > 0$ for all $n = 0, 1, \ldots, n_1$. Then

$$g_n(\lambda_0) \le \tilde{r}_n \quad \text{for } n \le n_0, \tag{65}$$

which follows immediately by induction using (58), (63), noting that G_n is increasing in n.

Note that there is an $n_1 \in \{1, 2, \ldots, n_0 - 1\}$ so that

$$\tilde{r}_n > 0 \quad \text{for } n \in \{n_1+1, \ldots, n_0\} \quad \text{and } \tilde{r}_{n_1} < 0. \tag{66}$$

Indeed (from (62)) when n decreases from n_0 the numerator and denominator in (64) increase up to 1, then decrease, until the numerator becomes negative, when n equals $n_1 = n_0 - k_1$, where k_1 is the integer with $k_1 - 1 < (\pi/2 + \theta)/\phi \le k_1$. Since $\phi \sim \epsilon_{n_0}$ ($\epsilon_{n_0} \to 0$) then $k_1 \sim (3\pi)/(4\epsilon_{n_0})$, and, using (62), clearly $k_1 \in \{1, \ldots, n_0 - 1\}$ (if n_0 is sufficiently large).

Then (65) and (66) contradict the assumption that $g_{n_1}(\lambda_0) > 0$, and *Step II* is proved.

Step III. The function g_{n_1} is meromorphic on $[0, 1)$, with $g_{n_1}(0+) = +\infty$. There is a smallest value of λ in $(0, \lambda_0)$, where g_{n_1} changes sign: this is either a zero, or a pole.

Assume it was a pole. Let $p \in (0, \lambda_0)$ be the first pole of g_{n_1}. Then g_{n_1} is positive and analytic on $(0, p)$, and $g_{n_1}(p-) = +\infty$, $g_{n_1}(p+) = -\infty$. Since $g_{n+1} = 1/(G_{n+1} - g_n)$ (see (58)) then $g_{n_1+1}(p-) = 0-$, hence g_{n_1+1} changes sign in $(0, p)$. But g_{n_1+1} has no zero in $(0, p)$ (otherwise at that zero g_{n_1} would have had a pole, from (58)). Then g_{n_1+1} has a pole, with a change of sign, from $+$ to $-$, in $(0, p)$. Now the argument can be repeated. It follows that for any $k > 0$, g_{n_1+k} has a pole in $(0, p)$, which contradicts the fact that the domain of analyticity of g_n increases to $(0, 1)$ as $n \to \infty$.

Therefore, the first change of sign of g_{n_1} is at a zero. Let ζ_1 be the smallest value in $(0, 1)$ such that $g_{n_1}(\zeta_1-) = 0+$, $g_{n_1}(\zeta_1+) = 0-$. Then from (58) we have $g_{n_1-1}(\zeta_1-) = -\infty$ and g_{n_1-1} changes sign in $(0, \zeta_1)$. Now the argument can be repeated. It follows that g_0 has a pole at a point ζ_{n_1} with $g_0(\zeta_{n_1}-) = -\infty$.

(iii) Since $g_0(\lambda)$ takes all the values when $\lambda \in (0, \zeta_{n_1})$ there exists $\lambda = \lambda_s \in (0, 1)$ such that (59) holds.

(iv) For $\lambda = \lambda_s$, since the solution of (i) satisfies $g_n(\lambda) = \lambda^{-1} + O(n^{-1/2})$ we have from (57), with the notation $V^{[s]} = V(\lambda_s)$, that $V_n^{[s]} = \prod_{k=0}^n g_k(\lambda_s)^{-1} V_0^{[s]} = O(\lambda_s^{n+o(n)})$ and thus $V_n^{[s]} - V_{n-1}^{[s]} = O(\lambda_s^{n+o(n)})$; then from (55) $z_n^{[s]} = O(\lambda_s^{n+o(n)})$.

