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# Hydrodynamic Limit of Brownian Particles Interacting with Short- and Long-Range Forces

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We investigate the time evolution of a model system of interacting particles moving in a $d$-dimensional torus. The microscopic dynamics is first order in time with velocities set equal to the negative gradient of a potential energy term $\Psi$ plus independent Brownian motions: $\Psi$ is the sum of pair potentials, $V(r)+\gamma^d J(\gamma r)$; the second term has the form of a Kac potential with inverse range $\gamma$. Using diffusive hydrodynamic scaling (spatial scale $\gamma^{-1}$, temporal scale $\gamma^{-2}$) we obtain, in the limit $\gamma \downarrow 0$, a diffusive-type integrodifferential equation describing the time evolution of the macroscopic density profile.

**KEY WORDS:** Interacting particle systems; hydrodynamic limit; nonlocal evolution equations.

## 1. INTRODUCTION

The transition from the microscopic dynamics of interacting particles to hydrodynamical type equations describing the coarse grained evolution of macroscopic variables, such as the diffusion equation for the density, is a basic problem of non-equilibrium statistical mechanics. While far from resolved for systems with realistic interactions there has been much progress recently on this problem for model systems. Like in real systems, the transition from microscopic to macroscopic evolutions in these models is based on a separation between microscopic and macroscopic scales. Setting $\varepsilon$ equal to the ratio of microscopic to macroscopic spatial scale and then looking at macroscopic times which are of order $\varepsilon^{-\alpha}$ microscopic time units, $\alpha = 1$ for Euler (non-dissipative) and $\alpha = 2$ for diffusive evolutions,

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we expect to obtain the macroscopic equations in the hydrodynamical scaling limit (HSL) $\varepsilon \downarrow 0$. We refer to the books of De Masi and Presutti,[3] and Spohn,[24] for a general background on this subject (see also the review article by Lebowitz, Presutti, and Spohn[16]).

To actually prove this HSL, one needs to show that during macroscopic evolutions the microscopic particle system can be well described, on the microscopic scale, by a local version of the equilibrium measure which is stationary under the dynamics. These measures depend on quantities conserved by the microscopic dynamics, such as the particle density, which then evolve on the slower hydrodynamic time scale according to the hydrodynamic equations. This requires good mixing or chaotic properties of the dynamics (as well as of all the relevant equilibrium states). This is particularly so for the case of diffusive scaling where longer times are involved. It is for this reason that the only model systems of interacting particles for which the HSL has been established in the diffusive limit are systems with stochastic dynamics. Thus the HSL for Ginzburg–Landau models was established first by Guo, Papanicolaou and Varadhan,[12] by applying entropy techniques. These techniques were further developed by Rezakhanlou,[21] to cover the case when the invariant measure is not a product measure and phase transitions may occur. These methods can be applied also to lattice gas models that satisfy the so called "gradient condition."[24] For lattice gases this condition is however not natural and the only known examples are when the invariant measure is a product measure or the spatial dimension is one.[13, 24] Very recently the diffusive HSL for non gradient lattice gases has been proved by Varadhan and Yau.[26] For systems of particles in the continuum the gradient condition is more natural, while a common technical problem in these models is the control of the local number of particles: the conservation law cannot prevent locally very high densities. The only continuum models treated with the entropy techniques quoted above are one dimensional systems of Brownian particles interacting via positive superstable short range potentials considered by Varadhan,[25] and Ornstein–Uhlenbeck interacting processes studied by Olla and Varadhan.[19] We should also mention here that the diffusive limit can be proven for a Hamiltonian system of non-interacting particles moving among a fixed array of convex hard scatterers: the Sinai billiard system with finite horizon in $d=2$.[1, 2, 17, 18]

In 1991 Yau,[27] proposed a new method for proving the HSL of interacting particle systems of gradient type, looking at the relative entropy and its rate of change w.r.t. local Gibbs states. This method can be applied also to continuum systems in higher dimension, e.g., in the derivation of the Euler equations from a Hamiltonian system with weak noise considered by Olla, Varadhan, and Yau.[20]

In the present paper we extend the work of Varadhan to Brownian particles with positive superstable short range potentials in all dimensions. In addition we also permit long range pair interactions of the Kac type in which the range parameter $\gamma^{-1}$ goes to infinity as the macro to micro spatial scale $\varepsilon^{-1}$. This extends previous work for such systems on a lattice.[9]

To be more precise, we consider a system of $N$ particles which evolve in time according to the non-inertial Brownian dynamics

$$\frac{dr_i}{d\tau} = -\frac{\partial \Psi}{\partial r_i}(r_1,..., r_N) + \mathscr{W}_i(\tau) \tag{1.1}$$

where $\mathscr{W}_i(\tau)$ is a stochastic Langevin force with Gaussian statistics having covariance $(\beta/2)\, \delta_{ij}\, \delta(\tau - \tau')\, \underline{1}$, $\underline{1}$ the unit $d$-dimensional tensor. The parameter $\beta$ is the inverse temperature of the canonical ensemble, $\mu \sim \exp[-\beta\Psi]$, which is the stationary measure for the evolution. The potential energy $\Psi$ is a sum of pair potentials,

$$\Psi(r_1,..., r_N) = \tfrac{1}{2} \sum_{i \neq j} [V(r_{ij}) + \gamma^d J(\gamma r_{ij})] \tag{1.2}$$

where $r_{ij} = r_i - r_j$ and the $r_i$, $i = 1,..., N$, are confined to a $d$-dimensional torus $\mathscr{T}^d_L$ of length $L$. We take $V(r)$ to have a finite range $R$ with $R < \gamma L$: $\gamma^{-1}$ is the range of the Kac potential which will be taken to be large compared to the inter-particle spacing $L/N^{1/d}$. Systems with interaction of form (1.2), with $V(r) \equiv 0$, $J(r) > 0$, and different types of dynamics, have been investigated numerically and analytically by Klein and coworkers as model of glassy dynamics.[11, 14, 15]

We observe that due to the prefactor $\gamma^{d+1}$ appearing in the force term due to the Kac potential, the dynamics defined by (1.1) is a weak perturbation of the one defined for $J = 0$, i.e. without long range interactions. Thus we may expect that for small $\gamma$'s the system reaches local equilibrium w.r.t. the short range potential on spatial scales smaller than $\gamma^{-1}$ at times of order $\gamma^{-2}$. The effect of the long range interaction on such states will then appear only in determining the macroscopic equation for the relevant parameters describing the local equilibrium.

In fact we shall take as our initial distribution something close to the local equilibrium distribution relative to the short range potential $V$ with a density which varies on the scale of $L \sim N^{1/d} \sim \gamma^{-1}$ and consider macroscopic times of order $\tau/\gamma^{-2}$. The HSL will then correspond to letting $\gamma \downarrow 0$. We will prove that in that limit the density profile on the macroscopic scales $x$ and $t$ will satisfy the following non-local integro-differential equation of the diffusive type:

$$\frac{\partial \rho}{\partial t}(t, x) = \nabla \cdot \left\{ D(\rho(t, x)) \, \nabla \rho(t, x) + \sigma(\rho(t, x)) \int_{\mathscr{T}^d} dy \, \nabla J(x-y) \, \rho(t, y) \right\} \tag{1.3}$$

where the integral is over the $d$-dimensional unit torus $\mathscr{T}^d$ and $\sigma(\rho) \equiv \beta\rho$ is the mobility of a system of interacting Brownian particles, which, due to the fact that the system is gradient, does not depend on the interactions.[24] The diffusion coefficient $D(\rho)$ is given explicitly in terms of the Helmholtz free energy density $a(\beta, \rho)$ associated to the "reference system" interacting only with the short range potential $V$, in such a way that the following "Einstein Relation" holds (ref. 24):

$$D(\rho) = \sigma(\rho) \frac{\partial \lambda}{\partial \rho} = \sigma(\rho) \frac{\partial^2 a}{\partial \rho^2} \tag{1.4}$$

where $\lambda$ is the chemical potential of the reference system at density $\rho$. As in the lattice case,[9] Eq. (1.3) can be rewritten in terms of the gradient flux associated to the classical local mean field free energy functional and the density dependent mobility $\sigma(\rho)$:

$$\frac{\partial \rho}{\partial t}(t, x) = \nabla \cdot \left\{ \sigma(\rho) \, \nabla \frac{\delta \mathscr{F}}{\delta \rho} \right\}(t, x) \tag{1.5}$$

where

$$\mathscr{F}(\rho) = \int_{\mathscr{T}^d} dx \, a(\beta, \rho(x)) + \tfrac{1}{2} \int_{\mathscr{T}^d} dx \int_{\mathscr{T}^d} dy \, J(x-y) \, \rho(x) \, \rho(y) \tag{1.6}$$

Our proof is based on Yau's method quoted above. The main restriction of this method is that the derivation of the HSL is valid only as long as the macroscopic equation has a smooth classical solution. Consequently, unlike the lattice case, we can no longer guarantee existence of global solutions. In fact, even if the initial datum is smooth and lies in the one phase region (for the reference system), we cannot guarantee that the time evolution will not develop singularities or create regions of high density where the reference system undergoes a phase transition and the diffusion coefficient $D(\rho)$ vanishes.

The outline of the rest of the paper is as follows. In Section 2 we give a precise description of our system and present the results. In Section 3 we prove the HSL by computing the relative entropy and its rate of change w.r.t. the local equilibrium states of the reference system. To do this we need a local ergodic theorem whose proof is sketched in Section 4 and large deviation estimates for the local Gibbs states which are the content of Section 5. A local existence theorem of classical solutions for the macroscopic equation is quite standard, a sketch of the proof is given at the end of Section 3.

## 2. NOTATION AND RESULTS

In this section we state our problem in a precise mathematical form using from the beginning the rescaled space and time variables, $x_i = \gamma r_i$, and $t = \gamma^2 \tau$. We also absorb $\beta/2$ into the Brownian motion term which remains invariant under this rescaling of space and time. In these units we consider a system of $N$ interacting Brownian motions $\underline{x}(t) = \{x_1(t),..., x_N(t)\}$ with state space $\mathscr{T}^d$, the $d$-dimensional unit torus, satisfying the following equations ($i = 1,..., N$):

$$dx_i = -\beta \left[ \gamma^{-1} \sum_{j:\, j \neq i} \nabla V(\gamma^{-1}(x_i - x_j)) + \gamma^d \sum_{j:\, j \neq i} \nabla J(x_i - x_j) \right] dt + \sqrt{2}\, dw_i \tag{2.1}$$

where $\{w_1,..., w_N\}$ are independent Brownian motions on $\mathscr{T}^d$, the parameter $\beta \geqslant 0$ is the inverse temperature, $J \in C^2(\mathscr{T}^d)$ and $V(r) \in C^1(\mathbb{R}^d)$ is a positive function of $|r|$, with compact support and such that $V(0) > 0$. The latter implies that $V$ is superstable. In Eq. (2.1), $\nabla V(\gamma^{-1}(x_i - x_j))$ and $\nabla J(x_i - x_j)$ are the gradients of the functions $V(\cdot)$ and $J(\cdot)$ w.r.t. their arguments, evaluated at the points $\gamma^{-1}(x_i - x_j)$ and $(x_i - x_j)$ respectively. We shall further assume that the number of particles $N$ depends on the scaling parameter $\gamma \in (0, 1]$ in such a way that $N\gamma^d \nearrow 1$ as $\gamma \downarrow 0$ (typically $N = [\gamma^{-d}]$).

The process $t \to \underline{x}(t)$ is a diffusion on $\mathscr{T}^{dN}$ with generator

$$L_\gamma = L_\gamma^{(0)} + U_\gamma \tag{2.2}$$

where

$$L_\gamma^{(0)} = \sum_i \Delta_i - \gamma^{-1} \sum_{i \neq j} \beta \nabla V(\gamma^{-1}(x_i - x_j)) \cdot \nabla_i \tag{2.3}$$

and

$$U_\gamma = -\gamma^d \sum_{i \neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i \tag{2.4}$$

In (2.3) and (2.4) $\Delta_i$ ($\nabla_i$) denotes the Laplacian (gradient) w.r.t. the $i$th particle component of $\underline{x} \in \mathscr{T}^{dN}$. Note that the diffusion $L_\gamma^{(0)}$ is reversible w.r.t.

$$\mu_\gamma(d\underline{x}) = \frac{1}{Z_\gamma} \exp\left[ -\frac{\beta}{2} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j)) \right] d\underline{x} \tag{2.5}$$

where $Z_\gamma$ is the normalization factor making $\mu_\gamma$ a probability measure on $\mathscr{T}^{dN}$.

If the initial distribution of the diffusion has a density $f_\gamma^{(0)}$ w.r.t. $\mu_\gamma$ then the density at any later time, $f_\gamma(t, \underline{x})$, satisfies the forward Fokker–Planck equation

$$\frac{\partial f_\gamma}{\partial t} = L_\gamma^* f_\gamma, \qquad f_\gamma|_{t=0} = f_\gamma^{(0)} \tag{2.6}$$

where $L_\gamma^*$ is the adjoint of $L_\gamma$ w.r.t. $\mu_\gamma$.

To state our result we need to introduce some thermodynamic quantities relative to the reference system, i.e., the system of particles interacting only via the short range (superstable) potential $V$. For any regular domain $\Lambda$ of $\mathbb{R}^d$ we define the grand canonical partition function

$$Z_\Lambda(\beta, \lambda) = e^{-|\Lambda|} \sum_{N=0}^{\infty} \frac{e^{\beta\lambda N}}{N!} \int_{\Lambda^N} dr_1 \cdots dr_N \exp\left[ -\frac{\beta}{2} \sum_{i \neq j} V(r_i - r_j) \right] \tag{2.7}$$

where $\lambda \in \mathbb{R}$ is the chemical potential. The pressure is defined by the limit

$$p(\beta, \lambda) = \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{\beta\,|\Lambda|} \log Z_\Lambda(\beta, \lambda) \tag{2.8}$$

which exists and defines a convex and continuous function of $\beta$ and $\lambda$, see refs. 22 and 23.

Setting the inverse temperature equal to some fixed value $\beta > 0$ (which we will sometimes omit) there exists, for the reference system, a non empty open set $\mathscr{U} \subseteq \mathbb{R}$ such that for any $\lambda \in \mathscr{U}$ there is a unique (infinite volume) Gibbs state. This is a point process on $\mathbb{R}^d$, invariant and ergodic w.r.t. space translations, satisfying the DLR equations relative to the potential $V$, see, e.g., refs. 6 and 22. The pressure is a smooth function of $\lambda \in \mathscr{U}$ and the average density of particles $\rho$, as a function of the chemical potential, is given by the smooth 1-1 map $\lambda \mapsto \rho(\lambda) = \partial_\lambda p(\beta, \lambda)$ of $\mathscr{U}$ onto $\mathscr{W} \doteq \partial_\lambda p(\beta, \mathscr{U}) \subseteq \mathbb{R}_+$. To make more symmetric the correspondence between the parameters $\lambda$ and $\rho$ we introduce the Helmholtz free energy $a(\beta, \rho)$ as the Legendre transform of the pressure:

$$a(\beta, \rho) \doteq \sup_{\lambda \in \mathbb{R}} \{\lambda\rho - p(\beta, \lambda)\} \tag{2.9}$$

and we recover the chemical potential as a function of the density by the smooth 1-1 map $\rho \mapsto \lambda(\rho) = \partial_\rho a(\beta, \rho)$ of $\mathscr{W}$ onto $\mathscr{U}$.

