Nonlinear Response of a Dissipative Bloch Particle in an Oscillating Field

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We study the Caldeira-Leggett model of a particle coupled to a heat bath moving in a periodic cosine potential. In the limit of small viscosity we obtain an integral equation for the nonlinear response of the system to a constant or a slowly oscillating field. The equation is derived by a resummation of the infinite series expansion in the strength of the cosine potential. When solved via a self-consistent approximation it gives an *analytic* expression for the response function. Applications to Josephson junctions driven by a low-frequency source are discussed.

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We study the nonlinear response of a dissipative quantum particle moving in a periodic cosine potential in which the dissipation is modeled by coupling the particle to a boson bath with a general dissipation spectrum $J(\omega)$, the so-called Caldeira-Leggett model [1]. This model has been extensively studied in the past decade as it is directly related to the macroscopic quantum behavior of an ultrasmall Josephson junction and also to other problems such as the diffusion of a heavy particle in a solid or electrons in a superlattice [2–6]. For a recent review and summary, see [7]. Our results should be particularly relevant to experiments in ultrasmall Josephson tunnel junctions or junction arrays driven by a low-frequency current source (e.g., a microwave).

In general as the effective mass of a macroscopic degree of freedom becomes small, the quantum nature of the universe manifests itself. The resulting behavior is complicated by the presence of dissipation, i.e., the inevitable coupling of the macroscopic degree of freedom to its environment. A suitable description of this coupling is, both conceptually and technically, more problematic in the quantum domain than in the classical one [7,8]. One such problem in our model arises from the fact that quantum mechanically the dissipation (and the external force when present) must be described via some Hamiltonian which destroys in general the periodicity of the bare system. The question then arises of whether the band properties due to the Bloch theorem will also be lost. This has been extensively discussed in the literature and there seems to be agreement that with proper modeling of the environment some of the Bloch properties of the system will still be present. In fact, most of the approaches to this problem are based on this assumption. Some of these make use of driven quasiclassical Langevin equations and explicitly utilize the band structures of the Bloch particle. Others introduce statistical methods such as the master equation for incoherent Zener tunnelings at the Brillouin zone boundaries. Still another approach is based on the tight-binding picture and incoherent tunnelings between adjoining sites. Despite the difficulties in these studies with the Bloch theorem, multiband effects, and coherences between tunnelings, they all yield observable predictions of a Coulomb blockade which leads to negative differential resistances in certain regimes [4,5,7,9–11].

In the present work the band picture is not used explicitly. We therefore avoid the above issues concerning the Bloch theorem. Nevertheless, the Bloch properties of the particle appear quite naturally at the end. Our approach starts from the many-body density matrix of the whole system [5,6,12,13], with the initial state of the particle localized at the origin. After tracing out of the environmental degrees of freedom, a real-time Wigner distribution [6,12] for the particle alone is obtained in terms of a series expansion in V_0 (the strength of the cosine potential). This approach also applies for the tight-binding model, where Δ (the tunneling matrix element between two nearest minima) is the appropriate expansion parameter (cf. [5] and below).

The main result of this Letter is to show explicitly how the resummation of the series can be carried out *analytically* in the small-viscosity limit which includes regimes that are not semiclassical. This extends the results of [13], where only stationary fields were considered: The present results also describe the time-dependent response of the system to an oscillating driving force such as a microwave, and this we believe can have useful applications for experiments. It is achieved as in [13] by first transforming the problem into an integral equation and then solving the equation via a self-consistent approximation. Previous results of negative differential resistance are recovered in a more general way.

We consider a particle with a Hamiltonian

$$\hat{H}_p = \frac{\hat{p}^2}{2m} + V_0\cos(k_0\hat{x}) - F(t)\hat{x} \tag{1}$$

coupled to a boson bath with a dissipation spectrum $J(\omega)$. Starting at $t=-\infty$ with the bath in equilibrium and the system localized at the origin, one finds for the response of the particle [12,13]

$$\langle \hat{x}(t)\rangle = F_0(t) + \frac{k_0}{2}\sum_{n=1}^{\infty}(-1)^n V_0^{2n}\int_{-\infty}^{t} dt_{2n}\, g(t-t_{2n})\int_{-\infty}^{t_{2n}} dt_{2n-1}\cdots$$
$$\times \int_{-\infty}^{t_2} dt_1 \sum_{\{\mu_j=\pm 1\}}(-i)\exp\left[i\sum_{k=1}^{2n}\mu_k F_0(t_k)k_0\right]\times F_1\times F_2\Bigg|_{\sum_{j=1}^{2n}\mu_j=0}. \quad (2)$$