(v) The substitution (variation of constants) $V_n = V_n^{[s]} v_n$ brings the recurrence (56) to a first order one: with the notation $\Delta_n = v_n - v_{n-1}$ we have $\Delta_n = V_{n-2}^{[s]}/V_n^{[s]} \Delta_{n-1}$ and the rest of the argument consists of straightforward estimates. □

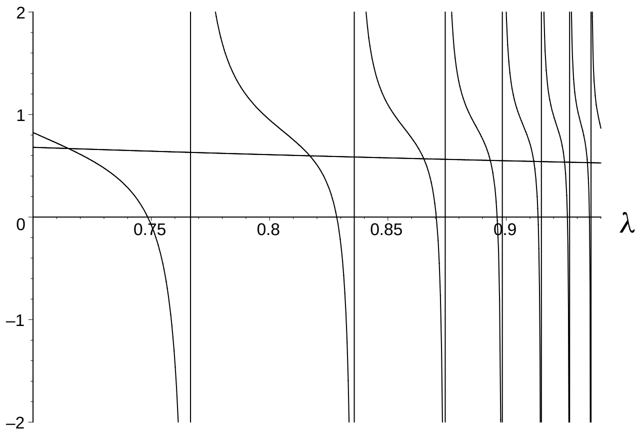

Fig. 1. Graph of g_0 given by (58) (discontinuous graph) and by (59) in a region near $\lambda = 1$, as functions of λ

6.3.3. Proof of Lemma 18.

Proof. Lemma 21(v) shows that Eq. (53) has a unique (up to a multiplicative constant) small solution, $z_n^{[s]} \sim \lambda_s^{n+o(n)}$ $(n \to \infty)$, while the general solution behaves like $z_n \sim \lambda_s^{-n+o(n)}$. Since $y_n \sim \sqrt{n} z_n$ the uniqueness of the l_2 solution is proven.

6.3.4. Examples of solutions. We will show next how concrete values λ_s satisfying Lemma 21 (iii) are relatively straightforwardly, and rigorously, found. One method is as follows. Note that the minimum/maximum of the function $a - b/x$, where x varies in an interval not containing zero is achieved at the endpoints. We thus take the recurrence (58) with initial conditions $g_{n_0} = \lambda^{-1} \pm \frac{1-\lambda^2}{\sqrt{n\omega}}$ and compute g_0 from these. The actual graph will be between these two, unless the condition mentioned is violated in between n_0 and 0. This graph is to be intersected with the graph of the initial condition (59).

We take for instance $\omega = 1.1$, $r = 0.45$, $s_p = 0.11$, $n_0 = 10$, for which the rigorous control is not too involved. The two graphs are very close to each other (within about 3.10^{-6} for $\lambda \in (0.3, 0.4)$) and cannot be distinguished from each-other in Fig. 1. A first intersection is seen at $\lambda \approx 0.327$; see Fig. 2.

6.4. Proof of Lemma 19. Denote $B = (I - \mathcal{J}_0)A$; we have $B = A - S$. Hence $B^* = \overline{A} - S$. Then $\mathrm{Ker}(B) = \mathrm{Ker}(B^*)$ (since $Az = Sz$ implies (47), so $Az = \overline{A}z$, and similarly, $\overline{A}z = z$ implies $Az = \overline{A}z$). So $\mathrm{Ker}[(1 - \mathcal{J}_0)A] = \mathrm{Ker}[A(1 - \mathcal{J}_0{}^*)]$ so that (since A is one-to-one) $A^{-1}\,\mathrm{Ker}(1 - \mathcal{J}_0) = \mathrm{Ker}(1 - \mathcal{J}_0{}^*)$, which proves the lemma.

6.5. Discussion of the singularities of solutions of (17). Let $\lambda = \lambda_s$. We have that $I - \mathcal{J}$ is invertible for $\Re p_0 > 0$, and is not invertible at $p_0 = is_p$ (Lemma 18). By the analytic Fredholm theorem (see e.g. [30]) $(I - \mathcal{J})^{-1}$ is meromorphic on a small neighborhood of is_p, therefore there exist $m \geq 1$ and operators $S_m, \ldots, S_1, R(p_0)$ so that:

$$(I - \mathcal{J})^{-1} = \frac{1}{\left(p_0 - is_p\right)^m} S_m + \cdots + \frac{1}{p_0 - is_p} S_1 + R(p_0), \qquad (67)$$