We consider the nonlinear non-local integro-differential equation (1.3) that we rewrite below in a more concise form:

$$\frac{\partial \rho}{\partial t}(t, x) = \nabla \cdot \{D(\rho) \nabla \rho + \sigma(\rho) \nabla J * \rho\}(t, x) \tag{2.10}$$

where " $*$ " denotes convolution on $\mathscr{T}^d$ and recall that $\sigma(\rho) = \beta\rho$. In the sequel we will use the capital letter $P$ to denote the pressure as a function of the density. Then $P'(\rho) = \rho\lambda'(\rho)$ so that the diffusion coefficient in (2.10) is $D(\rho) = \beta P'(\rho)$ (see (1.4)).

In the one phase region the pressure is a smooth, strictly increasing function of the density, so that $D(\rho)$ is smooth and strictly positive for any $\rho \in \mathscr{W}$. Then the following theorem holds, whose proof is sketched at the end of the next section.

**Theorem 2.1.** There exist locally classical solutions of (2.10) that lie inside the one phase region $\mathscr{W}$.

We fix such a solution $\rho(t, x)$, $0 \leqslant t \leqslant T$ $(T > 0)$. We may assume that there is a compact set $K_w \subset \mathscr{W}$ such that $\rho(t, x) \in K_w$ for any $(t, x) \in [0, T] \times \mathscr{T}^d$ and $\operatorname{dist}(K_w, \mathbb{R}_+ \setminus \mathscr{W}) \geqslant 2\delta_1$ for some $\delta_1 > 0$. Clearly $\lambda(t, x) \doteq \partial_\rho a(\beta, \rho(t, x))$ lies in the compact set $K_u \doteq \partial_\rho a(\beta, K_w)$ and $\operatorname{dist}(K_u, \mathbb{R} \setminus \mathscr{U}) \geqslant 2\delta_2$ for some $\delta_2 > 0$.

We introduce the local Gibbs state associated to the above macroscopic evolution $\rho(t, x)$ as the probability measure on $\mathscr{T}^{dN}$ which is absolutely continuous w.r.t. $\mu_\gamma(d\underline{x})$ with density

$$\hat{f}_\gamma(t, \underline{x}) = \frac{1}{C_\gamma(t)} \exp\left[\sum_i \beta\lambda(t, x_i)\right] \tag{2.11}$$

where $C_\gamma(t)$ is the normalization constant making $\hat{f}_\gamma$ a probability density.

Our main result is

**Theorem 2.2.** Let $f_\gamma$ be the solution of the Fokker–Plank equation (2.6) with an initial distribution $f_\gamma^{(0)}$ such that

$$\lim_{\gamma \downarrow 0} \gamma^d \int \mu_\gamma(d\underline{x})\, f_\gamma^{(0)}(\underline{x}) \log \frac{f_\gamma^{(0)}(\underline{x})}{\hat{f}_\gamma(0, \underline{x})} = 0 \tag{2.12}$$

Then, for any $\varphi \in C^\infty(\mathscr{T}^d)$, any $\delta > 0$, and any $t \in [0, T]$,

$$\lim_{\gamma \downarrow 0} \int_{A^t_{\delta, \varphi}} \mu_\gamma(d\underline{x})\, f_\gamma(t, \underline{x}) = 0 \tag{2.13}$$

where

$$A^t_{\delta, \varphi} \doteq \left\{ \underline{x} \in \mathscr{T}^{dN} : \left| N^{-1} \sum_i \varphi(x_i) - \int_{\mathscr{T}^d} dx\, \varphi(x)\, \rho(t, x) \right| > \delta \right\}$$

**Notation:** From now on we will write $f(t, \cdot) = f(t)$ for functions on $[0, T] \times \mathscr{T}^d$ or $[0, T] \times \mathscr{T}^{dN}$.

We will prove Theorem 2.2 by using the relative entropy method introduced by Yau.[27] We recall the basic *entropy estimate*: if $\mu$, $\nu$ are two probability measures on the same measurable space, then for any $F \in L^1(d\nu)$,

$$\int d\mu\, F \leqslant H(\mu \mid \nu) + \log \int d\nu \exp[F]$$

where $H(\mu \mid \nu)$ is the relative entropy of $\mu$ w.r.t. $\nu$ and, if $\mu \ll \nu$,

$$H(\mu \mid \nu) = \int d\mu \log \frac{d\mu}{d\nu}$$

For any $t \in [0, T]$ define the functional

$$H_\gamma(t) \doteq \gamma^d \int \mu_\gamma(d\underline{x})\, f_\gamma(t, \underline{x}) \log \frac{f_\gamma(t, \underline{x})}{\hat{f}_\gamma(t, \underline{x})} \tag{2.14}$$

Note that $H_\gamma(t)$ is $\gamma^d$ times the relative entropy of $f_\gamma(t)\, d\mu_\gamma$ w.r.t. $\hat{f}_\gamma(t)\, d\mu_\gamma$ and that the argument of the limit in the l.h.s. of (2.12) is exactly $H_\gamma(0)$. In the next section we will prove:

**Theorem 2.3.** Under the same hypothesis of Theorem 2.2, for any $t \in [0, T]$,

$$\lim_{\gamma \downarrow 0} H_\gamma(t) = 0 \tag{2.15}$$

The hydrodynamic limit (2.13) follows as a corollary of Theorem 2.3. To see this we note that it follows from the large deviation principle (LDP) for the local Gibbs states (2.11), see Section 5, that there is a $c(\delta, \varphi) > 0$ such that

$$\mathbb{E}^{\hat{f}_\gamma(t)}[\mathbb{1}_{A^t_{\delta, \varphi}}] \leqslant \exp[-c(\delta, \varphi)\, N]$$

where $\mathbb{E}^f[\,\cdot\,]$ denotes the expectation w.r.t. the measure $f\, d\mu_\gamma$ and $\mathbb{1}_\Gamma$ is the characteristic function of the set $\Gamma$. On the other hand, from the basic entropy estimate the following inequality holds (see, e.g., ref. 27):

$$\mathbb{E}^{f_\gamma(t)}[\mathbb{1}_{A^t_{\delta, \varphi}}] \leqslant \frac{\log 2 + \gamma^{-d} H_\gamma(t)}{\log(1 + \mathbb{E}^{\hat{f}_\gamma(t)}[\mathbb{1}_{A^t_{\delta, \varphi}}]^{-1})}$$

so that, for some $C > 0$, $\mathbb{E}^{f_\gamma(t)}[\mathbb{1}_{A^t_{\delta, \varphi}}] \leqslant C(N^{-1} + (N\gamma^d)^{-1}\, H_\gamma(t)) \to 0$ as $\gamma \downarrow 0$.

## 3. PROOF OF THEOREMS 2.3 AND 2.1

Because of the hypothesis (2.12) on the initial distribution, we only need a good estimate on the time derivative of $H_\gamma(t)$. By Lemma 3.1 of ref. 20, the following bound holds:

$$\frac{dH_\gamma}{dt} \leqslant \gamma^d \int \mu_\gamma(d\underline{x})\, f_\gamma(t, \underline{x})\, \hat{f}_\gamma(t, \underline{x})^{-1} \left(L^*_\gamma - \frac{\partial}{\partial t}\right) \hat{f}_\gamma(t, \underline{x}) \tag{3.1}$$

Recalling (2.2) and that $L^{(0)}_\gamma$ is reversible w.r.t. $\mu_\gamma(d\underline{x})$, (3.1) gives

$$\frac{dH_\gamma}{dt} \leqslant \gamma^d \int \mu_\gamma(d\underline{x})\, f_\gamma(t, \underline{x})\, \hat{f}_\gamma(t, \underline{x})^{-1} \left(L^{(0)}_\gamma - \frac{\partial}{\partial t}\right) \hat{f}_\gamma(t, \underline{x})$$

$$+ \gamma^d \int \mu_\gamma(d\underline{x})\, f_\gamma(t, \underline{x})\, \hat{f}_\gamma(t, \underline{x})^{-1}\, U^*_\gamma\, \hat{f}_\gamma(t, \underline{x}) \tag{3.2}$$

where $U_\gamma$ is defined in (2.4). By an explicit computation

$$\hat{f}_\gamma(t, \underline{x})^{-1}\left(L_\gamma^{(0)} - \frac{\partial}{\partial t}\right)\hat{f}_\gamma(t, \underline{x})$$

$$= \beta \sum_i \Big\{ \beta\, |\nabla\lambda|^2\,(t, x_i) + \Delta\lambda(t, x_i) - \dot{\lambda}(t, x_i)$$

$$- \gamma^{-1} \sum_{j\,:\,j \neq i} \beta \nabla V(\gamma^{-1}(x_i - x_j)) \cdot \nabla\lambda(t, x_i)\Big\} + \mathbb{E}^{\hat{f}_\gamma(t)}\Big[\sum_i \beta\dot{\lambda}(t, x_i)\Big] \tag{3.3}$$

where $\dot{\lambda}$ denotes the time derivative of $\lambda$. Integrating by parts one computes the action of the adjoint operator $U_\gamma^*$ and gets

$$\hat{f}_\gamma(t, \underline{x})^{-1}\, U_\gamma^* \hat{f}_\gamma(t, \underline{x}) = \gamma^d \beta \sum_{i \neq j} \Big\{ \beta \nabla J(x_i - x_j) \cdot \nabla\lambda(t, x_i) + \Delta J(x_i - x_j)$$

$$- \gamma^{-1} \sum_{k\,:\,k \neq i} \beta \nabla V(\gamma^{-1}(x_i - x_k)) \cdot \nabla J(x_i - x_j)\Big\} \tag{3.4}$$

In both (3.3) and (3.4) there is a term of the following type:

$$K_V(\underline{x}) = \sum_{i \neq k} \gamma^{-1}\, \nabla V(\gamma^{-1}(x_i - x_k)) \cdot \nabla\varphi(x_i)$$

for some smooth function $\varphi$ on $\mathcal{T}^d$ (actually $\varphi = \beta^2\lambda(t, \cdot)$ and $\varphi = \beta^2 J(\cdot - x_j)$ in (3.3) and (3.4) respectively). Since $\nabla V$ is an odd function,

$$K_V(\underline{x}) = \tfrac{1}{2} \sum_{i \neq k} \gamma^{-1}\, \nabla V(\gamma^{-1}(x_i - x_k)) \cdot (\nabla\varphi(x_i) - \nabla\varphi(x_k))$$

$$= -\tfrac{1}{2} \sum_{i \neq k} \sum_{\xi, \eta} (\nabla V)^\xi\, (\gamma^{-1}(x_i - x_k))\, D_{\xi\eta}\varphi(x_i)(\gamma^{-1}(x_i - x_k))^\eta + R_V(\underline{x}) \tag{3.5}$$

where $x^\xi$ is the $\xi$th component of $x \in \mathcal{T}^d$ and $D_{\xi\eta} = \partial^2/(\partial x^\xi\, \partial x^\eta)$. Since $\nabla V$ has compact support, we can estimate the reminder using Taylor expansion:

$$|R_V(\underline{x})| \leqslant r_\gamma(\varphi) \sum_{i \neq k} |\nabla V|\, (\gamma^{-1}(x_i - x_k))$$

with $r_\gamma(\varphi) \to 0$ as $\gamma \downarrow 0$.

Inserting (3.3) and (3.4) into (3.2) and using (3.5) we get

$$\frac{dH_\gamma}{dt} \leqslant \mathbb{E}^{f_\gamma(t)}[\beta \Phi_\gamma(\underline{x}, \lambda(t))] + \mathbb{E}^{\hat{f}_\gamma(t)}\left[\sum_i \beta \dot{\lambda}(t, x_i)\right] + \varepsilon_\gamma(t) \tag{3.6}$$

where

$$\begin{aligned}
\Phi_\gamma(\underline{x}, \lambda(t)) &= \gamma^d \sum_i \{\beta\, |\nabla\lambda|^2\,(t, x_i) + \Delta\lambda(t, x_i) - \dot{\lambda}(t, x_i)\} \\
&\quad + \gamma^d \sum_{i \neq j} \sum_{\xi, \eta} \frac{\beta}{2} (\nabla V)^\xi\, (\gamma^{-1}(x_i - x_j))\, D_{\xi\eta}\lambda(t, x_i)(\gamma^{-1}(x_i - x_j))^\eta \\
&\quad + \gamma^{2d} \sum_{i \neq j} \{\beta \nabla\lambda(t, x_i) \cdot \nabla J(x_i - x_j) + \Delta J(x_i - x_j)\} \\
&\quad + \gamma^{2d} \sum_{i \neq j} \sum_{k : k \neq i} \sum_{\xi, \eta} \frac{\beta}{2} (\nabla V)^\xi\, (\gamma^{-1}(x_i - x_k)) \\
&\quad \times D_{\xi\eta}\, J(x_i - x_j)(\gamma^{-1}(x_i - x_k))^\eta
\end{aligned} \tag{3.7}$$

while $\varepsilon_\gamma(t)$ satisfies the bound

$$|\varepsilon_\gamma(t)| \leqslant \gamma^d (r_\gamma(\beta^2 \lambda(t)) + N\gamma^d r_\gamma(\beta^2 J))\; \mathbb{E}^{f_\gamma(t)}\left[\sum_{i \neq j} |\nabla V|\; (\gamma^{-1}(x_i - x_j))\right]$$

To obtain the behavior of $\varepsilon_\gamma(t)$ when $\gamma \downarrow 0$ we use the following lemma which is proved in Section 4:

**Lemma 3.1.** If $W$ is a continuous function on $\mathbb{R}^d$ with compact support, there is $C_W > 0$ such that, for any $\gamma \in (0, 1]$ and any $t \in (0, T]$,

$$\mathbb{E}^{f_\gamma(t)}\left[\gamma^d \sum_{i \neq j} W(\gamma^{-1}(x_i - x_j))\right] \leqslant C_W$$

By applying the lemma with $W = |\nabla V|$ we conclude that

$$\lim_{\gamma \downarrow 0} \sup_{t \in [0, T]} |\varepsilon_\gamma(t)| = 0 \tag{3.8}$$

Moreover, by the LDP for the local Gibbs state, see Section 5, for any $t \in [0, T]$,

$$\lim_{\gamma \downarrow 0} \mathbb{E}^{\hat{f}_\gamma(t)}\left[\sum_i \dot{\lambda}(t, x_i)\right] = \int_{\mathscr{T}^d} dx\, \dot{\lambda}(t, x)\, \rho(t, x) \tag{3.9}$$

Now we want to write $\Phi_\gamma(\underline{x}, \lambda(t))$ in terms of local empirical quantities. Let $\Omega$ be the space of particle configurations on $\mathbb{R}^d$, i.e. $\omega \in \Omega$ is a subset of $\mathbb{R}^d$ which is locally finite (see Section 5 for more details). Given $x \in \mathcal{T}^d$, for any $\underline{x} \in \mathcal{T}^{dN}$ we construct a configuration $\omega_{x,\gamma} \in \Omega$ by setting

$$\omega_{x,\gamma} \doteq \{q_i = \gamma^{-1}(x_i - x) : |x_i - x| < 1/4\}$$

(since there is no risk of confusion, to simplify notation we omit the explicit dependence on $\underline{x}$ of $\omega_{x,\gamma}$). Clearly $\omega_{x,\gamma}$ is well defined in every compact set inside the cube of $\mathbb{R}^d$ of side $1/(2\gamma)$ and centered in the origin. So, if $F(\omega)$ is a local function on $\Omega$, $F(\omega_{x,\gamma})$ is well defined for any $\gamma$ small enough.