The functions F_0, F_1, and F_2 are

$$F_0(t)=\int_{-\infty}^{t} d\tau\, g(t-\tau)F(\tau)\,, \quad (3)$$

$$F_1=\prod_{k=1}^{2n-1}\frac{1}{\hbar}\sin\left[\frac{k_0^2\hbar}{2}\sum_{j=k+1}^{2n}\mu_j g(t_j-t_k)\right], \quad (4)$$

$$F_2=\exp\left[\frac{k_0^2}{2}\sum_{j,k=1}^{2n}\mu_j\mu_k C(t_j-t_k)\right], \quad (5)$$

where in the Ohmic damping limit $J(\omega)=\eta\omega$ (we shall return to the more general case below),

$$g(t)=\theta(t)[1-\exp(-\eta t/m)]/\eta\,, \quad (6)$$

$$C(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\hbar J(\omega)|g(\omega)|^2\coth\frac{\beta\hbar\omega}{2}[1-\cos\omega t]$$
$$=k_B T[|t|-mg(|t|)]/\eta+O(\eta)\,, \quad (7)$$

with $g(\omega)$ being the Fourier transform of $g(t)$ [cf. below (22)]. Note that the expression for the corresponding tight-binding model [5], though appearing similar in structure, is rather different in details. Mathematically it corresponds to roughly $m\to 0$ and $\eta\to 1/\eta$ of the continuous model; see [12,13]. Our method of solution is, however, not applicable to it. They work in different parameter regimes.

The coefficients of V_0 in (2) consist of multidimensional integrals over past times. Following [13] we divide $\{t_1,t_2,\ldots,t_{2n-1},t_{2n}\}$ into irreducible neutral charge clusters according to their "charges" $\{\mu_j\}$ (imagine a one-dimensional Coulomb system, viewing each t_j as the position of a particle carrying charge μ_j). In the limit of small η, the intercluster distances are very large ($\sim m/\eta$) while the size of a neutral charge cluster becomes nearly independent of the viscosity ($\sim\sqrt{m/kT}$). Moreover, the expressions for $g(t)$ and $C(t)$ at $\eta\to 0$ reduce to $g(t)=\theta(t)t/m$ and $C(t)=k_BTt^2/2m$. Note that at $t\to 0$ quantum correction comes in, giving an asymptotic behavior of $C(t)$ of the form $C(t)\sim\eta t^2\ln\beta\hbar/t$; cf. also [3]. This correction can, however, be quite safely ignored as η is small here. For a single cluster centered at $t=\tau$, we further approximate

$$\sum_k \mu_k F_0(t_k)\cong\sum_k\mu_k\tilde{F}(\tau)t_k\,, \quad (8)$$

where the "rescaled force" $\tilde{F}(t)$ is defined by

$$\tilde{F}(t)=\frac{d}{dt}\int_{-\infty}^{t} d\tau\, g(t-\tau)F(\tau)\,. \quad (9)$$

Setting $t/(m/k_0^2V_0)^{1/2}\to t$, $F/k_0V_0\to F$, $kT/V_0\to T$, $\eta/(mk_0^2V_0)^{1/2}\to\eta$, and $\hbar k_0/2(mV_0)^{1/2}\to\Omega_q$, where Ω_q is the dimensionless form of $\hbar k_0^2/2m$ (a measure of quantum effects in frequency) we introduce the following functions and operators (see [13]):

$$y_0(\xi,\gamma)=\frac{1}{4}\left[\frac{2\pi}{T(1-\gamma^2)}\right]^{1/2}\exp\left[-\frac{\xi^2}{2T(1-\gamma^2)}\right], \quad (10)$$

$$y_i(\xi,\gamma)=\hat{Q}_i(\xi)y_0(\xi,\gamma),\quad i=1,2\,, \quad (11)$$

where

$$\hat{Q}_i(\xi)=S_i\left[\frac{Z(\hat{d}_\xi^{+}+\hat{d}_\xi^{-}-2)}{4\Omega_q^2}\right]\left[\frac{\hat{d}_\xi^{+}-\hat{d}_\xi^{-}}{2\Omega_q}\right] \quad (12)$$

with $S_1(x)=\cos(x)$ and $S_2(x)=\sin(x)$, $Z\cong 0.80$ is a numerical factor, and $\hat{d}_\xi^{\pm}$ are lattice displacement operators, $\hat{d}_\xi^{\pm}\phi(\xi)=\phi(\xi\pm\Omega_q)$. These operators can be conveniently handled via Fourier transform and using the techniques developed in [13] the equivalent "driving force" arising from summing over a single irreducible cluster centered at τ reads