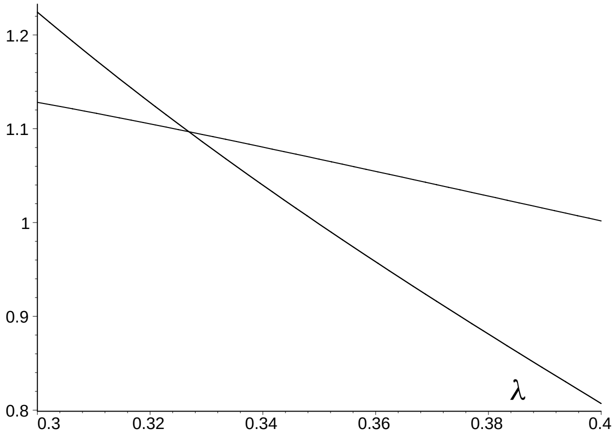

Fig. 2. Graphs of g_0 (steeper graph) and of the initial condition for g_0 (59)

where $R(p_0)$ is analytic at is_p, and $S_m \neq 0$ (since $I - \mathcal{J}_0$ is not invertible). Multiplying (67) by $I-\mathcal{J}$ to the left, respectively to the right, and writing $\mathcal{J} = \mathcal{J}_0 + (p_0 - is_p)\mathcal{J}_1(p_0)$ (where $\mathcal{J}_1(p_0)$ is analytic at is_p) we get that

$$R_1(p_0) = \frac{1}{\left(p_0 - is_p\right)^m} (I - \mathcal{J}_0)\, S_m + O\left(\left(p_0 - is_p\right)^{-m+1}\right),$$
$$R_2(p_0) = \frac{1}{\left(p_0 - is_p\right)^m} S_m\, (I - \mathcal{J}_0) + O\left(\left(p_0 - is_p\right)^{-m+1}\right),$$

where $R_{1,2}$ are analytic at $p_0 = is_p$. By the uniqueness of the series of the analytic functions (Banach space valued) $R_{1,2}$ we must then have

$$(I - \mathcal{J}_0)\, S_m = 0 = S_m\, (I - \mathcal{J}_0)\,. \tag{68}$$

The first equality in (68) implies $\mathrm{Ran}(S_m) \subset \mathrm{Ker}\,(I - \mathcal{J}_0) = \bigvee\{y_{\mathrm{Ker}}\}$ and since $S_m \neq 0$ then $\mathrm{Ran}(S_m) = \bigvee\{y_{\mathrm{Ker}}\}$, therefore $S_m y = \langle y, u\rangle y_{\mathrm{Ker}}$ for some $u \in l_2 \setminus \{0\}$. The second equality in (68) means $u \in \mathrm{Ran}\,(I - \mathcal{J}_0)^{\perp} = \mathrm{Ker}\left(I - \mathcal{J}_0^*\right)$.

By Lemma 19 then (up to a multiplicative constant) $u = A^{-1} y_{\mathrm{Ker}} = z_{\mathrm{Ker}}$, where z_{Ker} satisfies (46), hence (53),(54). The solution $y = (I - \mathcal{J})^{-1} f$ of (17) is then singular at $p_0 = is_p$ if $c =< f, z_{\mathrm{Ker}} >\neq 0$. For the example of Sect. 6.3.4 this latter condition can be checked by explicit calculation of the truncations to 10 terms and estimation of the remainder based on the contractivity bounds in the previous section. The result is $c = -1.953 \pm 0.001$. Thus the inhomogeneous equation has poles.

Lemma 22. *Let $Y(t)$ be analytic on $[0, \infty)$, with $\lim_{t\to\infty} Y(t) = 0$.*
Let $s \in \mathbb{R}$. Then

$$\lim_{a\downarrow 0} a \int_0^{\infty} e^{-(a+is)t}\, Y(t)\, dt = 0. \tag{69}$$

Corollary 23. *Let $Y(t)$ be as in Lemma 22. Let $y(p) = \int_0^{\infty} e^{-pt}\, Y(t)\, dt$. Assume that $y(p)$ is analytic on $i\mathbb{R}_+$, except for a set of isolated points. Then $y(p)$ does not have poles on $i\mathbb{R}_+$.*

Proof. I. We first show (69) for $s = 0$.