Let us introduce the cubes $D_n = \{q \in \mathbb{R}^d : |q^\xi| \leqslant n, \xi = 1,..., d\}$, $n \in \mathbb{N}$. For any local function $F$ we denote by $F_n$ its spatial average over the cube $D_n$, i.e.

$$F_n(\omega) \doteq \frac{1}{|D_n|} \int_{D_n} dr\, F(\tau_r \omega) \tag{3.10}$$

where $\tau_r$ is the space translation by $r$ ($\tau_r r' = r + r'$).

Let $\chi$ be any non negative function on $\mathbb{R}^d$ with compact support and total integral 1. We define the following local functions on $\Omega$:

$$R(\omega) \doteq \sum_{q \in \omega} \chi(q) \tag{3.11}$$

$$G^{\xi\eta}(\omega) \doteq \frac{\beta}{2} \sum_{\substack{q,\, q' \in \omega \\ q \neq q'}} \chi(q)(\nabla V)^\xi (q - q')(q - q')^\eta, \qquad \xi, \eta = 1,..., d \tag{3.12}$$

and let $R_n(\omega)$, $G_n^{\xi\eta}(\omega)$ be their averages over $D_n$. Observe that $R(\omega)$ is a natural version of local density for the configuration $\omega$, while $G^{\xi\eta}(\omega)$ is the local quantity appearing in the virial theorem (see below, before (3.19)).

**Lemma 3.2.** Let $\varphi$, $\psi$ be smooth functions on $\mathcal{T}^d$. Then

$$\limsup_{n\to\infty} \limsup_{\gamma \downarrow 0} \sup_{t\in[0,T]} \mathbb{E}^{f_\gamma(t)} \left| \gamma^d \sum_i \varphi(x_i) - \int_{\mathcal{T}^d} dx\, \varphi(x)\, R_n(\omega_{x,\gamma}) \right| = 0 \tag{3.13}$$

$$\limsup_{n\to\infty} \limsup_{\gamma \downarrow 0} \sup_{t\in[0,T]} \mathbb{E}^{f_\gamma(t)} \left| \gamma^d \sum_{i\neq j} \frac{\beta}{2} (\nabla V)^\xi (\gamma^{-1}(x_i - x_j))(\gamma^{-1}(x_i - x_j))^\eta\, \varphi(x_i) - \int_{\mathcal{T}^d} dx\, \varphi(x)\, G_n^{\xi\eta}(\omega_{x,\gamma}) \right| = 0 \tag{3.14}$$

$$\limsup_{n\to\infty}\limsup_{\gamma\downarrow 0}\sup_{t\in[0,T]}\mathbb{E}^{f_\gamma(t)}\Bigg|\gamma^{2d}\sum_{i\neq j}\varphi(x_i)\,\psi(x_i-x_j)$$

$$-\int_{\mathscr{T}^d}dx\int_{\mathscr{T}^d}dy\;\varphi(x)\,R_n(\omega_{x,\gamma})\,\psi(x-y)\,R_n(\omega_{y,\gamma})\Bigg|=0 \tag{3.15}$$

$$\limsup_{n\to\infty}\limsup_{\gamma\downarrow 0}\sup_{t\in[0,T]}\mathbb{E}^{f_\gamma(t)}$$

$$\times\Bigg|\gamma^{2d}\sum_{i\neq j}\sum_{k\,:\,k\neq i}\frac{\beta}{2}(\nabla V)^\xi\,(\gamma^{-1}(x_i-x_k))(\gamma^{-1}(x_i-x_k))^\eta$$

$$\times\,\psi(x_i-x_j)-\int_{\mathscr{T}^d}dx\int_{\mathscr{T}^d}dy\;G_n^{\xi\eta}(\omega_{x,\gamma})\,\psi(x-y)\,R_n(\omega_{y,\gamma})\Bigg|=0 \tag{3.16}$$

*Proof.* We have

$$\int_{\mathscr{T}^d}dx\;\varphi(x)\,R_n(\omega_{x,\gamma})=\gamma^d\sum_i\varphi_{n,\gamma}(x_i)$$

where, for $z\in\mathscr{T}^d$,

$$\varphi_{n,\gamma}(z)\doteq\gamma^{-d}\int_{\mathscr{T}^d}dx\;\varphi(x)\,\frac{1}{|D_n|}\int_{D_n}dq\;\chi(\gamma^{-1}(z-x)+q)$$

Then the expectation in the l.h.s. of (3.13) can be bounded by $\|\varphi-\varphi_{n,\gamma}\|_\infty$ that vanishes as $\gamma\downarrow 0$ for any $n\in\mathbb{N}$ because of the smoothness assumptions on $\varphi$. In an analogous way one can estimate the expectation in the l.h.s. of (3.14) by

$$\|\varphi-\varphi_{n,\gamma}\|_\infty\,\gamma^d\,\mathbb{E}^{f_\gamma(t)}\left[\sum_{i\neq j}\frac{\beta}{2}\,|\nabla V|\,(\gamma^{-1}(x_i-x_j))\,|\gamma^{-1}(x_i-x_j)|\right]$$

and (3.14) follows from Lemma 3.1.

Let us consider now (3.15). We have

$$\int_{\mathscr{T}^d}dx\int_{\mathscr{T}^d}dy\;\varphi(x)\,R_n(\omega_{x,\gamma})\,\psi(x-y)\,R_n(\omega_{y,\gamma})$$

$$=\gamma^{2d}\sum_{i\neq j}\varphi_{n,\gamma}(x_i)\,\psi_{n,\gamma}(x_i-x_j)+Err_1$$

where $\psi_{n,\gamma}$ is defined as $\varphi_{n,\gamma}$ and

$$|Err_1| \leqslant \|\varphi_{n,\gamma}\|_\infty \times \max\{|\psi_{n,\gamma}(x-z)-\psi_{n,\gamma}(y-z)| : x, y, z \in \mathscr{T}^d, |x-y| \leqslant \gamma(r+n)\}$$

with $r$ such that the support of $\chi$ is contained in the closed ball of radius $r$. Since $|\psi_{n,\gamma}(x-z)-\psi_{n,\gamma}(y-z)| \leqslant \|\nabla\psi\|_\infty \gamma(r+n)+2\|\psi-\psi_{n,\gamma}\|_\infty$, $Err_1$ vanishes as $\gamma\downarrow 0$. Since $|\varphi_{n,\gamma}(x_i)\,\psi_{n,\gamma}(x_i-x_j)-\varphi(x_i)\,\psi(x_i-x_j)| \leqslant \|\varphi\|_\infty \|\psi-\psi_{n,\gamma}\|_\infty + \|\psi\|_\infty \|\varphi-\varphi_{n,\gamma}\|_\infty$, (3.15) follows. In the same manner we compute

$$\int_{\mathscr{T}^d} dx \int_{\mathscr{T}^d} dy\, G_n^{\xi\eta}(\omega_{x,\gamma})\,\psi(x-y)\,R_n(\omega_{y,\gamma})$$

$$=\gamma^{2d}\sum_{i\neq j}\psi_{n,\gamma}(x_i-x_j)\sum_{k:k\neq i}\frac{\beta}{2}(\nabla V)^\xi(\gamma^{-1}(x_i-x_k))(\gamma^{-1}(x_i-x_k))^\eta+Err_2$$

with

$$|Err_2| \leqslant \max\{|\psi_{n,\gamma}(x-z)-\psi_{n,\gamma}(y-z)| : x, y, z \in \mathscr{T}^d, |x-y| \leqslant \gamma(r+n)\} \times \gamma^d \sum_{i\neq j} |\nabla V|(\gamma^{-1}(x_i-x_j))\,|\gamma^{-1}(x_1-x_j)|$$

Then (3.16) follows from Lemma 3.1. ∎

Collecting together (3.6), (3.8), (3.9) and applying Lemma 3.2 to $\mathbb{E}^{f_\gamma(t)}[\Phi_\gamma(\underline{x}, \lambda(t))]$, we obtain

$$\limsup_{n\to\infty}\ \limsup_{\gamma\downarrow 0}\ \sup_{t\in[0,T]}\left\{\frac{dH_\gamma}{dt}+\mathbb{E}^{f_\gamma(t)} \times\left[\beta\int_{\mathscr{T}^d} dx\,(\Psi_{\underline{x}}(t,x)-\dot{\lambda}(t,x)\,\rho(t,x))\right]\right\}\leqslant 0 \tag{3.17}$$

with

$$\begin{aligned}\Psi_{\underline{x}}(t,x) = {} & [\dot{\lambda}(t,x)-\beta\,|\nabla\lambda|^2(t,x)-\Delta\lambda(t,x)]\,R_n(\omega_{x,\gamma}) \\ & -\sum_{\xi,\eta} D_{\xi\eta}\lambda(t,x)\,G_n^{\xi\eta}(\omega_{x,\gamma}) \\ & -\beta\nabla\lambda(t,x)\cdot(\nabla J * R_n(\omega_{\cdot,\gamma}))(x)\,R_n(\omega_{x,\gamma}) \\ & -(\Delta J * R_n(\omega_{\cdot,\gamma}))(x)\,R_n(\omega_{x,\gamma}) \\ & -\sum_{\xi,\eta}(D_{\xi\eta} J * R_n(\omega_{\cdot,\gamma}))(x)\,G_n^{\xi\eta}(\omega_{x,\gamma})\end{aligned} \tag{3.18}$$

where, for any $\varphi \in C(\mathscr{T}^d)$,

$$(\varphi * R_n(\omega_{\cdot,\gamma}))(x) \doteq \int_{\mathscr{T}^d} dx\, \varphi(x-y)\, R_n(\omega_{y,\gamma})$$

Now we want to substitute the spatial average $G_n^{\xi\eta}(\omega_{x,\gamma})$ with a function of the empirical density $R_n(\omega_{x,\gamma})$. More precisely we would like to replace it by the average of $G^{\xi\eta}$ w.r.t. the Gibbs state with density equal to $R_n(\omega_{x,\gamma})$.

To do this we need to introduce some cutoffs. Let $K$ be a compact set such that

$$K_w \subset K \subset \mathscr{W}, \qquad \operatorname{dist}(K, \mathbb{R}_+ \backslash \mathscr{W}) \geqslant \delta_1, \qquad \operatorname{dist}(K_w, \mathbb{R}_+ \backslash K) \geqslant \delta_1$$

(recall that $K_w$ is the compact set inside the one phase region $\mathscr{W}$ where the solution $\rho(t,x)$ lies and that $\operatorname{dist}(K_w, \mathbb{R}_+ \backslash \mathscr{W}) \geqslant 2\delta_1$) and define the local function $u_n(\omega) \doteq \mathbb{1}_K(R_n(\omega))$. We denote also by $\phi_k$ the cutoff at the level $k \in \mathbb{R}_+$, i.e. $\phi_k(s) = s$ if $|s| \leqslant k$, $\phi_k(s) = \operatorname{sign}(s)\, k$ otherwise. Finally let $\hat{G}^{\xi\eta}(\rho)$, $\rho \in \mathscr{W}$, be the average of $G^{\xi\eta}(\omega)$ w.r.t. the unique Gibbs measure with density $\rho$. By the virial theorem, see, e.g., ref. 25,

$$\hat{G}^{\xi\eta}(\rho) = (\beta P(\rho) - \rho)\, \delta_{\xi\eta}$$

where $P(\rho)$ is the pressure as a function of the density $\rho$ introduced just after (2.10).

For any measurable function $m: \mathscr{T}^d \to \mathbb{R}_+$ we define the functional

$$\begin{aligned}\Omega(t,x,m) \doteq{}& (\dot{\lambda}(t,x) - \beta\, |\nabla\lambda|^2\, (t,x))\, m(x) - \beta\Delta\lambda(t,x)\, P(m(x))\\ &- \beta\nabla\lambda(t,x)\cdot(\nabla J * m)(x)\, m(x) - \beta(\Delta J * m)(x)\, P(m(x))\end{aligned} \tag{3.19}$$

Observe now that, since $P'(\rho) = \rho\lambda'(\rho)$, by integration by parts,

$$\begin{aligned}\int_{\mathscr{T}^d} dx\, P(\rho(t,x))\, \Delta\lambda(t,x) &= -\int_{\mathscr{T}^d} dx\, \rho(t,x)\, \lambda'(\rho(t,x))\, \nabla\rho(t,x)\cdot\nabla\lambda(t,x)\\ &= -\int_{\mathscr{T}^d} dx\, \rho(t,x)\, |\nabla\lambda(t,x)|^2\end{aligned}$$

and, analogously,

$$\int_{\mathscr{T}^d} dx\, P(\rho(t,x))(\Delta J * \rho(t))(x) = -\int_{\mathscr{T}^d} dx\, \rho(t,x)\, \nabla\lambda(t,x)\cdot(\nabla J * \rho(t))(x)$$

so that, for any $t \in [0, T]$,

$$\int_{\mathcal{T}^d} dx\, \dot{\lambda}(t, x)\, \rho(t, x) = \int_{\mathcal{T}^d} dx\, \Omega(t, x, \rho(t))$$

Then we can replace $\dot{\lambda}(t, x)\, \rho(t, x)$ by $\Omega(t, x, \rho(t))$ in (3.17).

We decompose now, for any $k > 0$,

$$\Psi_{\underline{x}}(t, x) - \Omega(t, x, \rho(t)) = \sum_{p=1}^{4} \Omega_p(t, x)$$

with

$$\begin{aligned}
\Omega_1(t, x) &= [\Omega(t, x, R_n(\omega_{\cdot, \gamma})) - \Omega(t, x, \rho(t))]\, u_n(\omega_{x, \gamma}) \\
\Omega_2(t, x) &= [\Psi^{(k)}_{\underline{x}}(t, x) - \Omega(t, x, R_n(\omega_{\cdot, \gamma}))]\, u_n(\omega_{x, \gamma}) \\
\Omega_3(t, x) &= [\Psi^{(k)}_{\underline{x}}(t, x) - \Omega(t, x, \rho(t))](1 - u_n(\omega_{x, \gamma})) \\
\Omega_4(t, x) &= [\Psi_{\underline{x}}(t, x) - \Psi^{(k)}_{\underline{x}}(t, x)]
\end{aligned} \tag{3.20}$$

where $\Psi^{(k)}_{\underline{x}}(t, x)$ is defined as $\Psi_{\underline{x}}(t, x)$ in (3.18) with $G^{\xi\eta}_n$ replaced by $(\phi_k \circ G^{\xi\eta})_n$.