$$f_{(1)}(\tau)=y_1(\tilde{F}(\tau),0)\,. \quad (13)$$

We now briefly outline the resummation over the neutral clusters. Going from left to right the distance $\tau_\alpha-\tau_{\alpha'}=\tau_{\alpha\alpha'}$ between an upper cluster (cluster α) and a lower cluster (cluster α') is in general of the order $1/\eta$, which is much larger than the intracluster distances of the order $T^{-1/2}$. The relevant functions connecting the two parts can then be approximated by

$$\sum_{j(\alpha)}\mu_{j(\alpha)}g(t_{j(\alpha)}-t_{j(\alpha')})\cong\dot{g}(\tau_{\alpha\alpha'})\sum_{j(\alpha)}\mu_{j(\alpha)}t_{j(\alpha)},\quad \sum_{j(\alpha),j(\alpha')}\mu_{j(\alpha)}\mu_{j(\alpha')}C(t_{j(\alpha)}-t_{j(\alpha')})\cong -T\left[\sum_{j(\alpha)}\mu_{j(\alpha)}t_{j(\alpha)}\right]\dot{g}(\tau_{\alpha\alpha'})\left[\sum_{j(\alpha')}\mu_{j(\alpha')}t_{j(\alpha')}\right],$$

where $\mu_{j(\alpha)}$ and $t_{j(\alpha)}$ are the "charges" and "positions" in the αth cluster. The two-cluster contribution to the driving force then has the form

$$f_{(2)}(\tau_2)=-\int_{-\infty}^{\tau_2}d\tau_1\left\{\sum_{i=1}^{2}y_i(\tilde{F}(\tau_1),0)\hat{Q}_i(\xi)y_1(\tilde{F}(\tau_2)-\xi\dot{g}(\tau_{21}),g(\tau_{21}))\right\}\Bigg|_{\xi=\tilde{F}(\tau_1)}. \quad (14)$$

The general procedure for the resummation of a multicluster term is as follows. In step I we go from the bottom to the top with fixed cluster lengths and intercluster distances. We first integrate over the intracluster degrees of freedom of the lowest cluster (cluster 1). It modifies the structures of other clusters. We then integrate over the second lowest one (cluster 2) with the modification from cluster 1. Both will modify the remaining clusters. But the effect of cluster 1 becomes indirect after adding the effect of cluster 2. [This is true only when $\dot{g}(t)$ is an exponential function. It is precisely the property that allows us to express the final result in terms of an integral equation.] This procedure can be continued to the top. In step II, we perform the resummation over V_0 for each cluster one by one from the top to the bottom. The final structure can be expressed in terms of an integral equation. Introduce an auxiliary function $y(\xi,\gamma:t,t')$ satisfying the equation

$$y(\xi,\gamma:t,t')=\xi-\sum_{i=1}^{2}y_i(\xi,\gamma)\hat{Q}_i(\xi')\int_{t'}^{t}d\tau\, y(\tilde{F}(\tau)-\xi'g(\tau-t'),\dot{g}(\tau-t'):t,\tau)|_{\xi'\to\tilde{F}(t')}. \tag{15}$$

Then the resummation yields the response

$$\langle k_0\hat{x}(t)\rangle=\int_{-\infty}^{t}dt'y(\tilde{F}(t'),0:t,t'). \tag{16}$$

We emphasize that the result of (15) and (16) is obtained from a complete resummation over the formally exact expansion in V_0. It therefore holds quite generally in the entire parameter space of T, F, and Ω_q provided $\Omega_q/\eta\gg 1$, $T/\eta^2\gg 1$, and $T[F/(dF/dt)]^2\gg 1$ (all quantities dimensionless).

Solving (15) appears rather difficult and we next discuss a self-consistent approximation. Write

$$y(\xi,\gamma:t,t')=\xi-\sum_{i=1}^{2}y_i(\xi,\gamma)A_i \tag{17}$$

with $A_i(t,t':[\tilde{F}(\tau)])$ satisfying

$$A_i(t,t':[\tilde{F}(\tau)])=g(t-t')\delta_{i,1}+\hat{Q}_i(\xi')\sum_{j=1}^{2}\left[\int_{t'}^{t}d\tau\, y_j(\tilde{F}(\tau)-\xi'g(\tau-t'),\dot{g}(\tau-t'):t,\tau)A_j(t,\tau:[\tilde{F}(\tau)])\right]\Bigg|_{\xi'\to\tilde{F}(t')}. \tag{18}$$