Separating the positive and negative parts of $\Re Y(t)$, $\Im Y(t)$ write $Y(t) = Y^{[1]}(t) - Y^{[2]}(t) + iY^{[3]}(t) - iY^{[4]}(t)$ with $Y^{[k]}(t)$ nonnegative, continuous, nonanalytic only on a discrete set, where the left and right derivatives exist, with $\lim_{t\to\infty} Y^{[k]}(t) = 0$. It is enough to show (69) for each $Y^{[k]}$. Let then Y be one of the $Y^{[k]}$'s. Denote $H(t) = \sup_{\tau\ge t} Y(\tau)$. The function H on $[0,\infty)$ has the same properties as Y and in addition is decreasing. Then H' exists a.e. and $H' \in L^1[0,\infty)$, since $\int_0^\infty |H'(\tau)|\,d\tau = -\lim_{t\to\infty}\int_0^t H'(\tau)\,d\tau = \lim_{t\to\infty} -H(t) + H(0) = H(0)$.

Then

$$a\int_0^\infty e^{-at}Y(t)\,dt \le a\int_0^\infty e^{-at}H(t)\,dt = -\int_0^\infty \frac{d}{dt}\left(e^{-at}\right)H(t)\,dt$$
$$= H(0) + \int_0^\infty e^{-at}H'(t)\,dt,$$

therefore

$$\lim_{a\downarrow 0} a\int_0^\infty e^{-at}Y(t)\,dt \le H(0) + \lim_{a\downarrow 0}\int_0^\infty e^{-at}H'(t)\,dt = 0,$$

which proves the lemma in this case.

II. Let now $s \in \mathbb{R}$ arbitrary. Then (69) follows from the result for $s = 0$ applied to the function $\tilde{Y}(t) = e^{-ist}Y(t)$. □

Proof of Theorem 2. In conclusion $Y(t)$ cannot tend to zero as $t \to \infty$ and complete ionization fails. □

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Appendix A. Proof of Lemma 3

(i) Consider $L^1_{loc}[0, A]$ endowed with the norm $\|F\|_\nu := \int_0^A |F(s)|e^{-\nu s}ds$, where $\nu > 0$. If f is continuous and $F, G \in L^1_{loc}[0, A]$, a straightforward calculation shows that

$$\|fF\|_\nu < \|F\|_\nu \sup_{[0,A]} |f|, \tag{A1}$$

$$\|F * G\|_\nu < \|F\|_\nu \|G\|_\nu, \tag{A2}$$

$$\|F\|_\nu \to 0 \quad \text{as } \nu \to \infty, \tag{A3}$$

where the last relation follows from the Riemann–Lebesgue lemma.

The integral equation (9) can be written as

$$Y = \eta + \mathcal{J}Y \quad \text{where } \mathcal{J}F := \eta(2i + M) * F. \tag{A4}$$

Since M is locally in L^1 and bounded for large x it is clear that for large enough B_2, (9) is contractive if $\nu > B_2$, for any A.

(ii) This is an immediate consequence of Lemma 3 and of elementary properties of the Laplace transform.

(iii) We have in $\mathbb{H}$,

$$\mathcal{L}M = \lim_{a\downarrow 0} \frac{2i}{\pi} \int_0^\infty \mathrm{d}x e^{-px} \int_0^\infty \frac{u^2 e^{-i(x-ia)(1+u^2)}}{1+u^2}\mathrm{d}u \tag{A5}$$

$$= \frac{i}{\pi} \int_{-\infty}^\infty \frac{u^2}{(1+u^2)(p+i(1+u^2))}\mathrm{d}u. \tag{A6}$$

For $\Re(p) > 0$ we push the integration contour through the upper half plane. At the two poles in the upper half plane $u^2 + 1$ equals 0 and ip respectively, so that

$$\frac{i}{\pi} \int_{-\infty}^\infty \frac{u^2}{(1+u^2)(p+i(1+u^2))}\mathrm{d}u$$
$$= \frac{i}{\pi}\left(\frac{(-1)}{(2i)(p)}\oint \frac{ds}{s} + \frac{u_0^2}{(ip)(2iu_0)}\oint \frac{ds}{s}\right) = -\frac{i}{p} + \frac{u_0}{p}, \tag{A7}$$

where u_0 is the root of $p + i(1+u^2) = 0$ in the *upper* half plane. Thus

$$\mathcal{L}M = -\frac{i}{p} + \frac{i\sqrt{1-ip}}{p} \tag{A8}$$

with the branch satisfying $\sqrt{1-ip} \to 1$ as $p \to 0$ in $\mathbb{H}$.