In Section 5 we will prove that there is $\delta_0 > 0$ such that

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \limsup_{\gamma \downarrow 0} \sup_{t \in [0, T]} \left\{ \mathbb{E}^{f_\gamma(t)} \left[ \int_{\mathcal{T}^d} dx\, \beta \Omega_p(t, x) \right] - \delta_0^{-1} H_\gamma(t) \right\} \leqslant 0, \qquad p = 3, 4 \tag{3.21}$$

On the other hand, the local ergodic theorem, see Section 4, implies that

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \limsup_{\gamma \downarrow 0} \int_0^T ds\, \mathbb{E}^{f_\gamma(s)} \left[ \int_{\mathcal{T}^d} dx\, \beta\, |\Omega_2(s, x)| \right] = 0 \tag{3.22}$$

From (3.17), (3.21) and (3.22) we get, for any $t \in [0, T]$,

$$H_\gamma(t) + \int_0^t ds \left\{ \mathbb{E}^{f_\gamma(s)} \left[ \int_{\mathcal{T}^d} dx\, \beta \Omega_1(s, x) \right] - 2\delta_0^{-1} H_\gamma(s) \right\} \leqslant o(n, \gamma) \tag{3.23}$$

with

$$\limsup_{n \to \infty} \limsup_{\gamma \downarrow 0} o(n, \gamma) = 0$$

Now, from the basic entropy estimate, for any $\delta > 0$ and any $s \in [0, T]$,

$$\mathbb{E}^{f_\gamma(s)}\left[\int_{\mathscr{T}^d} dx\, \beta\Omega_1(s,x)\right] \geqslant -\delta^{-1} H_\gamma(s) - \delta^{-1}\gamma^d \log \mathbb{E}^{\hat{f}_\gamma(s)} \exp\left[-\delta\gamma^{-d}\int_{\mathscr{T}^d} dx\, \beta\Omega_1(s,x)\right]$$

so that, from (3.23), for any $t \in [0, T]$,

$$H_\gamma(t) - \delta^{-1}\gamma^d \int_0^t ds \log \mathbb{E}^{\hat{f}_\gamma(s)} \exp\left[-\delta\gamma^{-d}\int_{\mathscr{T}^d} dx\, \beta\Omega_1(s,x)\right] - (2\delta_0^{-1} + \delta^{-1}) \int_0^t ds\, H_\gamma(s) \leqslant o(n,\gamma)$$

By applying the Gronwall Lemma to the last inequality we get

$$H_\gamma(t) \leqslant e^{(2\delta_0^{-1}+\delta^{-1})\,t}\left(o(n,\gamma) + \delta^{-1}\gamma^d \int_0^t ds \log \mathbb{E}^{\hat{f}_\gamma(s)} \times \exp\left[-\delta\gamma^d \int_{\mathscr{T}^d} dx\, \beta\Omega_1(s,x)\right]\right) \tag{3.24}$$

In Section 5 we will prove that

$$\limsup_{n\to\infty}\ \limsup_{\gamma\downarrow 0}\ \delta^{-1}\gamma^d \log \mathbb{E}^{\hat{f}_\gamma(s)} \exp\left[-\delta\gamma^{-d}\int_{\mathscr{T}^d} dx\, \beta\Omega_1(s,x)\right] \leqslant \delta^{-1}\Theta_\delta(s,\lambda) \tag{3.25}$$

where

$$\Theta_\delta(s,\lambda) \doteq \sup\left\{\int_{\mathscr{T}^d} dx\, [\delta\beta[\Omega(s,x,m) - \Omega(s,x,\rho(s))] \times \mathbb{1}_K(m(x)) - I_\beta(\lambda(s,x), m(x))];\ m : \mathscr{T}^d \to \mathbb{R}_+ \text{ integrable}\right\} \tag{3.26}$$

and (recall definitions (2.8) and (2.9))

$$I_\beta(\lambda, m) \doteq \beta(p(\beta, \lambda) + a(\beta, m) - \lambda m) \tag{3.27}$$

From (3.24) and (3.25), for any $t \in [0, T]$,

$$\limsup_{\gamma \downarrow 0} H_\gamma(t) \leqslant e^{(2\delta_0^{-1} + \delta^{-1})\, t} \int_0^t ds\, \delta^{-1} \Theta_\delta(s, \lambda) \tag{3.28}$$

We conclude the proof of Theorem 2.3 from (3.28) by showing that, for $\delta$ small enough, $\Theta_\delta(s, \lambda) = 0$ for any $s \in [0, T]$. We first note that, for $\lambda \in \mathscr{U}$, $m \mapsto I_\beta(\lambda, m)$ is strictly convex on $K$, non negative, and equal to 0 iff $m = \partial_\lambda p(\beta, \lambda)$ so that

$$-\int_{\mathscr{T}^d} dx\, I_\beta(\lambda(s, x), m(x)) \leqslant 0, \qquad = 0 \qquad \text{iff} \quad m(x) = \rho(s, x)$$

On the other hand the functional

$$\mathscr{G}_s(m) \doteq \int_{\mathscr{T}^d} dx\, \beta[\Omega(s, x, m) - \Omega(s, x, \rho(s))]\, \mathbb{1}_K(m(x))$$

is bounded on the class of functions considered in (3.26) and equal to 0 for $m = \rho(s)$. Then, for $\delta$ small enough, $\Theta_\delta(s, \lambda) = 0$ provided that

$$\frac{\delta \mathscr{G}_s}{\delta m}(\rho(s)) = 0$$

(observe that $\rho(s)$ is away from $\mathbb{R}_+ \setminus K$ because $\operatorname{dist}(K_w, \mathbb{R}_+ \setminus K) \geqslant \delta_1$). By an explicit computation,

$$\begin{aligned} \beta^{-1} \frac{\delta \mathscr{G}_s}{\delta m}(\rho(s)) &\\ = \dot{\lambda}(s) - \beta(|\nabla\lambda|^2(s) + \Delta\lambda(s)\, P'(\rho(s)) - \nabla\lambda(s) \cdot \nabla J * \rho(s) &\\ - \nabla J * (\rho(s)\, \nabla\lambda(s)) - P'(\rho(s))\, \Delta J * \rho(s) - \Delta J * P(\rho(s))) & \end{aligned}$$

But, recalling that $P'(\rho) = \rho\lambda'(\rho) = \beta^{-1} D(\rho)$,

$$|\nabla\lambda|^2 + \Delta\lambda P'(\rho) = \lambda'(\rho)(\nabla\rho \cdot \nabla\lambda + \rho\Delta\lambda) = \beta^{-1}\lambda'(\rho)\, \nabla \cdot (D(\rho)\, \nabla\rho)$$

and

$$\nabla\lambda \cdot \nabla J * \rho + \nabla J * (\rho \nabla\lambda) + (\Delta J * \rho)\, P'(\rho) + (\Delta J * P(\rho))$$
$$= \lambda'(\rho)(\nabla\rho \cdot \nabla J * \rho + \rho \Delta J * \rho) = \lambda'(\rho)\, \nabla \cdot (\rho \nabla J * \rho)$$

So that, for any $(s, x) \in [0, T] \times \mathscr{T}^d$,

$$\frac{\delta \mathscr{G}_s}{\delta m(x)}(\rho(s)) = \beta\lambda'(\rho(s, x)) \left[ \frac{\partial \rho}{\partial s} - \nabla \cdot \{ D(\rho)\, \nabla\rho + \beta\rho\nabla J * \rho \} \right] (s, x) = 0$$

since $\rho(s, x)$ satisfies (2.10) and $\sigma(\rho) = \beta\rho$. ∎

We conclude the section with the proof of Theorem 2.1.

*Proof of Theorem 2.1 (sketch).* Let $\rho_0 \in C^2(\mathscr{T}^d)$ be such that $\rho_0(\mathscr{T}^d) \subset \mathscr{W}$. By continuity and compactness we can find three compact intervals $I_i = [a_i, b_i]$, $i = 1, 2, 3$, such that $\rho_0(\mathscr{T}^d) \subseteq I_1 \subset I_2 \subset I_3 \subset \mathscr{W}$, $a_3 < a_2 < a_1 < b_1 < b_2 < b_3$. We construct two functions $\tilde{D}$, $\tilde{\sigma} \in C^1(\mathbb{R})$ with the following properties: $\tilde{D}(u) = D(u)$ and $\tilde{\sigma}(u) = \sigma(u)$ for $u \in I_2$, $c^{-1} \leqslant \tilde{D}(u) \leqslant c$ for some $c > 1$ and for any $u \in \mathbb{R}$, $\operatorname{supp}(\tilde{\sigma}) \subseteq I_3$. Then we consider the Cauchy problem

$$\frac{\partial \rho}{\partial t}(t, x) = \nabla \cdot \{ \tilde{D}(\rho)\, \nabla\rho + \tilde{\sigma}(\rho)\, \nabla J * \rho \}(t, x) \tag{3.29}$$

with initial datum $\rho_0$. Arguing exactly as in Theorem 4.1 and Remark 4.1 of ref. 10, we know that there exists a (unique) classical solution $\rho(t, x)$ of the Cauchy problem above (moreover it lies in the region $I_3$ at any time). Clearly we can find $T > 0$ such that $\rho(t, x) \in I_2$ for any $t \in [0, T]$ and $x \in \mathscr{T}^d$. From the choice of $\tilde{D}$ and $\tilde{\sigma}$ it follows that $\{\rho(t, x); (t, x) \in [0, T] \times \mathscr{T}^d\}$ is also a (local) classical solution of the original equation (2.10). ∎

## 4. LOCAL ERGODICITY AND ENTROPY BOUNDS

We start this section by proving (3.22). Since $D_{\xi\eta}\lambda$ and $D_{\xi\eta} J * R_n(\omega_{\cdot,\gamma})$ are bounded functions on $\mathscr{T}^d$, it is sufficient to prove that

$$\limsup_{k \to \infty}\, \limsup_{n \to \infty}\, \limsup_{\gamma \downarrow 0} \int_0^T dt\, \mathbb{E}^{f_\gamma(t)} \left[ \int_{\mathscr{T}^d} dx\, |(\phi_k \circ G^{\xi\eta})_n (\omega_{x,\gamma}) - \hat{G}^{\xi\eta}(R_n(\omega_{x,\gamma}))|\, u_n(\omega_{x,\gamma}) \right] = 0 \tag{4.1}$$

The main step in proving (4.1) is a local ergodic theorem for the measure $f_\gamma(t)\,d\mu_\gamma$:

**Theorem 4.1.** For any local, bounded and continuous function $F$,

$$\limsup_{n\to\infty}\ \limsup_{\gamma\downarrow 0}\int_0^T dt\ \mathbb{E}^{f_\gamma(t)}\left[\int_{\mathcal{T}^d} dx\ |F_n(\omega_{x,\gamma})-\hat{F}(R_n(\omega_{x,\gamma}))|\ u_n(\omega_{x,\gamma})\right]=0 \tag{4.2}$$

where $\hat{F}(\rho)$, $\rho\in\mathscr{W}$, is the average of $F$ w.r.t. the (unique) Gibbs measure with density $\rho$.

*Proof.* We introduce the translation invariant density

$$\bar{f}_\gamma(\underline{x})=\frac{1}{T}\int_0^T ds\int_{\mathcal{T}^d} dx\ f_\gamma(s,\tau_x\underline{x}) \tag{4.3}$$

so that (4.2) is equivalent to proving that

$$\limsup_{n\to\infty}\ \limsup_{\gamma\downarrow 0}\ \mathbb{E}^{\bar{f}_\gamma}[\,|F_n(\omega_{0,\gamma})-\hat{F}(R_n(\omega_{0,\gamma}))|\ u_n(\omega_{0,\gamma})]=0 \tag{4.4}$$

Let $n_F\in\mathbb{N}$ be such that $\operatorname{supp}(F)\subset D_{n_F}$ ($D_n$ as in Definition (3.10)) and let $\bar{n}=n+n_F$. Through the mapping $\underline{x}\mapsto\omega_{0,\gamma}$, for any $\gamma$ small enough it is well defined the projection

$$\Pi_{\bar{n}}:\mathcal{T}^{dN}\to\Omega|_{D_{\bar{n}}}:\Pi_{\bar{n}}(\underline{x})=\omega_{0,\gamma}|_{D_{\bar{n}}}$$

We have to characterize the family $\mathscr{F}_{\bar{n}}$ of the (weak) limit points of $\{\nu_\gamma=\Pi_{\bar{n}}(\bar{f}_\gamma d\mu_\gamma);\ \gamma\in(0,1]\}$. First of all we observe that, by translation invariance, for any finite region $\Lambda$ of $D_{\bar{n}}$,

$$\mathbb{E}^{\nu_\gamma}[N_\Lambda]=N\gamma^d\,|\Lambda|\leqslant|\Lambda| \tag{4.5}$$

This proves that the family $\{\nu_\gamma=\Pi_{\bar{n}}(\bar{f}_\gamma d\mu_\gamma);\ \gamma\in(0,1]\}$ is tight.

Let $\mu^{\bar{\omega}}_{m,\bar{n}}$ be the canonical Gibbs measure on the cube $D_{\bar{n}}$ with boundary conditions $\bar{\omega}\in\Omega$ and number of particles $m$. We prove that any $\nu\in\mathscr{F}_{\bar{n}}$ can be written as

$$\nu(d\omega)=\int\hat{\nu}(d\bar{\omega},dm)\ \mu^{\bar{\omega}}_{m,\bar{n}}(d\omega) \tag{4.6}$$

where $\hat{\nu}(d\bar{\omega},dm)$ is a measure supported on $\{m\leqslant|D_{\bar{n}}|\}$. Since any limit point satisfies an inequality like (4.5), we have only to prove (4.6) for some $\hat{\nu}$.

By a straightforward extension to higher dimensions of the analogous argument in Varadhan, [ref. 25, Lemma 7.5], we can reduce the problem to the estimate of a certain Dirichlet form. More precisely (4.6) follows if

$$\sup_{\gamma\in(0,1]} \gamma^d \int d\mu_\gamma \sum_i \frac{|\nabla_i \bar{f}_\gamma|^2}{\bar{f}_\gamma} < +\infty \tag{4.7}$$

The bound (4.7) is consequence of the following lemma:

**Lemma 4.2.** Let

$$\bar{f}_\gamma(t, \underline{x}) \doteq \frac{1}{t} \int_0^t ds \int_{\mathscr{T}^d} dx\, f_\gamma(t, \tau_x \underline{x}) \tag{4.8}$$

and define, for any density $f$,

$$\sigma_\gamma(f) \doteq \gamma^d \int d\mu_\gamma \sum_i \frac{|\nabla_i f|^2}{f}$$

Then there is $C > 0$ such that, for any $\gamma \in (0, 1]$ and any $t \in (0, T]$,

$$\sigma_\gamma(\bar{f}_\gamma(t)) \leqslant \frac{C}{t} \tag{4.9}$$

*Proof.* With an abuse of notation, denote by $H(f_\gamma(t) \mid 1)$ the relative entropy of $f_\gamma(t)\, d\mu_\gamma$ w.r.t. $d\mu_\gamma$. Observing that $L_\gamma 1 = 0$ we get, after some standard computations,

$$\frac{d}{dt} H(f_\gamma(t) \mid 1) = \frac{d}{dt} \int d\mu_\gamma\, f_\gamma(t) \log f_\gamma(t) = \int d\mu_\gamma\, L_\gamma^* f_\gamma(t) \log f_\gamma(t)$$

$$= -\gamma^{-d} \sigma_\gamma(f_\gamma(t)) - \gamma^d \int d\mu_\gamma \sum_{i\neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i f_\gamma(t) \tag{4.10}$$

Since for any $x \in \mathscr{T}^d \tau_x f_\gamma(t) = f_\gamma(t, \tau_x \cdot)$ solves the same Fokker–Planck equation, recalling definition (4.3), from (4.10) we get

$$\int_{\mathscr{T}^d} dx \frac{H(\tau_x f_\gamma(t) \mid 1) - H(\tau_x f_\gamma(0) \mid 1)}{t}$$

$$= -\frac{1}{t} \int_0^t ds \int_{\mathscr{T}^d} dx\, \gamma^{-d} \sigma_\gamma(\tau_x f_\gamma(s)) - \int d\mu_\gamma\, \gamma^d \sum_{i\neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i \bar{f}_\gamma(t) \tag{4.11}$$

Now, since $\sigma_\gamma(\cdot)$ is a convex functional,[4]

$$\sigma_\gamma(\bar{f}_\gamma(t)) \leqslant \frac{1}{t} \int_0^t ds \int_{\mathscr{T}^d} dx\, \sigma_\gamma(\tau_x f_\gamma(s)) \tag{4.12}$$