We now approximate the right-hand side by bringing A_j out of the integral at $\tau=t'$. (18) can then be solved self-consistently,

$$A_1=\frac{(1+C_{22})g(t-t')}{(1+C_{11})(1+C_{22})-C_{12}C_{21}}, \tag{19}$$

$$A_2=\frac{-C_{21}g(t-t')}{(1+C_{11})(1+C_{22})-C_{12}C_{21}}, \tag{20}$$

where the coefficients are

$$C_{ij}=\int_{t'}^{t}d\tau\,\hat{Q}_i(\xi)y_j(\tilde{F}(\tau)-\xi g(\tau-t'),\dot{g}(\tau-t'):t,\tau)|_{\xi\to\tilde{F}(t')}. \tag{21}$$

The above result can be generalized to more general dissipations. For an arbitrary $J(\omega)$ the Fourier transform of $g(t)$ is [12]

$$g(\omega)=\frac{1}{\omega^2}\left\{\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{2J(\omega')/\omega'}{(\omega+i0^+)^2-\omega'^2}-m\right\}^{-1}. \tag{22}$$

Though the mapping between summing over the clusters and solving the integral equation is rigorous only when $\dot{g}(t)$ is an exponential function, the integral equation (thus the self-consistent solution) may still be a good approximation since the distances between the clusters are large. The equation remains formally the same, but one may need to replace the dimensionless temperature by an effective one,

$$\tilde{T}=\frac{2m}{V_0\beta}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega J(\omega)|g(\omega)|^2. \tag{23}$$

This concludes our general analysis. Upon substituting an arbitrary external force $F(t)$, it yields the analytical expression for the response. The result has many interesting features and more detailed studies will be presented elsewhere.

For a constant force the dominant contribution comes from $t-t'\to\infty$. In this case, *the self-consistency is exact*. Figure 1 plots as an example the resulting stationary velocities for selected parameters, which illustrate nicely the effect of the Bloch states. For a *free* particle (i.e., no cosine potential) driven by the external force, the peaks (located at $\Omega_q/2$ and $3\Omega_q/2$) in the curves correspond to having terminal wave vectors at the Brillouin zone boundaries. Thus these peaks are related to the Bragg scatter-

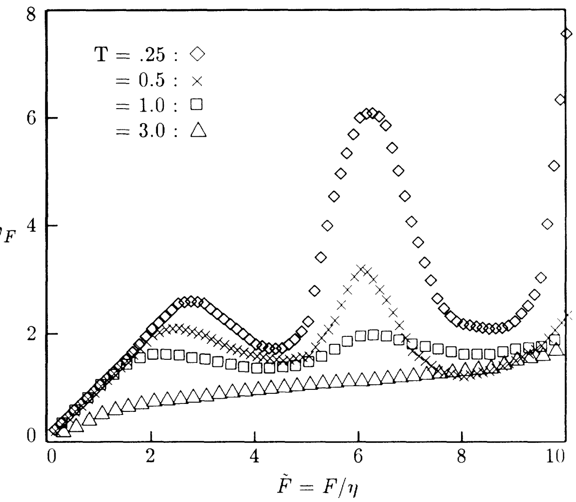

FIG. 1. The curves of the nonlinear stationary velocity vs the rescaled force for different temperature T at $\Omega_q = 4.0$ and $\eta \rightarrow 0$.

ing with the periodic potential [13]. Increase in F could then lead the system to either go up to the next band via Zener tunnelings or remain at the lower band with a lower velocity. In a realistic situation one needs to add finite temperature and quantum coherence effects. It is these effects which are treated here, in the context of the Caldeira-Leggett model in a unified way. Note that in Fig. 1 there are multisegments of negative differential regimes. This means that at low temperatures higher bands also play roles. But it is very sensitive to the temperature and the viscosity.

We close with a brief discussion of applications to a small Josephson junction. The intrinsic parameters of the latter are $m = (\hbar/2e)^2 C$, $k_0 = 1$, and $V_0 = E_J = I_e \hbar/2e$. Thus the dimensionless quantities read (assuming the usual resistively shunted junction model; cf. below) $F = I/I_c$, $\eta = (\hbar/2eI_cCR^2)^{1/2}$, and $\Omega_q = (2E_C/E_J)^{1/2}$, where E_C represents the Coulomb energy of a single electron sitting at the junction. The curves of stationary velocity versus external force correspond to the full I-V curve in a current bias setup. A semiquantitative fit of our theory with experiment [14] was found in [13]. Finally, due to the pure harmonicity of the environments our result should be particularly useful when the effects of leads or other electromagnetic couplings are dominant [15].

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