Thus, the analytic continuation of $\sqrt{1-ip}$ in $\mathbb{H}\cup\partial\mathbb{H}$ in our calculations is as follows:

Remark 24. As p varies in $\mathbb{H}$, $1-ip$ belongs to the lower half plane $-i\mathbb{H}$ so that $\sqrt{1-ip}$ varies in the fourth quadrant, and in particular $\Im\sqrt{1-ip} < 0$. If $p \in i\mathbb{R}$ and $-ip \geq -1$ then $\sqrt{1-ip}$ is real and nonnegative, while if $-ip < -1$ and $\sqrt{1-ip}$ has zero real part and negative imaginary part.

To show (15) note that for $\Re(p) > 0,\ \omega > 0$ we have

$$\mathcal{L}\left(e^{\pm i\omega}M\right) = -\frac{i}{p \mp i\omega} + \frac{i\sqrt{1-ip\mp\omega}}{p\mp i\omega},$$
$$(\text{with } \sqrt{1-ip-\omega} = -i\sqrt{\omega - 1 + ip} \ \text{ if } \ \omega > 1) \tag{A9}$$

The branch of the square root was discussed in Remark 24. This concludes the proof of Lemma 3 (iii).

Appendix B. Discussion of the Genericity Condition (g)

A thorough analysis of the properties of the shift operator is provided by the treatise [29]. We provide here an independent discussion, meant to give an insight on the interesting analytic properties involved in this condition.

Let $C = (C_0, C_1, \ldots, C_n, \ldots) \in l_2(\mathbb{N})$ and the operator T defined as before by $TC = (C_1, C_2, \ldots)$. We want to see for which such vectors, the system of equations

$$(\mathbf{z}, T^j\mathbf{C}) = 0, \ \ j = 0, 1, \ldots \tag{B1}$$

has nontrivial solutions $\mathbf{z}$ in l_2. We can associate to $\mathbf{z}$ and $\mathbf{C}$ analytic functions in the unit disk, $Z(x)$ and $C(x)$ by

$$C(x) = \sum_{k=0}^{\infty} C_k x^k \quad Z(x) = \sum_{k=0}^{\infty} z_k x^k. \tag{B2}$$

These functions, extend to L^2 functions on the unit circle. The system of equations (B1) can be written as

$$z_0 C(x) + z_1 x^{-1}(C(x)C(0)) + \dots \\ + z_n \left[x^{-n} C(x) - x^{-n} \sum_{k=0}^{n-1} \frac{x^k}{k!} C^{(k)}(0) \right] + \dots = 0. \tag{B3}$$

Using Cauchy's formula, we can the difference in square brackets in (B3) as

$$\frac{1}{2\pi i} \oint_{|s|=1} \frac{C(s)}{s^n (s-x)} ds, \tag{B4}$$

and thus (B1) becomes

$$\oint_{|s|=1} \frac{C(s) Z(1/s)}{s-x} ds = 0. \tag{B5}$$

The functions C for which this equation has nontrivial solutions Z relate to the Beurling inner functions [29] and are very "rare".

Examples. (i) Let $|\lambda| < 1$ and $C_n = \lambda^n$, i.e. $C(x) = (1 - \lambda x)^{-1}$. This is related to the function (11). Taking advantage of the analyticity of Z outside the unit circle, we can push the contour of integration towards $s = \infty$, collecting the residue at $x = \lambda^{-1}$; we see that Eq. (B5) holds iff $Z(\lambda) = 0$, i.e., for a space of analytic functions of codimension one.