On the other hand, by Cauchy–Schwartz inequality,

$$\begin{aligned}
&-\int d\mu_\gamma\, \gamma^d \sum_{i \neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i \bar{f}_\gamma(t) \\
&\quad \leqslant \sqrt{\int d\mu_\gamma\, \bar{f}_\gamma(t)\, \gamma^d \sum_i \left| \sum_{j:\, j \neq i} \beta \nabla J(x_i - x_j) \right|^2} \sqrt{\sigma_\gamma(\bar{f}_\gamma(t))} \\
&\quad \leqslant C_1 \gamma^{-d} \sqrt{\sigma_\gamma(\bar{f}_\gamma(t))}
\end{aligned} \tag{4.13}$$

with $C_1 = \beta\, \|\nabla J\|_\infty$. Also, since $d\mu_\gamma$ is $\tau_x$-invariant,

$$\begin{aligned}
H(\tau_x f_\gamma(0) \mid 1) &= H(f_\gamma(0) \mid 1) \\
&= \gamma^{-d} H_\gamma(0) - \log C_\gamma(0) + \int d\mu_\gamma f_\gamma(0, \underline{x}) \sum_i \beta\lambda(0, x_i) \\
&\leqslant \gamma^{-d} H_\gamma(0) + 2\beta\, \|\lambda(0, \cdot)\|_\infty N \leqslant C_2 \gamma^{-d}
\end{aligned} \tag{4.14}$$

for some $C_2 > 0$ (in the first bound we used Jensen's inequality, in the second one the assumption (2.12) on the initial distribution). Collecting together (4.11), (4.12), (4.13) and (4.14), recalling also that the relative entropy is a positive function, we get

$$-\frac{C_2}{t} \gamma^{-d} \leqslant -\gamma^{-d} \sigma_\gamma(\bar{f}_\gamma(t)) + C_1 \gamma^{-d} \sqrt{\sigma_\gamma(\bar{f}_\gamma(t))}$$

so that, for any $\gamma \in (0, 1]$,

$$\sigma_\gamma(\bar{f}_\gamma(t)) \leqslant C_1 \sqrt{\sigma_\gamma(\bar{f}_\gamma(t))} + \frac{C_2}{t} \tag{4.15}$$

But (4.15) implies that $\sigma_\gamma(\bar{f}_\gamma(t)) \leqslant C/t$ with $C$ the positive solution of $x = C_1 \sqrt{Tx} + C_2$. The lemma is proven. ∎

Now we conclude the proof of Theorem 4.1. Using (4.6), the l.h.s. of (4.4) can be bounded by

$$\limsup_{n \to \infty} \sup_{\mu \in \mathscr{G}_1} \mathbb{E}^\mu[\,|F_n(\omega) - \hat{F}(R_n(\omega))|\, \mathbb{1}_K(R_n(\omega))\,]$$

where $\mathcal{G}_1$ is the class of Gibbs states with density $\rho \leqslant 1$. The characteristic function $\mathbb{1}_K$ reduces the problem to computing the above limit in the one phase region. The limit is then zero by the law of large numbers for the unique Gibbs state of given density $\rho \in K$.

Finally, the limit (4.1) follows easily from Theorem 4.1. In fact (4.2) with $F = \phi_k \circ G^{\xi\eta}$ implies that the l.h.s. of (4.1) can be bounded by

$$\lim_{k \to \infty} T \sup_{\rho \in K} \{ | \widehat{\phi_k \circ G^{\xi\eta}}(\rho) - \hat{G}^{\xi\eta}(\rho) | \} \tag{4.16}$$

But, for any $\rho \in K$,

$$\widehat{\phi_k \circ G^{\xi\eta}}(\rho) - \hat{G}^{\xi\eta}(\rho) = \mathbb{E}^{\mu_\rho}[(\phi_k \circ G^{\xi\eta})(\omega) - G^{\xi\eta}(\omega)]$$

where $\mu_\rho$ is the (unique) Gibbs state with chemical potential $\lambda = \partial_\rho a(\beta, \rho)$. We observe now that $\phi_k \circ G^{\xi\eta} \to G^{\xi\eta}$ pointwise as $k \to \infty$. Moreover $|\phi_k \circ G^{\xi\eta}| \leqslant |G^{\xi\eta}| \leqslant c N_B^2$ for some $c > 0$ and some finite subset $B$ of $\mathbb{R}^d$. Recalling that, by superstability, $\mathbb{E}^{\mu_\rho}[N_B^2(\omega)] < \infty$, the limit (4.16) is zero by the Dominated Convergence Theorem.

We conclude the section by proving Lemma 3.1:

*Proof of Lemma 3.1.* Since $V$ is positive and superstable, there is a constant $\tilde{C}_W$ such that, for any $\underline{x} \in \mathcal{T}^{dN}$,

$$\sum_{i \neq j} W(\gamma^{-1}(x_1 - x_j)) \leqslant \tilde{C}_W \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j))$$

The previous inequality is a straightforward extension to higher dimensions of the analogous one derived in the proof of Lemma 4.2 of ref. 25, so we omit the details. Then it is enough to prove the lemma for $W = V$. From the basic entropy inequality, recalling definition (2.5) of the reference measure $\mu_\gamma$,

$$\begin{aligned} &\mathbb{E}^{f_\gamma(t)}\left[\gamma^d \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j))\right] \\ &\quad \leqslant -\frac{2\gamma^d}{\beta} \log \int d\underline{x} \exp\left[-\frac{\beta}{2} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j))\right] + \frac{2\gamma^d}{\beta} H(f_\gamma(t) \mid 1) \end{aligned}$$

and, by Jensen inequality,

$$\begin{aligned} &-\frac{2\gamma^d}{\beta} \log \int d\underline{x} \exp\left[-\frac{\beta}{2} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j))\right] \\ &\quad \leqslant \gamma^d \int d\underline{x} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j)) \leqslant \|V\|_\infty \end{aligned}$$

Then we are left with an estimate of the relative entropy $H(f_\gamma(t) \mid 1)$. Recalling (4.10) and that $\sigma_\gamma(\cdot)$ is a positive functional, we can bound

$$H(f_\gamma(t) \mid 1) \leqslant H(f_\gamma(0) \mid 1) - \gamma^d \int_0^t ds \int d\mu_\gamma \sum_{i \neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i f_\gamma(s) \quad (4.17)$$

Since $\mu_\gamma(d\underline{x})$ is translation invariant, recalling (4.8) and using (4.13), we have

$$-\gamma^d \int_0^t ds \int d\mu_\gamma \sum_{i \neq j} \beta \nabla J(x_i - x_j) \cdot \nabla_i f_\gamma(s) \leqslant C_1 t \gamma^{-d} \sqrt{\sigma_\gamma(\bar{f}_\gamma(t))} \quad (4.18)$$

The r.h.s. of (4.18) can be bounded using (4.9). Then, recalling (4.14), from (4.17) and (4.18) we finally get, for some $\tilde{C} > 0$,

$$\frac{2\gamma^d}{\beta} H(f_\gamma(t) \mid 1) \leqslant \tilde{C} \quad (4.19)$$

The lemma is proved. ∎

## 5. LARGE-DEVIATION ESTIMATES AND REMOVAL OF THE CUTOFFS

Part of the large deviation estimates of this section are contained in the theory developed in refs. 7, 8 and 20. We will sometimes refer to these papers for proofs and details.

Let us start with some elementary facts in the theory of point processes. A configuration of particles in $\mathbb{R}^d$ can be represented by a locally finite subset $\omega$ of $\mathbb{R}^d$. Sometimes it can be useful to look at $\omega$ as a Radon point measure on $\mathbb{R}^d$ via the map $\omega \mapsto \sum_{q \in \omega} \delta(\cdot - q)$. We denote by $\Omega$ the set of all such configurations. $\Omega$ can be made into a Polish space under the vague topology $\tau_\Omega$, defined as the smallest topology making continuous the mappings $\omega \mapsto \int \omega(dq)\, g(q) = \sum_{q \in \omega} g(q)$ for any $g : \mathbb{R}^d \to \mathbb{R}$ which is continuous and compactly supported. The natural $\sigma$-algebra $\mathscr{F}$ on $\Omega$ is the one generated by the counting variables $N_B : \omega \to \operatorname{card}(\omega \cap B)$ for $B$ any Borel subset of $\mathbb{R}^d$. It can be proven that $\mathscr{F}$ is the Borel $\sigma$-algebra relative to the topology $\tau_\Omega$.

A point process on $\mathbb{R}^d$ is a probability measure $Q$ on $(\Omega, \mathscr{F})$. We denote by $\mathscr{M}$ the set of all point processes $Q$ with finite expected number of particles $\mathbb{E}^Q[N_B]$ in any bounded Borel set $B \subset \mathbb{R}^d$. $\mathscr{M}$ can be equipped with the topology $\tau_w$ of weak convergence based on the topology $\tau_\Omega$. However it is useful to introduce a finer topology on $\mathscr{M}$, called the topology $\tau_{\mathscr{L}}$ of local convergence. Let $\mathscr{L}$ be the class of measurable functions $F$ on $\Omega$ that are local and tame, i.e. for any such $F$ there are a bounded set $B$ and a constant $c>0$ such that $F(\omega)=F(\omega \cap B)$ and $|F(\omega)| \leqslant c(1+N_B(\omega))$. Then $\tau_{\mathscr{L}}$ is defined as the weak* topology on $\mathscr{M}$ relative to $\mathscr{L}$, i.e. the smallest one making continuous the mappings $Q \mapsto Q(F) \doteq \mathbb{E}^Q[F]$ for any $F \in \mathscr{L}$. Note that $\tau_{\mathscr{L}}$ is strictly finer than $\tau_w$, as follows observing that the mappings $Q \mapsto Q(N_B)$ are $\tau_{\mathscr{L}}$-continuous for any bounded Borel set $B$.

Let $\mathscr{M}_\tau$ be the set of all the stationary point processes $Q$ in $\mathscr{M}$, i.e. those $Q$ such that $Q=Q \circ \tau_x^{-1}$ for any $x \in \mathbb{R}^d$. $\mathscr{M}_\tau$ is $\tau_{\mathscr{L}}$-closed and is assumed equipped with the induced topology. For any $Q \in \mathscr{M}$ there is a Radon measure $\nu_Q(dx)$ on $\mathbb{R}^d$ such that $Q(N_B)=\nu_Q(B)$ for any Borel set $B \subset \mathbb{R}^d$. If $Q \in \mathscr{M}_\tau$ then $\nu_Q(dx)=\rho(Q)\, dx$ for some positive number $\rho(Q)$, called the intensity of $Q$.

Let $P$ be the Poisson point process on $\mathbb{R}^d$ with intensity $\rho(P)=1$ (i.e. $P$ is such that for any collection of disjoint bounded subsets $B_1,..., B_n$, the counting variables $N_{B_1},..., N_{B_n}$ are independent and Poisson distributed with parameters $|B_1|,..., |B_n|$). We introduce the entropy density $h(Q|P)$ of $Q \in \mathscr{M}_\tau$ w.r.t. $P$ as follows. For any $B \subset \mathbb{R}^d$ we denote by $\pi_B$ the projection on $B$, and let $D_n$, $n \in \mathbb{N}$, be as in definition (3.10). Denote by $\mathscr{H}_n(Q \mid P)$ the relative entropy of $Q \circ \pi_{D_n}^{-1}$ w.r.t. $P \circ \pi_{D_n}^{-1}$. It is easy to see that $n \mapsto \mathscr{H}_n(Q \mid P)$ is a super-additive functional, so we can define

$$h(Q \mid P)=\lim_{n \to \infty} |D_n|^{-1}\, \mathscr{H}_n(Q \mid P)=\sup_n |D_n|^{-1}\, \mathscr{H}_n(Q \mid P) \tag{5.1}$$

Let now $V$ be a positive and superstable finite range potential like the one introduced in Section 2. The associated Hamiltonian in $D_n$ with free boundary conditions is

$$H_n(\omega)=\frac{1}{2} \sum_{\substack{q,\, q' \in \omega \cap D_n \\ q \neq q'}} V(q-q'), \qquad \omega \in \Omega \tag{5.2}$$

For each $n \in \mathbb{N}$ and $Q \in \mathscr{M}_\tau$ let

$$\Phi_n(Q) \doteq |D_n|^{-1}\, Q(H_n)$$

be the expected energy per volume in $D_n$. By our assumptions on the potential, $\Phi_n$ is well defined and positive. Moreover, see [ref. 8, Thm. 1], the limit $\Phi(Q)=\lim_{n \to \infty} \Phi_n(Q)$ exists and satisfies

$$\Phi(Q)=\begin{cases} Q(U) & \text{if} \quad Q \in \mathscr{M}_{\tau, 2} \\ +\infty & \text{otherwise} \end{cases} \tag{5.3}$$

where

$$\mathcal{M}_{\tau,2} \doteq \{Q \in \mathcal{M}_\tau : Q(N_B^2) < +\infty \text{ for any bounded } B \subset \mathbb{R}^d\}$$

and

$$U(\omega) = \frac{1}{2} \sum_{\substack{q, q' \in \omega \\ q \neq q'}} \chi(q)\, V(q - q') \tag{5.4}$$

In (5.4) $\chi$ is any non negative function on $\mathbb{R}^d$ with compact support and total integral 1 (it is easy to verify that $Q(U)$ does not depend on $\chi$ if $Q$ is a stationary point process).

Finally define, for any $\lambda \in \mathbb{R}$ and $\beta \geqslant 0$,

$$K_{\beta,\lambda}(Q) = \beta\Phi(Q) + h(Q \mid P) - \beta\lambda\rho(Q) \tag{5.5}$$

The following proposition establishes the basic properties of the functionals introduced above (see refs. 7, 8 and 20).

**Proposition 5.1.** The functionals $\Phi, K_{\beta,\lambda} : \mathcal{M}_\tau \to [0, +\infty]$ are lower semicontinuous relative to $\tau_{\mathcal{L}}$ (and then also relative to any coarser topology on $\mathcal{M}_\tau$, e.g. $\tau_w$). Moreover $K_{\beta,\lambda}$ has $\tau_{\mathcal{L}}$-compact level sets.

We can formulate now the LDP for Gibbsian point processes. The form we need here is strictly contained in the results of the papers quoted above:

**Theorem 5.2.** Let $F \in \mathcal{L}$ and $F_n$ be as in definition (3.10). Then, for any $\lambda \in \mathbb{R}$ and $\beta \geqslant 0$, there exists the limit

$$\lim_{n\to\infty} |D_n|^{-1} \log \mathbb{E}^P \exp[\beta\lambda N_{D_n} - \beta H_n + |D_n|\, F_n]$$

$$= \sup_{Q \in \mathcal{M}_\tau} \{Q(F) - K_{\beta,\lambda}(Q)\} \tag{5.6}$$

and the r.h.s. of (5.6) is finite.

**Remark.** In ref. 8 the principle is formulated in terms of functionals of the stationary empirical field $R_{n,\omega}$ obtained by replacing the configuration $\omega$ with the periodic continuation $\omega^{(n)}$ of its restriction to $D_n$, but the same result holds also for the spatial average $F_n$ by standard arguments that we do not describe here, see e.g. ref. 7.

An immediate consequence of the above principle is the existence of the pressure $p(\beta, \lambda)$ and a variational formula for it. In fact, taking $F \equiv 0$ in (5.6) and recalling (2.7) and (2.8), we get

$$p(\beta, \lambda) = -\min_{Q \in \mathscr{M}_\tau} \beta^{-1} K_{\beta, \lambda}(Q) \tag{5.7}$$

and, comparing with (2.9), we recover the Helmholtz free energy as

$$a(\beta, \rho) = \min_{Q \in \mathscr{M}_\tau : \rho(Q) = \rho} \{\Phi(Q) + \beta^{-1} h(Q \mid P)\} \tag{5.8}$$

where we used the fact that the r.h.s. of (5.8) is a convex function of $\rho$ (which is true since $\Phi$ and $h(\cdot \mid P)$ are affine functionals on $\mathscr{M}_\tau$).

Now we discuss the LDP for local Gibbs states of the type defined in (2.11). We forget the dependence on $t \in [0, T]$ and we deal with a generic smooth map $x \mapsto \lambda(x)$ of $\mathscr{T}^d$ into the one phase region $\mathscr{U}$ such that the corresponding density $x \mapsto \rho(x) \in \mathscr{W}$ satisfies

$$\int_{\mathscr{T}^d} dx\, \rho(x) = 1 \tag{5.9}$$

We denote by $\hat{f}_\gamma(\underline{x})$ the density w.r.t. $\mu_\gamma(d\underline{x})$ of the associated local Gibbs state (observe that the functions $x \mapsto \lambda(t, x)$ introduced in Section 2 satisfy the conditions above for any $t \in [0, T]$). The following theorem is contained in Section 5 of ref. 20.

**Theorem 5.3.** Let $A \in \mathscr{L}$ and $\varphi \in C(\mathscr{T}^d)$. Then

$$\begin{aligned}
&\lim_{\gamma \downarrow 0} \gamma^d \log \int d\underline{x} \exp\Bigg[ \sum_i \beta\lambda(x_i) - \frac{\beta}{2} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j)) \\
&\qquad + \gamma^{-d} \int_{\mathscr{T}^d} dx\, \varphi(x)\, A(\omega_{x,\gamma}) \Bigg] \\
&\quad = \sup \Bigg\{ \int_{\mathscr{T}^d} dx\, [\varphi(x)\, Q_x(A) \\
&\qquad - K_{\beta, \lambda(x)}(Q_x)];\ \{Q_x\} \subset \mathscr{M}_\tau : \int_{\mathscr{T}^d} dx\, \rho(Q_x) = 1 \Bigg\}
\end{aligned} \tag{5.10}$$

Taking $A \equiv 0$ in (5.10) and recalling (5.7), we get

$$\lim_{\gamma \downarrow 0} \gamma^d \log \int d\underline{x} \exp\left[\sum_i \beta\lambda(x_i) - \frac{\beta}{2} \sum_{i \neq j} V(\gamma^{-1}(x_i - x_j))\right]$$

$$= \sup\left\{\int_{\mathcal{T}^d} dx\,[-K_{\beta,\lambda(x)}(Q_x)];\ \{Q_x\} : \int_{\mathcal{T}^d} dx\,\rho(Q_x) = 1\right\}$$

$$= \int_{\mathcal{T}^d} dx\,\beta p(\beta, \lambda(x)) \tag{5.11}$$

where in the last equality we used the fact that the supremum of the integrand is reached at $\rho(Q_x) = \rho(x)$ satisfying (5.9).

Theorem 5.3 is not sufficient for our purposes since to prove (3.25) we need also an upper bound for the Laplace asymptotics of non local functionals. The following theorem gives the required estimate.

**Theorem 5.4.** Let $A, F, G \in \mathcal{L}$ with $F$ bounded and let $\varphi \in C(\mathcal{T}^d)$ and $\psi \in C(\mathcal{T}^d \times \mathcal{T}^d)$. Then:

$$\limsup_{\gamma \downarrow 0} \gamma^d \log \mathbb{E}^{\hat{f}_\gamma} \exp \gamma^{-d}\left[\int_{\mathcal{T}^d} dx\,\varphi(x)\,A(\omega_{x,\gamma})\right.$$

$$\left. + \int_{\mathcal{T}^d} dx \int_{\mathcal{T}^d} dy\,\psi(x,y)\,F(\omega_{x,\gamma})\,G(\omega_{y,\gamma})\right]$$

$$\leqslant \sup\left\{\int_{\mathcal{T}^d} dx \int_{\mathcal{T}^d} dy\,[\varphi(x)\,Q_x(A) + \psi(x,y)\,Q_x(F)\,Q_y(G)\right.$$

$$\left. - K_{\beta,\lambda(x)}(Q_x) - \beta p(\beta,\lambda(x))];\ \{Q_x\} \subset \mathcal{M}_\tau : \int_{\mathcal{T}^d} dx\,\rho(Q_x) = 1\right\} \tag{5.12}$$

*Proof.* For any configuration of particles $\omega$ on $\mathcal{T}^d$ define

$$T(\omega) = \sum_{z \in \omega} \beta\lambda(z) - \frac{\beta}{2} \sum_{\substack{z, z' \in \omega \\ z \neq z'}} V(\gamma^{-1}(z - z')) + \gamma^{-d} \int_{\mathcal{T}^d} dx\,\varphi(x)\,A(\omega_{x,\gamma})$$

$$+ \gamma^{-d} \int_{\mathcal{T}^d} dx \int_{\mathcal{T}^d} dy\,\psi(x,y)\,F(\omega_{x,\gamma})\,G(\omega_{y,\gamma}) \tag{5.13}$$

Setting

$$Z_N^c \doteq \int d\underline{x} \exp[T(\underline{x})]$$

and making use of (5.11) it is enough to prove that

$$\limsup_{\gamma\downarrow 0} \gamma^d \log Z_N^c \leqslant \sup \left\{ \int_{\mathscr{T}^d} dx \int_{\mathscr{T}^d} dy \, [\varphi(x)\, Q_x(A) + \psi(x, y)\, Q_x(F)\, Q_y(G) \right.$$
$$\left. - K_{\beta, \lambda(x)}(Q_x)]; \{Q_x\} \subset \mathscr{M}_\tau : \int_{\mathscr{T}^d} dx\, \rho(Q_x) = 1 \right\} \quad (5.14)$$

First of all we observe that for any partition of $\mathscr{T}^d$ into cubes of side $2\epsilon$ there are uniform approximations of $\lambda$, $\varphi$, and $\psi$ that are constant on this partition. Let $T_\epsilon(\cdot)$, $Z_{N,\epsilon}^c$ be defined as $T(\cdot)$, $Z_N^c$ with $\lambda$, $\varphi$, and $\psi$ replaced by their approximations. Let $D_{\ell_0}$ be a cube (as in definition (3.10)) such that the functions $A$, $F$ and $G$ depend only on $\omega \cap D_{\ell_0}$ and, for some $c > 0$, $A$, $G \leqslant c(1 + N_{D_{\ell_0}})$. Then one easily bounds

$$|T(\underline{x}) - T_\epsilon(\underline{x})| \leqslant O_1(\epsilon)\, \gamma^{-d} \int_{\mathscr{T}^d} dx\, (1 + N_{D_{\ell_0}}(\omega_{x,\gamma})) \leqslant O_1(\epsilon)\, |D_{\ell_0}|\, (1 + N) \quad (5.15)$$

with $O_1(\epsilon) \to 0$ as $\epsilon \downarrow 0$ (uniformly in $\gamma$), so that

$$\limsup_{\gamma\downarrow 0} \gamma^d \log Z_N^c \leqslant \limsup_{\gamma\downarrow 0} \gamma^d \log Z_{N,\epsilon}^c + O_1(\epsilon)\, |D_{\ell_0}|$$

On the other hand, also the error made in replacing the r.h.s. of (5.14) with its $\epsilon$-approximation (i.e. the one defined with the piecewise constant approximations of $\lambda$, $\varphi$, and $\psi$) is easily bounded by $O_2(\epsilon)\, |D_{\ell_0}|$ with $O_2(\epsilon) \to 0$ as $\epsilon \downarrow 0$. Thus, with no loss of generality we can prove (5.14) assuming $\lambda$, $\varphi$, and $\psi$ constant on some cubic partition of $\mathscr{T}^d$.

Let $P_\gamma$ be the Poisson point process on $\mathscr{T}^d$ with intensity $\gamma^{-d}$ and define, for any $\mu \in \mathbb{R}$,

$$Z_\mu(\gamma) \doteq \mathbb{E}^{P_\gamma} \exp[T(\omega) + \mu N(\omega)] = e^{-\gamma^{-d}} \sum_{n=0}^{\infty} \frac{\gamma^{-dn} e^{\mu n}}{n!} Z_n^c$$

Since

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{e^{-N} N^N}{N!} = 0$$

then (recall that $N\gamma^d \nearrow 1$ as $\gamma \downarrow 0$)

$$\limsup_{\gamma \downarrow 0} \gamma^d \log Z_N^c \leqslant \inf_{\mu \in \mathbb{R}} \limsup_{\gamma \downarrow 0} \{\gamma^d \log Z_\mu(\gamma) - \mu\} \tag{5.16}$$

From (5.16) we get (5.14) if

$$\begin{aligned}\limsup_{\gamma \downarrow 0} &\gamma^d \log Z_\mu(\gamma) \\ &\leqslant \sup_{\{Q_x\} \subset \mathscr{M}_\tau} \int_{\mathscr{T}^d} dx \int_{\mathscr{T}^d} dy \, \{\varphi(x)\, Q_x(A) \\ &\quad + \psi(x, y)\, Q_x(F)\, Q_y(G) - K_{\beta, \lambda(x)}(Q_x) + \mu\rho(Q_x)\}\end{aligned} \tag{5.17}$$

By redefining $\lambda(x)$ as $\lambda(x)+\mu$, it is enough to prove (5.17) for $\mu=0$. Divide $\mathscr{T}^d$ into disjoint boxes $B_\sigma$ of center $\sigma$ and side $(2\gamma\ell)$ and let $D_{\ell_0}$ be the cube as in (5.15). Set

$$B_{\sigma, \ell_0} = \{x \in B_\sigma : \text{dist}(x, \mathscr{T}^d \backslash B_\sigma) \geqslant 2\gamma\ell_0\} \tag{5.18}$$

and define

$$\begin{aligned}T_{\sigma\sigma'}(\omega) = (2\gamma\ell)^d \Bigg[ &\beta\lambda(\sigma)\, N_{B_\sigma} - \frac{\beta}{2} \sum_{\substack{z, z' \in \omega \cap B_\sigma \\ z \neq z'}} V(\gamma^{-1}(z-z')) \\ &+ \gamma^{-d} \varphi(\sigma) \int_{B_{\sigma, \ell_0}} dx\, A(\omega_{x, \gamma}) \Bigg] \\ &+ \gamma^{-d} \psi(\sigma, \sigma') \int_{B_{\sigma, \ell_0}} dx \int_{B_{\sigma', \ell_0}} dy\, F(\omega_{x, \gamma})\, G(\omega_{y, \gamma})\end{aligned} \tag{5.19}$$

Since the potential is positive, we obtain an upper bound on $T(\omega)$ by neglecting the interaction between different boxes. Then

$$T(\omega) \leqslant \sum_{\sigma, \sigma'} T_{\sigma\sigma'}(\omega) + \mathscr{R}(\omega) \tag{5.20}$$

where $\mathscr{R}$ takes into account the errors due to the integration on the smaller cubes $B_{\sigma, \ell_0}$ in (5.19). Because of the choice of $\ell_0$ we easily bound, for some $C > 0$,

$$\mathscr{R} \leqslant C\gamma^{-d} \int_{\bigcup (B_\sigma \backslash B_{\sigma, \ell_0})} dx\, (1 + N_{D_{\ell_0}}(\omega_{x, \gamma}))$$

We decompose $\bigcup (B_\sigma \backslash B_{\sigma, \ell_0})$ into a union of disjoint cubes of side $(2\gamma\ell_0)$. Calling $\hat{x}_\alpha$ the centers of these cubes, we have

$$\mathscr{R} \leqslant C \sum_{\alpha=1}^{n} \int_{D_{\ell_0}} dr (1 + N_{D_{\ell_0}}(\tau_{r+\gamma^{-1}\hat{x}_\alpha} \omega_{0,\gamma}))$$

with $n \sim (\gamma\ell)^{-d} (\ell/\ell_0)^{d-1}$ so that, by Jensen inequality,

$$\limsup_{\ell \to \infty} \limsup_{\gamma \downarrow 0} \gamma^d \log \mathbb{E}^{P_\gamma} \exp[\mathscr{R}]$$

$$\leqslant \limsup_{\ell \to \infty} \limsup_{\gamma \downarrow 0} \gamma^d n \log \mathbb{E}^{P} \exp[C(1 + N_{D_{\ell_0}})] = 0 \tag{5.21}$$

where we used the independence and stationarity of $P_\gamma$ and that $\mathbb{E}^P \exp[aN_B] < \infty$ for any $a > 0$ and any bounded set $B \subset \mathbb{R}^d$. From (5.20) and (5.21) we get

$$\limsup_{\gamma \downarrow 0} \gamma^d \log Z_0(\gamma) \leqslant \limsup_{\ell \to \infty} \limsup_{\gamma \downarrow 0} \gamma^d \log \mathbb{E}^{P_\gamma} \exp\left[\sum_{\sigma, \sigma'} T_{\sigma\sigma'}(\omega)\right] \tag{5.22}$$

Now define the following local function on $\Omega \times \Omega$:

$$S_{\sigma\sigma'}(\omega_1, \omega_2) = \beta\lambda(\sigma) R(\omega_1) - \beta U(\omega_1) + \varphi(\sigma) A(\omega_1) + \psi(\sigma, \sigma') F(\omega_1) G(\omega_2)$$

where $R$ and $U$ are defined in (3.11) and (5.4) respectively. Expanding variables in the r.h.s. of (5.22) and neglecting errors that can be proved to be of order $O(1/\ell)$ uniformly in $\gamma \in (0, 1]$ as done in (5.21), one easily obtains (introducing a new parameter $\varepsilon = 2\ell\gamma$)

$$\limsup_{\gamma \downarrow 0} \gamma^d \log Z_0(\gamma) \leqslant \limsup_{\ell \to \infty} \limsup_{\varepsilon \downarrow 0} \mathscr{X}(\ell, \varepsilon) \tag{5.23}$$

with

$$\mathscr{X}(\ell, \varepsilon) \doteq \varepsilon^d |D_\ell|^{-1} \log \mathbb{E}^{\otimes_\sigma P_\sigma} \exp\left[\varepsilon^d \sum_{\sigma, \sigma'} |D_\ell| (S_{\sigma\sigma'})_\ell (\omega_\sigma, \omega_{\sigma'})\right] \tag{5.24}$$

where $\bigotimes_\sigma P_\sigma$ is the product measure on $\Omega^{(\varepsilon^{-d})} = \Omega \times \cdots \times \Omega$ of $\varepsilon^{-d}$ independent Poisson processes $P_\sigma$ on $\mathbb{R}^d$ and, analogously to (3.10), for any local function $M$ on $\Omega \times \Omega$, we defined

$$M_\ell(\omega_1, \omega_2) \doteq \frac{1}{|D_\ell|^2} \int_{D_\ell} dr \int_{D_\ell} dr' \, M(\tau_r \omega_1, \tau_{r'} \omega_2)$$