(ii) Let $\lambda_n = 1/n$. Then $C(x) = \ln(1 - x)$, and by taking $s = 1/t$ in (B5) we get

$$\frac{1}{x} \oint_{|t|=1} \frac{Z(t) \ln(t-1)}{(t - x^{-1})t} dt - \frac{1}{x} \oint_{|t|=1} \frac{\ln(t) Z(t)}{t(t - x^{-1})} dt = 0. \tag{B6}$$

By making a cut on $[1, \infty)$ for the log we see that the integrand in the first integral is analytic in the unit circle and thus the integral vanishes. We decompose the second integral by partial fractions and we get

$$\oint_{|t|=1} \frac{\ln(t) Z(t)}{t} dt - \oint_{|t|=1} \frac{\ln(t) Z(t)}{(t-y)} dt = 0, \tag{B7}$$

where $y = x^{-1}$. The first integral is a constant, C. By pushing the contour of integration inwards, we see that the second integral extends analytically for small $y \neq 0$. For such y we thus have

$$\oint_{|t|=1} \frac{\ln(t)(Z(t) - Z(y))}{(t-y)} dt + Z(y) \oint_{|t|=1} \frac{\ln(t)}{(t-y)} dt = -C. \tag{B8}$$

Now the contour of integration can be pushed to the sides of the interval [0, 1] collecting the difference between the branches of the log. We get

$$\int_0^1 \frac{Z(t) - Z(y)}{t - y} dt + Z(y) \int_0^1 \frac{1}{t - y} dt = 0. \tag{B9}$$

Thus $\phi(y) + Z(y)\ln(-y) = C$ with ϕ and Z analytic in the unit circle, thus $\ln(-y)$ is analytic unless $Z = 0$. This shows $C_n = 1/n$ is generic.