To analyze the limit in the r.h.s. of (5.23) we need first to introduce some cutoffs. Given $k_+, k_- \in \mathbb{N}$, let $\mathscr{X}_{k_+k_-}(\ell, \varepsilon)$ be defined as $\mathscr{X}(\ell, \varepsilon)$ in (5.24) with $S_{\sigma\sigma'}$ replaced by

$$S_{\sigma\sigma'}^{k_+k_-} \doteq \begin{cases} k_+ & \text{if} \quad S_{\sigma\sigma'} > k_+ \\ S_{\sigma\sigma'} & \text{if} \quad -k_- \leqslant S_{\sigma\sigma'} \leqslant k_+ \\ -k_- & \text{if} \quad S_{\sigma\sigma'} < -k_- \end{cases}$$

Then, by definition of relative entropy,

$$\mathscr{X}_{k_+k_-}(\ell, \varepsilon) \leqslant \sup_Q \left\{ \mathbb{E}^Q \left[ \varepsilon^{2d} \sum_{\sigma, \sigma'} (S_{\sigma\sigma'}^{k_+k_-})_\ell (\omega_\sigma, \omega_{\sigma'}) \right] - \varepsilon^d \, |D_\ell|^{-1} \, H\left( Q \,\middle|\, \bigotimes_\sigma (P_\sigma \circ \pi_{D_{\ell+\bar{\ell}}}^{-1}) \right) \right\} \tag{5.25}$$

where the supremum is taken over point processes $Q$ on $D_{\ell+\bar{\ell}} \times \cdots \times D_{\ell+\bar{\ell}}$. The parameter $\bar{\ell}$ is chosen so large that $S_{\sigma\sigma'}$ is $D_{\bar{\ell}}$-measurable. The variational formula for the relative entropy gives also the explicit form of the measure $\bar{Q}$ where the supremum in the r.h.s. of (5.25) is achieved, see e.g. [ref. 5, Prop. 1.4.2]: $\bar{Q} \ll \bigotimes_\sigma (P_\sigma \circ \pi_{D_{\ell+\bar{\ell}}}^{-1})$ and

$$\frac{d\bar{Q}}{d(\bigotimes_\sigma (P_\sigma \circ \pi_{D_{\ell+\bar{\ell}}}^{-1}))} = \mathscr{N}_\varepsilon \exp \left[ \varepsilon^d \sum_{\sigma, \sigma'} |D_\ell| \, (S_{\sigma\sigma'}^{k_+k_-})_\ell (\omega_\sigma, \omega_{\sigma'}) \right] \tag{5.26}$$

($\mathscr{N}_\varepsilon$ the normalization constant). For any $x \in \mathscr{T}^d$ let $\sigma(x)$ be such that $x \in B_{\sigma(x)}$, and denote by $Q_\sigma$, $Q_{\sigma\sigma'}$ the $\sigma$th, $\{\sigma, \sigma'\}$th marginals of $Q$. From (5.26) it is easy to prove that, for any $x, y \in \mathscr{T}^d$, $x \neq y$,

$$\lim_{\varepsilon \downarrow 0} D_{\text{var}} \left( \bar{Q}_{\sigma(x)\,\sigma(y)}(d\omega_1, d\omega_2), \int \bar{Q}(d\bar{\omega}) \, \bar{Q}_{\sigma(x)}(d\omega_1 \mid \bar{\omega}_{\{\sigma(x)\}^c}) \, \bar{Q}_{\sigma(y)}(d\omega_2 \mid \bar{\omega}_{\{\sigma(y)\}^c}) \right) = 0$$

where $D_{\text{var}}(\cdot, \cdot)$ denotes the variation distance between measures. Moreover

$$H\left( Q \,\middle|\, \bigotimes_\sigma (P_\sigma \circ \pi_{D_{\ell+\bar{\ell}}}^{-1}) \right) \geqslant \sum_\sigma H(Q_\sigma \mid P_\sigma \circ \pi_{D_{\ell+\bar{\ell}}}^{-1})$$

Since $\lambda$, $\varphi$, $\psi$ are piecewise constant, as $\varepsilon$ is small enough, for any $x, y \in \mathcal{T}^d$, the function $S_{\sigma(x)\,\sigma(y)}$ is independent on the partition $\{B_\sigma\}$ and equal to

$$S_{xy}(\omega_1, \omega_2) \doteq \beta\lambda(x)\, R(\omega_1) - \beta U(\omega_1) + \varphi(x)\, A(\omega_1) + \psi(x, y)\, F(\omega_1)\, G(\omega_2)$$

Thus the limit as $\varepsilon \downarrow 0$ of the r.h.s. of (5.25) is easily bounded and we get

$$\limsup_{\varepsilon \downarrow 0} \mathcal{X}_{k_+ k_-}(\ell, \varepsilon)$$

$$\leqslant \sup_{\{Q'_x\}} \int_{\mathcal{T}^d} dx \int_{\mathcal{T}^d} dy\, \{ Q'_x \otimes Q'_y(S^{k_+ k_-}_{xy}) - |D_\ell|^{-1} H(Q'_x \mid P \circ \pi^{-1}_{D_{\ell+\bar{\ell}}})\} \tag{5.27}$$

where the supremum is taken over all the collections $\{Q'_x; x \in \mathcal{T}^d\}$ of point processes on $D_{\ell+\bar{\ell}}$. Now the proof follows in a standard way, see e.g. refs. 7 and 20. We extend $Q'_x$ to a point process $Q''_x$ on $\mathbb{R}^d$ by taking independent copies on all disjoint cubes translated of $D_{\ell+\bar{\ell}}$. Then we obtain from it a stationary process $Q^{(\ell)}_x$ by setting

$$Q^{(\ell)}_x = |D_{\ell+\bar{\ell}}|^{-1} \int_{D_{\ell+\bar{\ell}}} dr\, \tau_r Q''_x$$

From convexity of the relative entropy and the independence properties of $P$ one easily proves that $h(Q^{(\ell)}_x \mid P) \leqslant |D_{\ell+\bar{\ell}}|^{-1} H(Q'_x \mid P \circ \pi^{-1}_{D_{\ell+\bar{\ell}}})$. Moreover, for any bounded local function $M$ on $\Omega \times \Omega$, $|Q^{(\ell)}_x \otimes Q^{(\ell)}_y(M) - Q'_x \otimes Q'_y(M_\ell)| \mapsto 0$ as $\ell \to \infty$. Then from (5.27) we finally obtain

$$\limsup_{\ell \to \infty} \limsup_{\varepsilon \downarrow 0} \mathcal{X}_{k_+ k_-}(\ell, \varepsilon) \leqslant \mathcal{W}(\{S^{k_+ k_-}_{xy}\}) \tag{5.28}$$

where, for any collection $\{M_{xy}\}$ of local functions on $\Omega \times \Omega$, we defined

$$\mathcal{W}(\{M_{xy}\}) \doteq \sup_{\{Q_x\} \subset \mathcal{M}_\tau} \int_{\mathcal{T}^d} dx \int_{\mathcal{T}^d} dy\, \{Q_x \otimes Q_y(M_{xy}) - h(Q_x \mid P)\}$$

To remove the cutoffs we can argue as in the proof of Theorem 5.2 in ref. 20, then we just sketch the argument. Let $S^{k_+}_{\sigma\sigma'} = \min\{S_{\sigma\sigma'}; k_+\}$ and define $S^{k_+}_{xy}$ and $\mathcal{X}_{k_+}(\ell, \varepsilon)$ accordingly. Using the boundness of $S^{k_+ k_-}_{\sigma\sigma'}$ and that $h(\cdot \mid P)$ has compact level sets, one can prove that $\lim_{k_- \to \infty} \mathcal{W}(\{S^{k_+ k_-}_{xy}\}) \leqslant \mathcal{W}(\{S^{k_+}_{xy}\})$. Then, since $\mathcal{X}_{k_+}(\ell, \varepsilon) \leqslant \mathcal{X}_{k_+ k_-}(\ell, \varepsilon)$, from (5.28) we get

$$\limsup_{\ell \to \infty} \limsup_{\varepsilon \downarrow 0} \mathcal{X}_{k_+}(\ell, \varepsilon) \leqslant \mathcal{W}(\{S^{k_+}_{xy}\}) \tag{5.29}$$

Now we are left with the upper cutoff. Recalling definition (5.24), by Holder inequality, for any $p$, $q>1$ such that $p^{-1}+q^{-1}=1$, and any $k_+\in\mathbb{N}$,

$$\begin{aligned}\mathscr{X}(\ell,\varepsilon)\leqslant&\frac{\varepsilon^d\,|D_\ell|^{-1}}{p}\log\mathbb{E}^{\otimes_\sigma P_\sigma}\exp\left[\varepsilon^d\sum_{\sigma,\sigma'}|D_\ell|\,(pS^{k_+}_{\sigma\sigma'})_\ell\,(\omega_\sigma,\omega_{\sigma'})\right]\\&+\frac{\varepsilon^d\,|D_\ell|^{-1}}{q}\log\mathbb{E}^{\otimes_\sigma P_\sigma}\exp\left[q\varepsilon^d\sum_{\sigma,\sigma'}|D_\ell|\,((S_{\sigma\sigma'})_\ell\right.\\&\left.-(S^{k_+}_{\sigma\sigma'})_\ell)(\omega_\sigma,\omega_{\sigma'})\right]\end{aligned}\qquad(5.30)$$

To bound the first term in the r.h.s. of (5.30) we can use (5.29) with $S^{k_+}_{\sigma\sigma'}$ replaced by $pS^{k_+}_{\sigma\sigma'}$. To bound the second term we recall first that there are $C>0$ and a bounded subset $B$ of $\mathbb{R}^d$ such that $S_{\sigma\sigma'}(\omega_1,\omega_2)\leqslant C(1+N_B(\omega_1)+N_B(\omega_2))$ for any $\sigma$, $\sigma'$, so that

$$\begin{aligned}(S_{\sigma\sigma'}&-S^{k_+}_{\sigma\sigma'})(\omega_1,\omega_2)\\&\leqslant[\,C(1+N_B(\omega_1)+N_B(\omega_2))-k_+]^+\\&\leqslant\tfrac{1}{2}[\,C+2CN_B(\omega_1)-k_+]^+ +\tfrac{1}{2}[\,C+2CN_B(\omega_2)-k_+]^+\end{aligned}$$

Then, setting

$$Y_{\ell,k_+}(\omega)\doteq\exp\left[\frac{q}{2}\int_{D_\ell}dr\,[\,C+2CN_B(\tau_r\omega)-k_+]^+\right]$$

we have

$$\begin{aligned}\mathbb{E}^{\otimes_\sigma P_\sigma}&\exp\left[q\varepsilon^d\sum_{\sigma,\sigma'}|D_\ell|\,((S_{\sigma\sigma'})_\ell-(S^{k_+}_{\sigma\sigma'})_\ell)(\omega_\sigma,\omega_{\sigma'})\right]\\&\leqslant\mathbb{E}^{\otimes_\sigma P_\sigma}\left[\prod_{\sigma,\sigma'}Y_{\ell,k_+}(\omega_\sigma)^{\varepsilon^d}\,Y_{\ell,k_+}(\omega_{\sigma'})^{\varepsilon^d}\right]\\&=\mathbb{E}^P[\,Y_{\ell,k_+}(\omega)^{\varepsilon^d}]^{\varepsilon^{-2d}}\leqslant\mathbb{E}^P[\,Y_{\ell,k_+}(\omega)]^{\varepsilon^{-d}}\end{aligned}$$

so that, from (5.30),

$$\begin{aligned}&\limsup_{\ell\to\infty}\limsup_{\varepsilon\downarrow0}\mathscr{X}(\ell,\varepsilon)\\&\qquad\leqslant\lim_{k_+\to\infty}\mathscr{W}(\{pS^{k_+}_{xy}\})+\lim_{k_+\to\infty}\lim_{\ell\to\infty}\frac{1}{q\,|D_\ell|}\log\mathbb{E}^P[\,Y_{\ell,k_+}(\omega)]\end{aligned}\qquad(5.31)$$

The second term in the r.h.s. of (5.31) is zero, see [20, Thm. 5.2]. Then, since the first term in (5.31) is bounded by $\mathscr{W}(\{pS_{xy}\})$, in the limit $p \downarrow 1$ we finally get

$$\limsup_{\ell \to \infty} \limsup_{\varepsilon \downarrow 0} \mathscr{X}(\ell, \varepsilon) \leqslant \mathscr{W}(\{S_{xy}\}) \tag{5.32}$$

But (recall the above definition of $S_{xy}$) $\mathscr{W}(\{S_{xy}\})$ is equal to the r.h.s. of (5.17) with $\mu = 0$, which then follows from (5.23) and (5.32). The theorem is proved. ∎

Now we can give the missing proofs of Section 3.

*Proof of (3.21).* From the basic entropy inequality, for any $\delta_0 > 0$,

$$\mathbb{E}^{f_\gamma(t)}\left[\int_{\mathscr{T}^d} dx\, \beta \Omega_p(t, x)\right] - \delta_0^{-1} H_\gamma(t)$$

$$\leqslant \delta_0^{-1} \gamma^d \log \mathbb{E}^{\hat{f}_\gamma(t)} \exp\left[\delta_0 \gamma^{-d} \int_{\mathscr{T}^d} dx\, \beta \Omega_p(t, x)\right] \tag{5.33}$$

As before we forget the dependence on $t \in [0, T]$ and we deal with a generic smooth map $x \mapsto \lambda(x) \in K_u$ such that the corresponding density $\rho(x) = \partial_\lambda p(\beta, \lambda(x)) \in K_w$ satisfies (5.9) (the compact sets $K_u$ and $K_w$ have been introduced just after Theorem 2.1). Let $\hat{f}_\gamma$ be the density of the corresponding local Gibbs state. From (5.33) it is enough to prove that, for $\delta_0$ small enough and uniformly in the choice of $x \mapsto \lambda(x)$ above,

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \limsup_{\gamma \downarrow 0} \delta_0^{-1} \gamma^d \log \mathbb{E}^{\hat{f}_\gamma}$$

$$\times \exp\left[\delta_0 \gamma^{-d} \int_{\mathscr{T}^d} dx\, \beta \Omega_p(x)\right] = 0, \qquad p = 3, 4 \tag{5.34}$$

where $\Omega_p(x)$ is defined as in (3.20) but relative to the map $x \mapsto \lambda(x)$. We analyze the cases $p = 3, 4$ separately:

($p = 3$). From the smoothness of $x \mapsto \lambda(x)$ and since $|\varphi * R_n(\omega_{\cdot,\gamma})| \leqslant \|\varphi\|_\infty$, for any $\varphi \in C(\mathscr{T}^d)$ and any $\omega_{\cdot,\gamma}$, there is $C_1 > 0$ such that

$$\Omega_3(x) \leqslant C_1(1 + R_n(\omega_{x,\gamma}) + (\phi_k \circ |G|)_n (\omega_{x,\gamma}))(1 - u_n(\omega_{x,\gamma})) \tag{5.35}$$

where

$$|G|\,(\omega) \doteq \frac{\beta}{2} \sum_{\substack{q,\, q' \in \omega \\ q \neq q'}} \chi(q)\, |\nabla V|\, (q-q')\, |q-q'| \tag{5.36}$$

From Theorem 5.3, (5.11) and (5.35) we get the following bound:

$$\begin{aligned}
&\limsup_{k\to\infty}\ \limsup_{n\to\infty}\ \limsup_{\gamma\downarrow 0} \delta_0^{-1}\gamma^d \log \mathbb{E}^{\hat{f}_\gamma} \exp\left[\delta_0 \gamma^{-d} \int_{\mathscr{T}^d} dx\, \beta\Omega_3(x)\right] \\
&\quad \leqslant \limsup_{k\to\infty}\ \limsup_{n\to\infty}\ \sup \left\{ \int_{\mathscr{T}^d} dx\, [\delta_0 \beta C_1 Q_x((1+R_n+(\phi_k \circ |G|)_n)(1-u_n)) \right. \\
&\qquad \left. - K_{\beta,\,\lambda(x)}(Q_x) - \beta p(\beta, \lambda(x))];\ \{Q_x\} \subset \mathscr{M}_\tau : \int_{\mathscr{T}^d} dx\, \rho(Q_x) = 1 \right\}
\end{aligned} \tag{5.37}$$

For any $Q_x \in \mathscr{M}_\tau$ let $\int \nu_x(de)\, Q_e$ be its ergodic decomposition. Recalling that $K_{\beta,\lambda}(\cdot)$ is an affine functional, the r.h.s. of (5.37) becomes

$$\begin{aligned}
&\limsup_{k\to\infty}\ \limsup_{n\to\infty}\ \sup \left\{ \int_{\mathscr{T}^d} dx \int \nu_x(de) \right. \\
&\qquad \times [\delta_0 \beta C_1 Q_e((1+R_n+(\phi_k \circ |G|)_n)(1-u_n)) \\
&\qquad \left. - K_{\beta,\,\lambda(x)}(Q_e) - \beta p(\beta, \lambda(x))];\ \{\nu_x\} : \int_{\mathscr{T}^d} dx \int \nu_x(de)\, \rho(Q_e) = 1 \right\}
\end{aligned} \tag{5.38}$$

Since $|(1+R_n+(\phi_k \circ |G|)_n)(1-u_n)| \leqslant 1+k+R_n$ and, for any $\{\nu_x\}$ in the supremum above, $\int dx \int \nu_x(de)\, Q_e(1+k+R_n) \leqslant 2+k$, we can pass to the limit $n\to\infty$ inside the supremum and apply the Dominated Convergence Theorem. Then we can drop the constraint so that (5.38) is bounded by

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathscr{T}^d} dx\, \sup_{\nu} \left\{ \int \nu(de) [\delta_0 \beta C_1 (1+\rho(Q_e) + Q_e(\phi_k \circ |G|))(1-\mathbb{1}_K(\rho(Q_e))) \right. \\
&\qquad \left. - K_{\beta,\,\lambda(x)}(Q_e)] - \beta p(\beta, \lambda(x)) \right\}
\end{aligned} \tag{5.39}$$

By arguing as in the proof of Lemma 4.2 of ref. 25 (see also the proof of Lemma 3.1 in Section 4), since $V$ is positive and superstable, there is $C_2 > 0$ such that

$$|G|\,(\omega) \leqslant C_2 \beta U(\omega) \tag{5.40}$$

where $U(\omega)$ is defined as in (5.4) with an appropriate choice (depending on $|G|$) of the function $\chi$. Then, from (5.3) it follows that, for any $Q \in \mathscr{M}_\tau$,

$$Q(\phi_k \circ |G|) \leqslant C_2 \beta \Phi(Q)$$

so that, setting $\delta = \delta_0 \max\{\beta C_1 C_2; C_1\}$, (5.39) can be bounded by

$$\begin{aligned}
&\int_{\mathscr{T}^d} dx \sup_{\nu} \int \nu(de) \{ (\delta\beta + \delta\beta\rho(Q_e) + \delta\beta\Phi(Q_e) - K_{\beta, \lambda(x)}(Q_e) - \beta p(\beta, \lambda(x))) \\
&\quad \times (1 - \mathbb{1}_K(\rho(Q_e))) - (\beta p(\beta, \lambda(x)) + K_{\beta, \lambda(x)}(Q_e)) \, \mathbb{1}_K(\rho(Q_e)) \} \\
&\leqslant \int_{\mathscr{T}^d} dx \sup_{m} \{ (\delta\beta(1 + (\lambda(x) + \delta)\, m) + \beta(1-\delta)\, p(\beta(1-\delta), \lambda(x) + \delta) \\
&\quad - \beta p(\beta, \lambda(x)) - I_{\beta(1-\delta)}(\lambda(x) + \delta, m))(1 - \mathbb{1}_K(m)) - I_\beta(\lambda(x), m)\, \mathbb{1}_K(m) \}
\end{aligned}$$

In the last inequality we used (5.5), (5.8) and introduced the functional $I_\beta(\lambda, m)$ defined in (3.27). Observe that, for any $\lambda \in K_u$, $m \mapsto I_\beta(\lambda, m)$ is non negative, convex, and strictly positive for $m \notin K$. Recall also that the pressure $p(\beta, \lambda)$ is a continuous function of its variables. Then, by continuity, the superior is 0 for $\delta$ (i.e. $\delta_0$) small enough.

($p = 4$). Since $x \mapsto D_{\xi\eta} \lambda(x)$ is bounded and $|D_{\xi\eta} J * R_n(\omega_{\cdot, \gamma})| \leqslant \|DJ\|_\infty$, recalling definitions (3.12) and (5.36), there is $C_4 > 0$ such that

$$\int_{\mathscr{T}^d} dx \, \Omega_4(x) \leqslant C_4 \int_{\mathscr{T}^d} dx \, [|G|_n(\omega_{x,\gamma}) - k]^+ = C_4 \int_{\mathscr{T}^d} dx \, [|G| \, (\omega_{x,\gamma}) - k]^+$$

Then (5.34) for $p = 4$ is proved if, for $\delta_0$ small enough,

$$\limsup_{k \to \infty} \limsup_{\gamma \downarrow 0} \delta_0^{-1} \gamma^d \log \mathbb{E}^{\hat{f}_\gamma} \exp\left[ \delta_0 \beta C_4 \gamma^{-d} \int_{\mathscr{T}^d} dx \, [|G| \, (\omega_{x,\gamma}) - k]^+ \right] = 0 \tag{5.41}$$

Arguing as in the proof of Theorem 5.4, (5.41) follows if, for any $\delta$ small enough,

$$\limsup_{k \to \infty} \limsup_{\gamma \downarrow 0} \gamma^d \log \mathbb{E}^{P_\gamma} \exp[T^{(\delta, k)}(\omega)] - \int_{\mathscr{T}^d} dx \, \beta p(\beta, \lambda(x)) = 0 \tag{5.42}$$

where $P_\gamma$ is the Poisson process on $\mathcal{T}^d$ with intensity $\gamma^{-d}$ and

$$T^{(\delta,k)}(\omega)=\sum_{z\in\omega}\beta\lambda(z)-\frac{\beta}{2}\sum_{\substack{z,z'\in\omega\\ z\neq z'}}V(\gamma^{-1}(z-z'))$$

$$+\delta\gamma^{-d}\int_{\mathcal{T}^d}dx\,[\,|G|\,(\omega_{x,\gamma})-k]^+ \tag{5.43}$$

Let $\ell_0$ be such that $U(\omega)=U(\omega\cap D_{\ell_0})$ with $U$ chosen as in (5.40). As in the proof of Theorem 5.4 divide $\mathcal{T}^d$ into disjoint boxes $B_\sigma$ of side $(2\gamma\ell)$ and assume $\lambda(x)$ constant on this partition for small $\gamma$'s. Let $B_{\sigma,\ell_0}$ be as in (5.18). From (5.40) we can estimate:

$$\gamma^{-d}\int_{\mathcal{T}^d\setminus\bigcup B_{\sigma,\ell_0}}dx\,[\,|G|\,(\omega_{x,\gamma})-k]^+\leqslant\frac{\beta C_2}{2}\sum_{\substack{z,z'\in\omega\setminus\bigcup B_{\sigma,2\ell_0}\\ z\neq z'}}V(\gamma^{-1}(z-z')) \tag{5.44}$$

Restricting to $\delta<C_2^{-1}$, neglecting the interaction between different boxes and using the independence properties of $P_\gamma$, from (5.44) we get

$$\mathbb{E}^{P_\gamma}\exp[T^{(\delta,k)}(\omega)]\leqslant\prod_\sigma\mathbb{E}^{P_\gamma}\exp[T_\sigma^{(\delta,k)}] \tag{5.45}$$

where

$$T_\sigma^{(\delta,k)}=\beta\lambda(\sigma)\,N_{B_\sigma}-\frac{\beta}{2}\sum_{\substack{z,z'\in\omega\cap B_\sigma\\ z\neq z'}}V(\gamma^{-1}(z-z'))$$

$$+\frac{\beta\delta C_2}{2}\sum_{\substack{z,z'\in\omega\cap(B_\sigma\setminus B_{\sigma,2\ell_0})\\ z\neq z'}}V(\gamma^{-1}(z-z'))$$

$$+\delta\gamma^{-d}\int_{B_{\sigma,\ell_0}}dx\,[\,|G|\,(\omega_{x,\gamma})-k]^+$$

Now, expanding variables, we can rewrite:

$$\mathbb{E}^{P_\gamma}\exp[T_\sigma^{(\delta,k)}(\omega)]=Z_{D_\ell}^{(\delta)}(\beta,\lambda(\sigma))\,\mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}}\exp\left[\delta\int_{D_{\ell-\ell_0}}dr\,[\,|G|\,(\tau_r\omega)-k]^+\right] \tag{5.46}$$

where $\mu_{D_\ell}^{(\delta,\sigma)}$ is the grand canonical measure on $D_\ell$ with chemical potential $\lambda(\sigma)$ and interaction energy

$$H_\ell(\omega) - \frac{\delta C_2}{2} \sum_{\substack{q,\, q' \in \omega \cap D_{\ell,\ell_0} \\ q \neq q'}} V(q-q')$$

where $D_{\ell,\ell_0} \doteq D_\ell \setminus D_{\ell-2\ell_0}$, the Hamiltonian $H_\ell(\omega)$ was defined in (5.2), and $Z_{D_\ell}^{(\delta)}(\beta, \lambda(\sigma))$ is the corresponding partition function. From (5.45) and (5.46) we get

$$\begin{aligned}
&\gamma^d \log \mathbb{E}^{P_\gamma} \exp[T^{(\delta,k)}(\omega)] \\
&\quad \leqslant \gamma^d |D_\ell| \sum_\sigma |D_\ell|^{-1} \log Z_{D_\ell}^{(\delta)}(\beta, \lambda(\sigma)) \\
&\qquad + \gamma^d |D_\ell| \sum_\sigma |D_\ell|^{-1} \log \mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}} \exp\left[\delta \int_{D_{\ell-\ell_0}} dr\, [|G|(\tau_r\omega) - k]^+\right]
\end{aligned} \tag{5.47}$$

Since $V$ is superstable and positive,[23] there is a positive constant $b$ such that, for any $\omega \in \Omega$ and any bounded region $B \subset \mathbb{R}^d$,

$$\frac{1}{2} \sum_{\substack{q,\, q' \in \omega \cap B \\ q \neq q'}} V(q-q') \geqslant \frac{b N_B(\omega)^2}{|B|}$$

Then we can estimate

$$\begin{aligned}
&Z_{D_\ell}^{(\delta)}(\beta, \lambda(\sigma)) \\
&\quad \leqslant \mathbb{E}^P \exp\left[\beta\lambda(\sigma) N_{D_{\ell-2\ell_0}} - \beta H_{\ell-2\ell_0} - \frac{\beta(1-\delta C_2)\, b N_{D_{\ell,\ell_0}}^2}{|D_{\ell,\ell_0}|} + \lambda(\sigma) N_{D_{\ell,\ell_0}}\right] \\
&\quad \leqslant \exp\left[\frac{\lambda(\sigma)^2}{4\beta b(1-\delta C_2)} |D_{\ell,\ell_0}|\right] Z_{D_{\ell-2\ell_0}}(\beta, \lambda(\sigma))
\end{aligned}$$

so that

$$\limsup_{\ell \to \infty} \limsup_{\gamma \downarrow 0} \gamma^d |D_\ell| \sum_\sigma |D_\ell|^{-1} \log Z_{D_\ell}^{(\delta)}(\beta, \lambda(\sigma)) \leqslant \int_{\mathcal{T}^d} dx\, \beta p(\beta, \lambda(x)) \tag{5.48}$$

On the other hand, from (5.40),

$$\mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}} \exp\left[\delta \int_{D_{\ell-\ell_0}} dr [|G|(\tau_r\omega) - k]^+\right] \leqslant \mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}} \exp[\delta C_2 \beta H_\ell(\omega)] \tag{5.49}$$

and, again from superstability, the r.h.s. of (5.49) is finite for any $\ell$ if $\delta$ is small enough. Then, by the Dominated Convergence Theorem,

$$\limsup_{k\to\infty} \mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}} \exp\left[\delta \int_{D_{\ell-\ell_0}} dr\,[|G|\,(\tau_r\omega)-k]^+\right] = 1$$

so that

$$\limsup_{\ell\to\infty}\ \limsup_{k\to\infty}\ \limsup_{\gamma\downarrow 0} \gamma^d\,|D_\ell| \sum_\sigma |D_\ell|^{-1} \log \mathbb{E}^{\mu_{D_\ell}^{(\delta,\sigma)}}$$
$$\times \exp\left[\delta \int_{D_{\ell-\ell_0}} dr\,[|G|\,(\tau_r\omega)-k]^+\right] = 0 \tag{5.50}$$

From (5.47), (5.48) and (5.50) we get (5.42) (in fact the l.h.s. of (5.42) cannot be negative). ▮

*Proof of (3.25).* Recalling definition (3.20), by Theorem 5.4 the l.h.s. of (3.25) can be bounded by

$$\limsup_{n\to\infty} \sup \Bigg\{ \int_{\mathscr{T}^d} dx \int_{\mathscr{T}^d} dy\, [\delta\beta[(\dot\lambda - \beta\,|\nabla\lambda|^2)(s,x)\, Q_x(R_n u_n) - \beta\Delta\lambda(s,x)$$
$$\times Q_x(P(R_n)\,u_n) - \beta\nabla\lambda(s,x)\cdot\nabla J(x-y)\, Q_x(R_n u_n)\, Q_y(R_n)$$
$$-\beta\Delta J(x-y)\, Q_x(P(R_n)\,u_n)\, Q_y(R_n) - \Omega(s,x,\rho(s))\, Q_x(u_n)]$$
$$-K_{\beta,\lambda(s,x)}(Q_x) - \beta p(\beta,\lambda(s,x))];\ \{Q_x\}\subset\mathscr{M}_\tau : \int_{\mathscr{T}^d} dx\, \rho(Q_x) = 1 \Bigg\} \tag{5.51}$$

For any $Q_x\in\mathscr{M}_\tau$ let $\int \nu_x(de)\, Q_e$ be its ergodic decomposition. Just as argued after (5.38), we can pass to the limit through the supremum and apply the Dominated Convergence Theorem. Then, calling $\bar\nu_x(dm)$ the distribution of $\rho(Q_e)$ under $\nu_x(de)$, (5.51) can be bounded by

$$\sup \Bigg\{ \int \prod_{z\in\mathscr{T}^d} \bar\nu_z(dm(z)) \int_{\mathscr{T}^d} dx\, [\delta\beta[\Omega(s,x,m) - \Omega(s,x,\rho(s))]\, \mathbb{1}_K(m(x))$$
$$-I_\beta(\lambda(s,x), m(x))];\ \{\bar\nu_x\} : \int_{\mathscr{T}^d} dx \int \bar\nu_x(dm)\, m < \infty \Bigg\} \tag{5.52}$$

from which (3.25) follows immediately. ▮

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