References

1. Simon, B.: Schrödinger Operators in the Twentieth Century. J. Math. Phys. **41**, 3523 (2000)
2. Cycon, H.L., Froese, R.G., Kirsch, W. and Simon, B.: *Schrödinger Operators*. Berlin–Heidelberg–New York: Springer-Verlag, 1987
3. Pillet, C.-A.: Some results on the quantum dynamics of a particle in a Markovian potential. Commun. Math. Phys. **102**, 237–254 (1985) and **105**, 259 (1986)
4. Yajima, K.: Existence of Solutions for Schrödinger Evolution Equations. Commun. Math. Phys. **110**, 415–426 (1987)
5. Fring, A., Kastrykin, V. and Schrader, R.: Ionization Probabilities through Ultra-Intense Fields in the Extreme Limit. J. Phys. Math. Gen. **30**, 8599–8610 (1997)
6. Soffer, A. and Weinstein, M.I.: Nonautonomous Hamiltonians. J. Stat. Phys. **93**, 359–391 (1998)
7. Soffer, A. and Costin, O.: Resonance Theory for Schrödinger Operators. Submitted to Commun. Math. Phys.
8. Landau, L.D. and Lifschitz, E. M.: *Quantum Mechanics – Nonrelativistic Theory*. 2nd ed. New York: Pergamon, 1965
9. Cohen-Tannoudji, C., Duport-Roc, J. and Arynberg, G.: *Atom-Photon Interactions*. New York: Wiley, 1992
10. Fermi, E.: Notes on Quantum Mechanics. Chicago: The University of Chicago Press, 1961, p. 100
11. Koch, P.M. and van Leeuven, K.A.H.: The Importance of Resonances in Microwave “Ionization” of Excited Hydrogen Atoms. Phys. Repts. **255**, 289–403 (1995)
12. Casatti, G., Chirikovand, B. and Shepelyansky, D. L.: Relevance of Classical Chaos in Quantum Mechanics: The Hydrogen Atom in a Monochromatic Field. Phys. Repts. **154**, 77–123 (1987)
13. Patolige, R.M. and Shaheshaft, R.: Multiphoton Processes in an Intense Laser Field: Harmonic Generation and Total Ionization Rates for Atomic Hydrogen. Phys. Rev. A **40**, 3061–3079 (1990)
14. Buchleitner, A., Delande, D. and Gay, J.-C.: Microwave Ionization of Three Dimensional Hydrogen Atoms in a Realistic Numerical Experiment. J. Opt. Soc. Am. B **12**, 505–519 (1995)
15. Benenti, G., Casati, G., Maspero, G. and Shepelyansky, D.L.: Quantum Poincaré Recurrences for Hydrogen Atom in a Microwave Field. Preprint, physics/9911200
16. Schwendner, P., Seyl, F. and Schinke, R.: Photodissociation of Ar_2^+ in Strong Laser Fields, Chem. Phys. **217**, 233–244 (1997)
17. Guerin, S. and Jauslin, H.-R.: Laser-Enhanced Tunneling through Resonant Intermediate Levels. Phys. Rev. A **55**, 1262–1275 (1997)
18. Eberly, J.M. and Kulander, K.C.: Atomic-Stabilization by Super-Intense Lasers. Science **262** 1233 (1993)
19. Su, Q., Irving, B.P. and Eberly, J.H.: Ionization Modulation in Dynamic Stabilization. Laser Physics **7**, 568 (1997)
20. Figueira de Morisson Faria, C., Fring, A. and Schrader, R.: Analytical Treatment of Stabilization. Preprint, physics/9808047, v2
21. Barash, D., Orel, A.E. and Vemuri, V.R.: A Genetic Search in Frequency Space for Stabilizing Atoms by High-Intensity Laser Fields. J. Comp. Info. Tech. CIT 8, **2**, 103–113 (2000)
22. Rokhlenko, A. and Lebowitz, J.L.: Ionization of a Model Atom by Perturbation of the Potential. J. Math. Phys. **41**, 3511–3523 (2000)
23. Costin, O., Lebowitz, J.L. and Rokhlenko, A.: Exact Results for the Ionization of a Model Quantum System. J. Phys. A. **33**, 6311–6319 (2000) physics/9905038, and work in preparation
24. Costin, O., Lebowitz, J.L. and Rokhlenko, A.: To appear in: *Proceedings of the CRM meeting “Nonlinear Analysis and Renormalization Group”*, American Mathematical Society Publications (2000), math-ph/0002003
25. Demkov, Yu.N. and Ostrovskii, V.N.: *Zero Range Potentials and Their Application in Atomic Physics*. New York: Plenum, 1988; Albeverio, S., Gesztesy, F., Høegh-Krohn, R. and Holden, H.: *Solvable Models in Quantum Mechanics*. Berlin–Heidelberg–New York: Springer-Verlag, 1988

26. Susskind, R.M., Cowley, S.C. and Valeo, E.J.: Multiphoton Ionization in a Short Range Potential: A Nonperturbative Approach. Phys. Rev. A **42**, 3090–3106 (1990)
27. Scharf, G., Sonnemoser, K. and Wreszinski, W.F.: Sensitive Multiphoton Ionization. Phys. Rev. A **44**, 3250–3265 (1991)
28. LaGattuta, K.J.: Laser-Assisted Scattering from a One-Dimensional δ-function potential: An Exact Solution. Phys. Rev. A **49** No. 3, 1745–1751 (1994)
29. Nikol'skii, N.K.: *Treatise on the Shift Operator.* Berlin–Heidelberg–New York: Springer-Verlag, 1986
30. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics, Vol. I.* New York: Academic Press, 1979
31. Martin, P.A.: Scattering with Time Periodic Potentials and Cyclic States. Preprint 1999, Texas
32. Kovar, T. and Martin, P.A.: Scattering with a periodically kicked interaction and cyclic states. Preprint (1999)
33. Belissard, J.: Stability and Instability in Quantum Mechanics. In: *Trends and Developments in the Eighties*, Albeverio, S. and Blanchard, Ph. eds., Singapore: World Scientific, 1985, pp. 1–106
34. Jauslin, H.R. and Lebowitz, J.L.: Spectral and Stability Aspects of Quantum Chaos. Chaos **1**, 114–121 (1991)
35. Costin, O. and Costin, R.D.: Rigorous WKB for Discrete Schemes with Smooth Coefficients. SIAM J. Math. Anal. **27** no. 1, 110–134 (1996)

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